Energy Derivatives

Lecture Notes
LSE

VERSION: April 19, 2006

Prof.Dr. Rüdiger Kiesel
Department of Statistics, LSE
and
Department of Financial Mathematics
University of Ulm

email:ruediger.kiesel@uni-ulm.de

Short Description.

Content.
Within the last few years the markets for commodities, in particular energy-related commodities, has changed substantially. New regulations and products have resulted in a spectacular growth in spot and derivative trading. In particular, electricity markets have changed fundamentally over the last couple of years. Due to deregulation energy companies are now allowed to trade not only the commodity electricity, but also various derivatives on electricity on several Energy Exchanges (such as the EEX).

Specific topics

1. Basic Principles of Commodity Markets, models for forwards and futures.
2. Stylized facts of electricity markets; statistical analysis of spot and futures markets.
3. Spot and Forward Market Models for Electricity, mathematical models based on Lévy processes (including a short intro to such processes).
4. Special derivatives for the Electricity markets.

Literature.

– Eydeland, A. Wolyniec, K.: *Energy and Power Risk Management*, Wiley 2003

– Geman, H.: *Commodities and Commodity Derivatives*, Wiley 2005.

course webpage.
www.mathematik.uni-ulm.de/finmath
email: ruediger.kiesel@uni-ulm.de.

Contents

Chapter 1

Fundamentals

1.1 Markets and Price Processes

Since the deregulation of electricity markets in the end of the 1990s, power can be traded at exchanges like the Nordpool or the European Energy Exchange (EEX). All exchanges have established spot and futures markets.
The spot market usually is organised as an auction, which manages the distribution of power in the near future, i. e. one day ahead. Empirical studies, such as Knittel and Roberts (2001) using hourly prices in the California power market, show that spot prices exhibit seasonalities on different time scales, a strong mean-reversion and are very volatile and spiky in nature. Because of inherent properties of electricity as an almost non-storable commodity such a price behaviour has to be expected, see Geman (2005).
Due to the volatile behaviour of the spot market and to ensure that power plants can be deployed optimally, **power forwards and futures** are traded. Power exchanges established the trade of forwards and futures early on and by now large volumes are traded. A power forward contract is characterized by a fixed delivery price per MWh, a delivery period and the total amount of energy to deliver. Especially the length of the delivery period and the exact time of delivery determine the value and statistical characteristics of the contract vitally. One can observe, that contracts with a long delivery period show less volatile prices than those with short delivery. These facts give rise to a a term structure of volatility in most power forward markets, which has to be modelled accurately in order to be able to price options on futures. Figure 1.1 gives an example of such a term structure for futures traded at the EEX. Additionally, seasonalities can be observed in the forward curve within a year. Monthly contracts during winter months show higher prices than comparable contracts during the summer (cp. Figure 1.2).
Aside from spot and forward markets, valuing options is an issue for market participants. While some research has been done on the valuation of options on spot power, hardly any results can be found on options on forwards and futures. Both types impose different problems for the valuation.
Spot options fail most of the arbitrage and replication arguments, since power is almost non-storable. Some authors take the position to find a realistic model to describe the prices of spot prices and then value options via risk-neutral expectations (cp. de Jong and Huisman (2002), Benth, Dahl, and Karlsen (2004), Burger, Klar, Müller, and Schindlmayr (2004)). Other ideas explicitly take care of the special situation in the electricity production and use power plants to replicate certain contingent claims (cp. Geman and Eydeland (1999)).
Forward and futures options are heavily influenced by the length of their delivery period and their time to maturity. In Clewlow and Strickland (1999), for example, a one-factor model is presented, that tries to fit the term structure of volatility, but that does not incorporate a delivery period, since it is constructed for oil and gas markets.
As an example let us have a look at the EEX spot market. Here we have the following structure

- the EEX spot market is a day-ahead auction for single hours of the following day

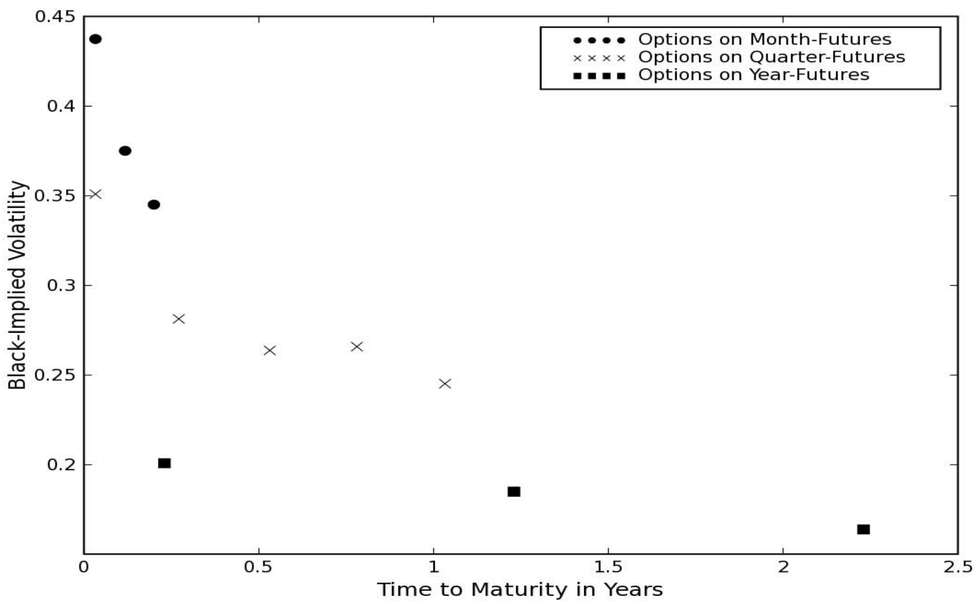


Figure 1.1: Implied volatilities of futures with different maturities and delivery periods, Sep. 14

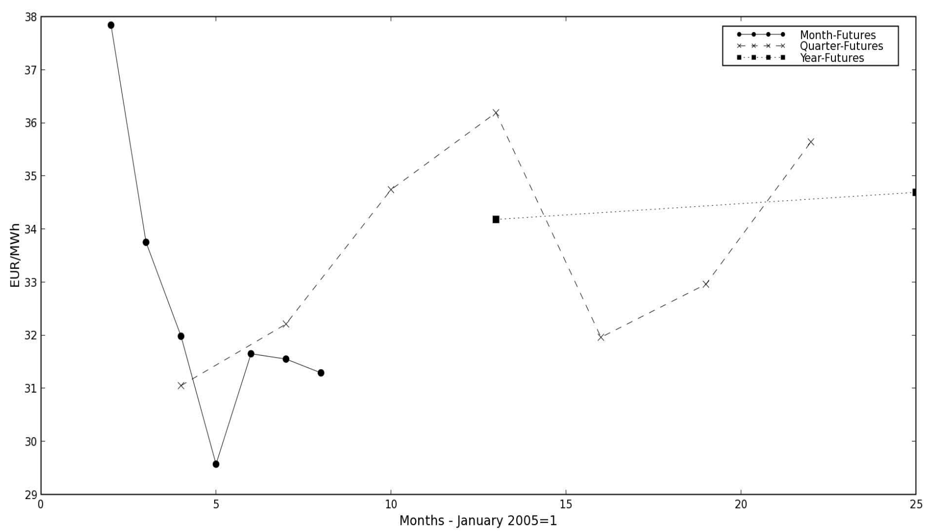
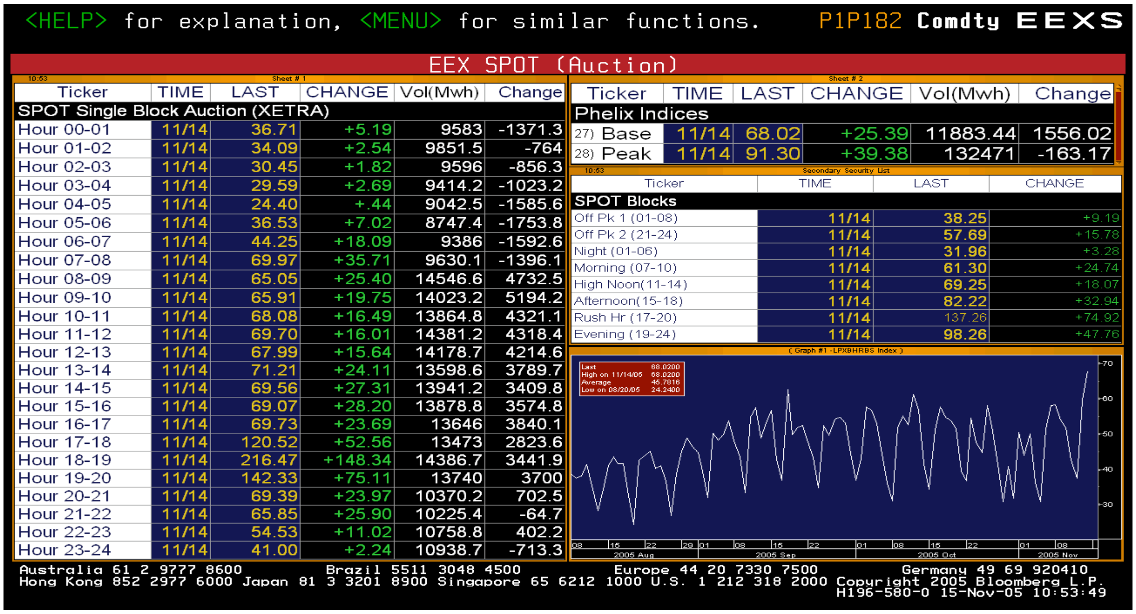


Figure 1.2: Forward prices of futures with different maturities and delivery periods, Feb. 18



<HELP> for explanation, <MENU> for similar functions. P1P182 Comdty EEXS

EEX SPOT (Auction)

Ticker	TIME	LAST	CHANGE	Vol(Mwh)	Change
SPOT Single Block Auction (XETRA)					
Hour 00-01	11/14	36.71	+5.19	9583	-1371.3
Hour 01-02	11/14	34.09	+2.54	9851.5	-764
Hour 02-03	11/14	30.45	+1.82	9596	-856.3
Hour 03-04	11/14	29.59	+2.69	9414.2	-1023.2
Hour 04-05	11/14	24.40	+.44	9042.5	-1585.6
Hour 05-06	11/14	36.53	+7.02	8747.4	-1753.8
Hour 06-07	11/14	44.25	+18.09	9386	-1592.6
Hour 07-08	11/14	69.97	+35.71	9630.1	-1396.1
Hour 08-09	11/14	65.05	+25.40	14546.6	4732.5
Hour 09-10	11/14	65.91	+19.75	14023.2	5194.2
Hour 10-11	11/14	68.08	+16.49	13864.8	4321.1
Hour 11-12	11/14	69.70	+16.01	14381.2	4318.4
Hour 12-13	11/14	67.99	+15.64	14178.7	4214.6
Hour 13-14	11/14	71.21	+24.11	13598.6	3789.7
Hour 14-15	11/14	69.56	+27.31	13941.2	3409.8
Hour 15-16	11/14	69.07	+28.20	13878.8	3574.8
Hour 16-17	11/14	69.73	+23.69	13646	3840.1
Hour 17-18	11/14	120.52	+52.56	13473	2823.6
Hour 18-19	11/14	216.47	+148.34	14386.7	3441.9
Hour 19-20	11/14	142.33	+75.11	13740	3700
Hour 20-21	11/14	69.39	+23.97	10370.2	702.5
Hour 21-22	11/14	65.85	+25.90	10225.4	-64.7
Hour 22-23	11/14	54.53	+11.02	10758.8	402.2
Hour 23-24	11/14	41.00	+2.24	10938.7	-713.3

Ticker	TIME	LAST	CHANGE	Vol(Mwh)	Change
Phelix Indices					
27) Base	11/14	68.02	+25.39	11883.44	1556.02
28) Peak	11/14	91.30	+39.38	132471	-163.17

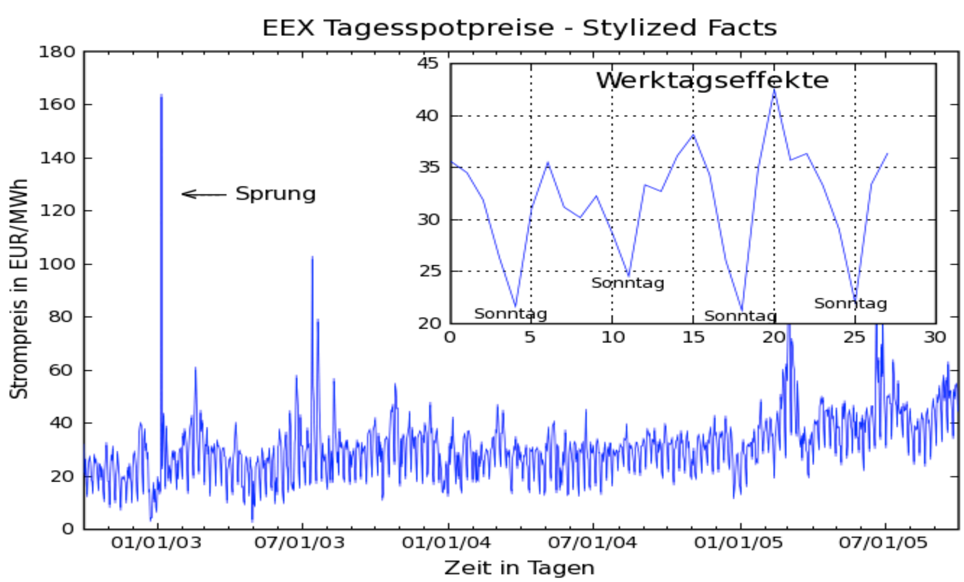
Ticker	TIME	LAST	CHANGE
SPOT Blocks			
Off Pk 1 (01-08)	11/14	38.25	+9.19
Off Pk 2 (21-24)	11/14	57.69	+15.78
Night (01-06)	11/14	31.96	+3.28
Morning (07-10)	11/14	61.30	+24.74
High Noon(11-14)	11/14	69.25	+18.07
Afternoon(15-18)	11/14	82.22	+32.94
Rush Hr (17-20)	11/14	137.26	+74.92
Evening (19-24)	11/14	98.26	+47.76

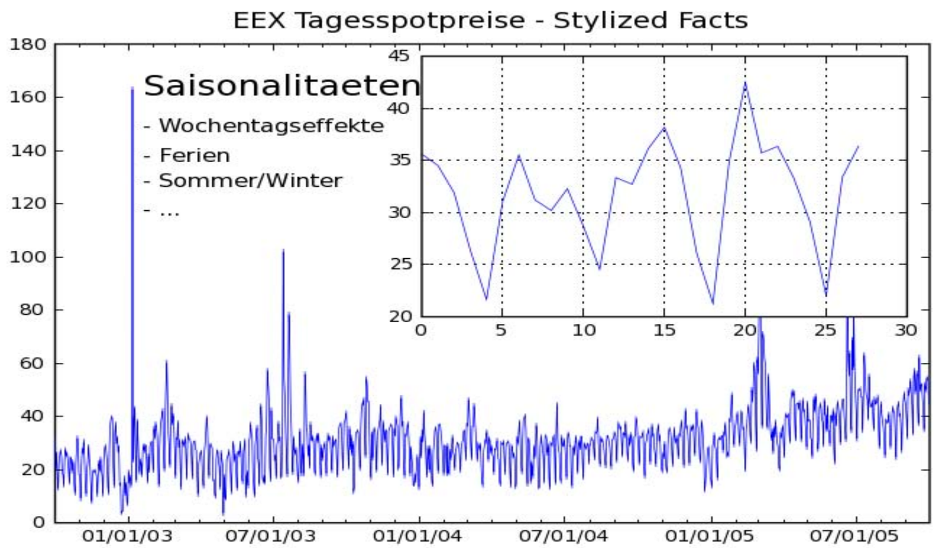
Australia 61 2 9777 8600 Brazil 5511 3048 4500 Europe 44 20 7330 7500 Germany 49 69 920410
Hong Kong 852 2977 6000 Japan 81 3 3201 8900 Singapore 65 6212 1000 U.S. 1 212 318 2000 Copyright 2005 Bloomberg L.P.
H196-580-0 15-Nov-05 10:53:49

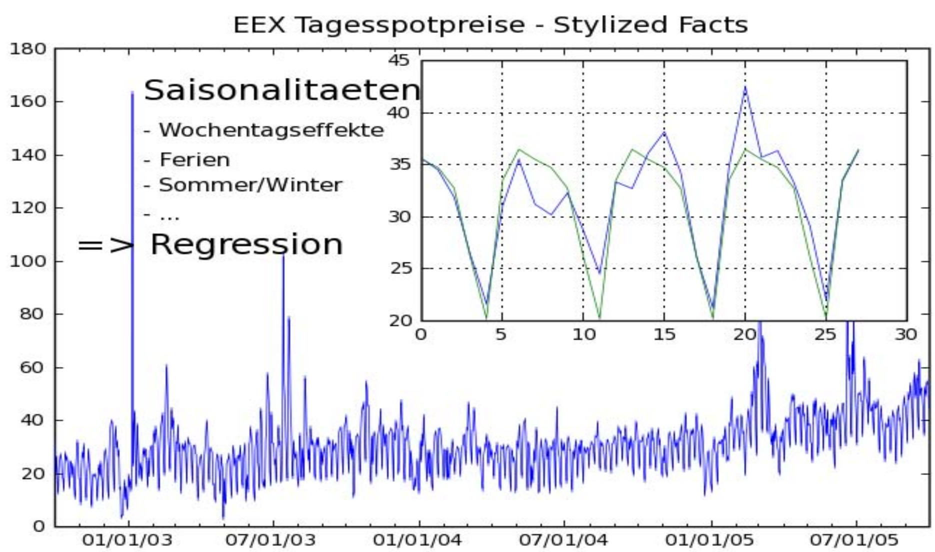
Figure 1.3: Bloomberg screen for energy spot prices

- participants submit their price offer/bit curves, the EEX system prices are equilibrium prices that clear the market.
- EEX day prices are the average of the 24-single hours.
- on fridays the hours for the whole weekend are auctioned.
- similar structures can be found on other power exchanges(Nord Pool, APX, etc.).

The following are examples of price processe



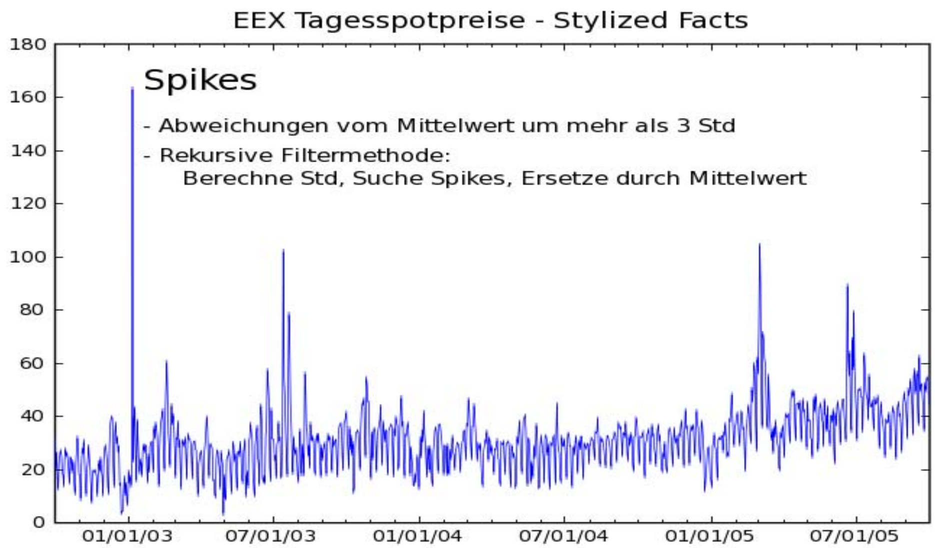


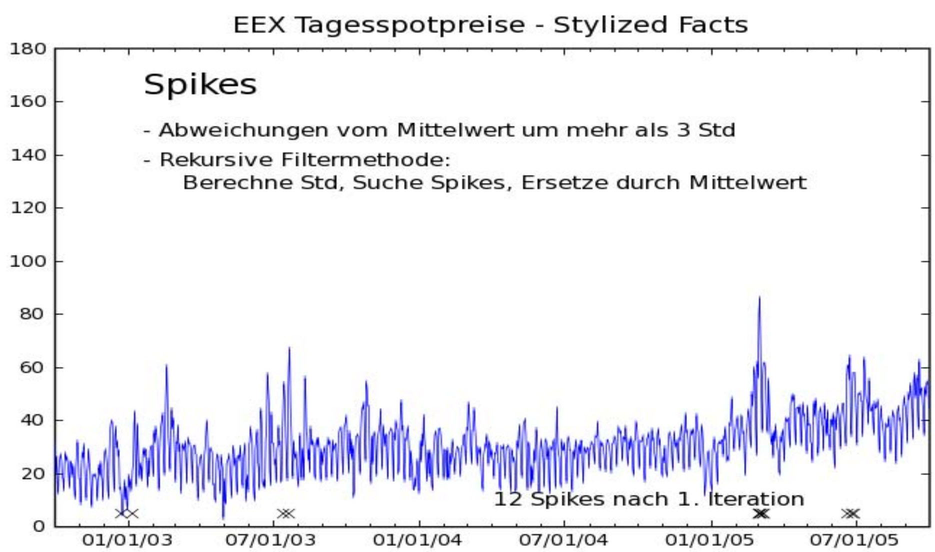


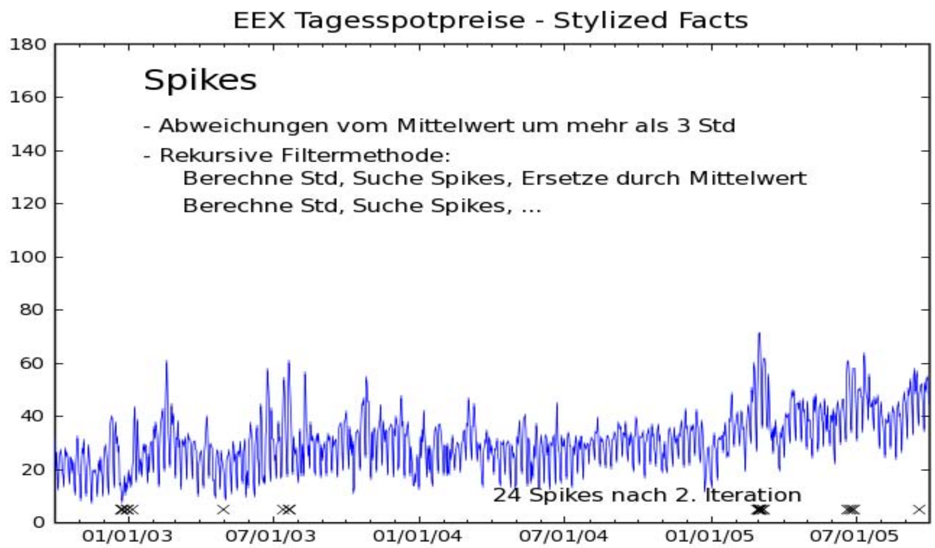
To analyse seasonalities one can preform a regression analysis. This can be done by standard methods assuming a model for the mean, e. g.

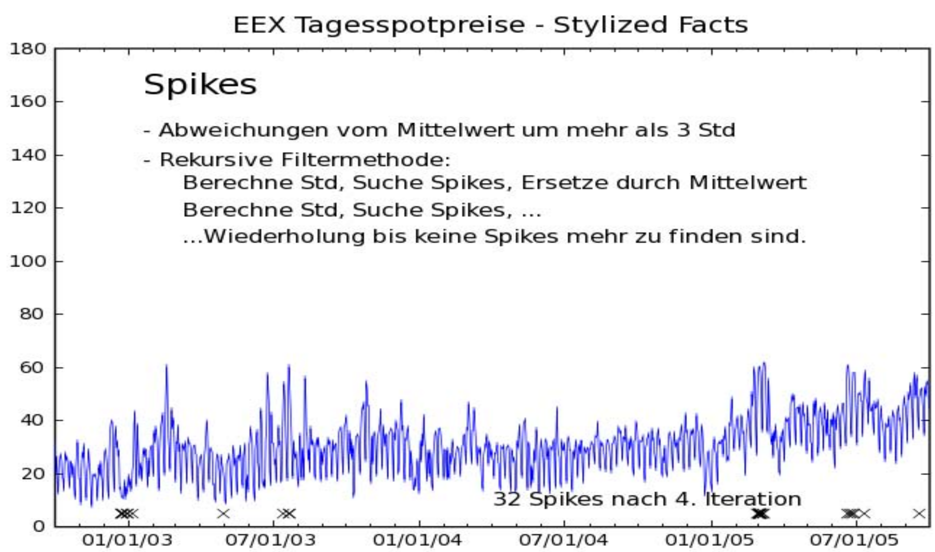
$$
\begin{aligned}
S_t = \; & \beta_1 \cdot 1(\text{if } t \in \text{Mondays}) + \ldots + \beta_7 \cdot 1(\text{if } t \in \text{Sundays}) \\
& + \text{other calendar day effects} \\
& + \beta_8 \cdot t \text{ for long term linear trend} \\
& + \beta_9 \sin(\frac{2\pi}{365}(t-c)) \text{ for summer/winter seasonality} \\
& + \ldots
\end{aligned}
$$

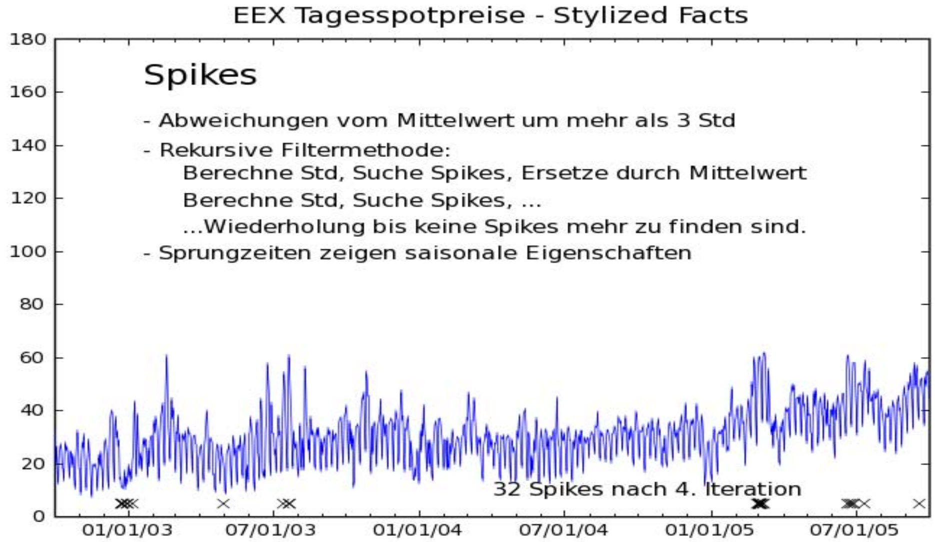
The unknown parameters $\beta_1, \ldots, \beta_p$ can be estimated easily by Least-Squares-Methods.
We also observe spikes


EEX Tagesspotpreise - Stylized Facts
Spikes
- Abweichungen vom Mittelwert um mehr als 3 Std
- Rekursive Filtermethode:
Berechne Std, Suche Spikes, Ersetze durch Mittelwert
180
160
140
120
100
80
60
40
20
0
01/01/03
07/01/03
01/01/04
07/01/04
01/01/05
07/01/05


EEX Tagesspotpreise - Stylized Facts
Spikes
- Abweichungen vom Mittelwert um mehr als 3 Std
- Rekursive Filtermethode:
Berechne Std, Suche Spikes, Ersetze durch Mittelwert
12 Spikes nach 1. Iteration
180
160
140
120
100
80
60
40
20
0
01/01/03
07/01/03
01/01/04
07/01/04
01/01/05
07/01/05


EEX Tagesspotpreise - Stylized Facts
Spikes
- Abweichungen vom Mittelwert um mehr als 3 Std
- Rekursive Filtermethode:
Berechne Std, Suche Spikes, Ersetze durch Mittelwert
Berechne Std, Suche Spikes, ...
24 Spikes nach 2. Iteration
180
160
140
120
100
80
60
40
20
0
01/01/03
07/01/03
01/01/04
07/01/04
01/01/05
07/01/05

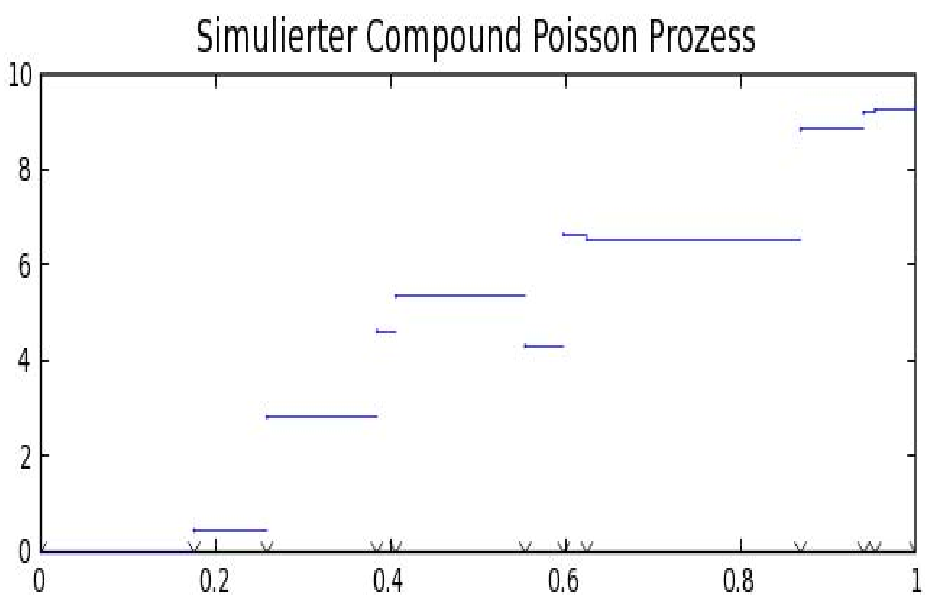

EEX Tagesspotpreise - Stylized Facts
Spikes
- Abweichungen vom Mittelwert um mehr als 3 Std
- Rekursive Filtermethode:
Berechne Std, Suche Spikes, Ersetze durch Mittelwert
Berechne Std, Suche Spikes, ...
...Wiederholung bis keine Spikes mehr zu finden sind.
32 Spikes nach 4. Iteration
180
160
140
120
100
80
60
40
20
0
01/01/03
07/01/03
01/01/04
07/01/04
01/01/05
07/01/05

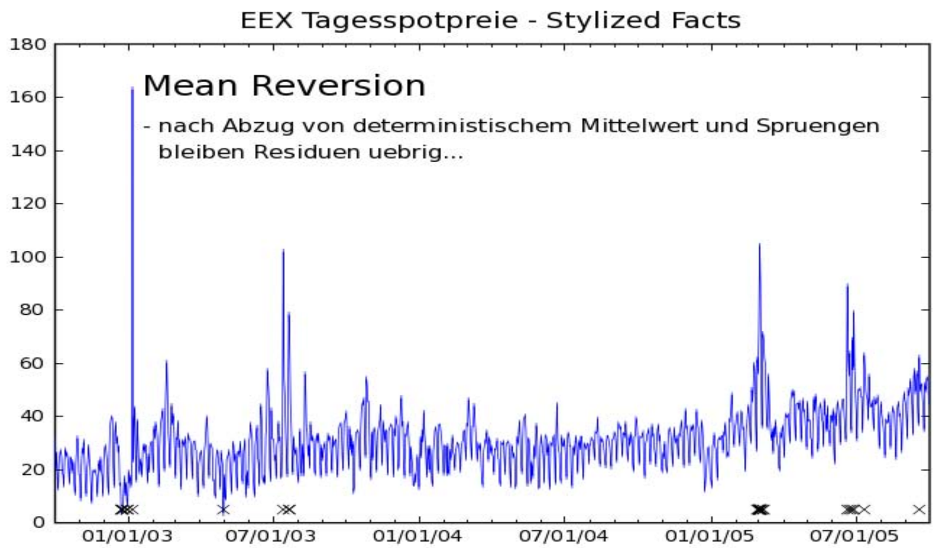


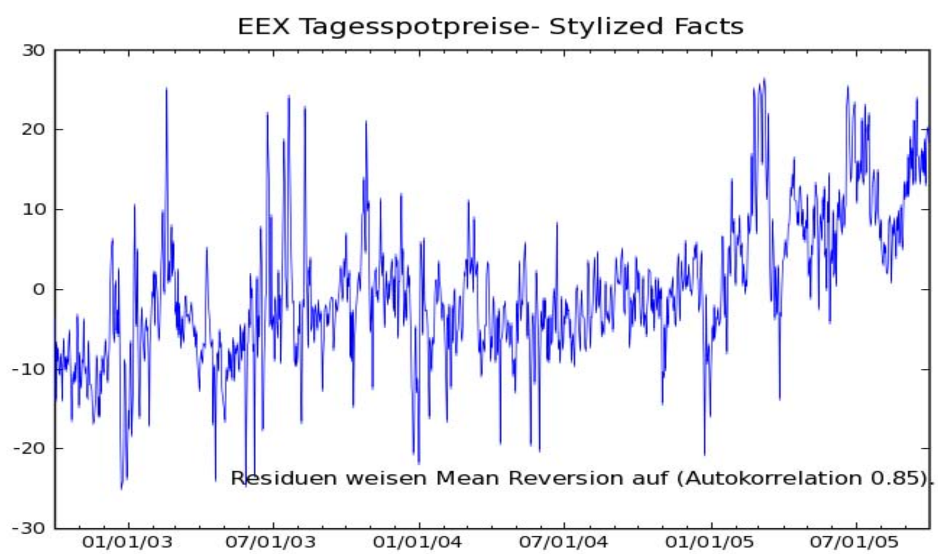
Spikes are often modelled by jumps. The process of jumps is often modelled by a compound poisson process

$$CP_t := \sum_{i=1}^{N_t} J_i$$

N_t is a Poisson process with intensity λ, which randomly jumps by 1 unit, so it counts how many jumps occurred up to time t. J_i is the random jump size of the ith jump.







1.2 Basic Products and Structures

We mostly have been dealing so far with derivatives based on underlying assets – stock – existing, and available for trading, now. It frequently happens, however, that the underlying assets relevant in a particular market will instead be available at some time in the future, and need not even exist now. Obvious examples include crop commodities – wheat, sugar, coffee etc. – which might not yet be planted, or be still growing, and so whose eventual price remains uncertain – for instance, because of the uncertainty of future weather. The principal factors determining yield of crops such as cereals, for instance, are rainfall and hours of sunshine during the growing season. Oil, gas, coals are another example of a commodity widely traded in the future, and here the uncertainty is more a result of political factors, shipping costs etc. Our focus here will be on electricity later on. Financial assets, such as currencies, bonds and stock indexes, may also be traded in the future, on exchanges such as the London International Financial Futures and Options Exchange (LIFFE) and the Tokyo International Financial Futures Exchange (TIFFE), and we shall restrict attention to financial futures for simplicity.

We thus have the existence of two parallel markets in some asset – the *spot market*, for assets traded in the present, and the *futures market*, for assets to be realized in the future. We may also consider the combined spot-futures market.

We now want briefly look at the most important products.

1.2.1 Forwards

A *forward contract* is an agreement to buy or sell an asset S at a certain future date T for a certain price K. The agent who agrees to buy the underlying asset is said to have a *long* position, the other agent assumes a *short* position. The settlement date is called *delivery date* and the specified price is referred to as *delivery price*. The *forward price* $F(t, T)$ is the delivery price which would make the contract have zero value at time t. At the time the contract is set up, $t = 0$, the forward price therefore equals the delivery price, hence $F(0, T) = K$. The forward prices $F(t, T)$ need not (and will not) necessarily be equal to the delivery price K during the life-time of the contract.

The payoff from a long position in a forward contract on one unit of an asset with price $S(T)$ at the maturity of the contract is

$$S(T) - K.$$

Compared with a call option with the same maturity and strike price K we see that the investor now faces a downside risk, too. He has the obligation to buy the asset for price K.

1.2.2 Futures Markets

Futures prices, like spot prices, are determined on the floor of the exchange by supply and demand, and are quoted in the financial press. *Futures contracts*, however – contracts on assets traded in the futures markets – have various special characteristics. Parties to futures contracts are subject to a daily settlement procedure known as *marking to market*. The initial deposit, paid when the contract is entered into, is adjusted daily by *margin payments* reflecting the daily movement in futures prices. The underlying asset and price are specified in the contract, as is the delivery date. Futures contracts are highly liquid – and indeed, are intended more for trading than for delivery. Being assets, futures contracts may be the subject of *futures options*.

We shall as before write $t = 0$ for the time when a contract, or option, is written, t for the present time, T for the expiry time of the option, and T^* for the delivery time specified in the futures (or forward) contract. We will have $T^* \geq T$, and in general $T^* > T$; beyond this, T^* will not affect the pricing of options with expiry T.

Swaps

A *swap* is an agreement whereby two parties undertake to exchange, at known dates in the future, various financial assets (or cash flows) according to a prearranged formula that depends on the value of one or more underlying assets. Examples are currency swaps (exchange currencies) and interest-rate swaps (exchange of fixed for floating set of interest payments).

1.3 Basic Pricing Relations

1.3.1 Storage, Inventory and Convenience Yield

The theory of storage aims to explain the differences between spot and Futures (Forward) prices by analyzing why agents hold inventories. Inventories allow to meet unexpected demand, avoid the cost of frequent revisions in the production schedule and eliminate manufacturing disruption. This motivates the concept of convenience yield as a benefit, that accrues to the owner of the

physical commodity but not to the holder of a forward contract. Thus the convenience yield is comparable to the dividend yield for stocks. A modern view is to view storage (inventory) as a timing option, that allows to put the commodity to the market when prices are high and hold it when the prices are low.
We will model the convenience yield y

- expressed as a rate, meaning that the benefit in a monetary amount for the holder of the commodity will be equal to $S(t)ydt$ over the interval $(t, t+dt)$, if $S(t)$ is the spot price at time t;
- defined as the difference between the positive gain attached to the physical commodity minus the cost of storage. Hence the convenience yield may be positive or negative depending on the period, commodity and cost of storage.

In recent literature the convenience yield is often modelled as a random variable, which allows to explain various shapes of forward curves over time.

1.3.2 Futures Prices and Expectation of Future Spot Prices

The rational expectation hypothesis (REH) (mainly used in the context of interest rates) states that the current futures price $f(t,T)$ for a commodity (interest rate) with delivery a time $T > t$ is the best estimator for the price $S(T)$ of the commodity. In mathematical terms

$$f(t,T) = \mathbb{E}[S(T)|\mathcal{F}_t]. \tag{1.1}$$

where $\mathcal{F}_t$ represents the information available at time t. The REH has been statistically tested in many studies for a wide range of commodities (resulting most of the time in a rejection).
When equality in (1.1) does not hold futures prices are biased estimators of future spot prices. If

- $>$ holds, then $f(t,T)$ is an up-ward biased estimate, then risk-aversion among market participants is such that buyers are willing to pay more than the expected spot price in order to secure access to the commodity at time T (political unrest);
- $<$ holds, then $f(t,T)$ is an down-ward biased estimate, this may reflect a perception of excess supply in the future.

No general theory for the bias has been developed. It may depend on the specific commodity, the actual forecast of the future spot price by market participant, and on the risk aversion of the participants.

1.3.3 Spot-Forward Relationship in Commodity Markets

Under the no-arbitrage assumption we have

$$F(t,T) = S(t)e^{(r-y)(T-t)} \tag{1.2}$$

where r is the interest rate at time t for maturity T and y is the convenience yield. We start by proving this relationship for stocks as underlying

Non-dividend paying stocks

Consider the portfolio

	t	T
buy stock	$-S(t)$	delivery
borrow to finance	$S(t)$	$-S(t)e^{r(T-t)}$
sell forward on S		F(t,T)

All quantities are known at t, the time t cash-flow is zero, so the cash-flow at T needs to be zero so we have

$$F(t,T) = S(t)e^{r(T-t)} \tag{1.3}$$

dividend paying stocks

Assume a continuous dividend κ, so we have a dividend rate of $\kappa S(t)dt$, which is immediately reinvested in the stock. We thus have a growth rate of $e^{g(T-t)}$ over the period of the quantity of stocks detained. Thus we only have to buy $e^{-g(T-t)}$ shares of stock S at time time. Replace in the above portfolio and obtain

$$F(t,T) = S(t)e^{(r-g)(T-t)} \tag{1.4}$$

storable commodity

Here the convenience yield plays the role of the dividend an we obtain (1.2).
In case of a linear rate this relationship is of the form

$$F(t,T) = S(t)\left\{1 + (r-y)(T-t)\right\}.$$

With the decomposition $y = y_1 - c$ with y_1 the *benefit from the physical commodity* and c the *storage cost* we have

$$F(t,T) = S(t)\left\{1 + r(T-t) + c(T-t) - y_1(T-t)\right\}.$$

Observe that (1.2) implies

(i) spot and forward are redundant (one can replace the other) and form a linear relationship (unlike options)

(ii) with two forward prices we can derive the value of $S(t)$ and y

(iii) knowledge of $S(t)$ and y allows us to construct the whole forward curve

(iv) for $r - y < 0$ we have backwardation; for $y - r > 0$ we have contango.

1.3.4 Futures Pricing Relations

We start by discussing the subtle but important issue of the difference of the price of a Futures contract i.e. at which we can buy or sell the contract today (for payment at maturity) and the value of a position build in the past and containing this contract.
So consider a Futures contract with a fixed maturity T and a designated underlying.
The price $f(0,T)$ of this contract is defined as the Euro amount the buyer of the contract agrees to pay at date T in order to take delivery of the underlying at date T.
A day later (at t_1) the price of the same contract is $f(t_1,T)$ and may (and will) be different from $f(0,T)$.
So the buyer (the long position) is facing a loss/gain equal to

$$f(t_1,T) - f(0,T)$$

and needs to pay a margin call equal to this amount to the clearing house (Futures exchange). Assuming the position is not closed until maturity T we get

$$f(T,T) - f(0,T) = \underbrace{f(T,T) - f(t_{n-1},T)}_{\text{last day}} + \ldots + \underbrace{f(t_1,T) - f(0,T)}_{\text{first day}}.$$

So the left-hand side represents the profit and loss of a long position P in the Futures contract initiated at time 0. Denoting by $V_p(t)$ the market value of this position at any date t between 0 and T, we know $V_p(0) = 0$ (since this is how the contract is priced).
Also by the convergence assumption $f(T,T) = S(T)$ since it is equivalent to buy a commodity on the spot market and a Futures contract that matures immediately.

In order to find the value $V_p(t)$ of a portfolio containing a Futures contract purchased at date $t=0$ for delivery at T consider the portfolio consisting of P and a short position P' in a Futures contract entered in at time t.
Payoff at T is

$$\begin{array}{lll} P & : & -f(0,T) & \text{(buy commodity)} \\ P' & : & f(t,T) & \text{(sell commodity)} \\ P''=P+P' & : & f(t,T)-f(0,T) & \end{array}.$$

So $V_{P+P'}(T) = f(t,T) - f(0,t)$ and P'' is riskless at time t since the value of all cash flows is known, so by no arbitrage

$$V_{P''}(t) = e^{r(T-t)}(f(t,T) - f(0,T)).$$

Since the value $V_{P'}(t)$ is zero (recall no payment needed to enter a Futures contact) we have

$$V_P(t) = V_{P''}(t) - V_{P'}(t) = e^{-r(T-t)}(f(t,T) - f(0,T)).$$

So the value of a futures contract entered in at 0 at time t is

$$V_P(t) = e^{-r(T-t)}(f(t,T) - f(0,T)) \tag{1.5}$$

Despite their fundamental differences, futures prices $f(t,T)$ and the corresponding forward prices $F(t,T)$, are closely linked. We use the notation $p(t,T)$ for the bond price process.

Proposition 1.3.1. *If the bond price process $p(t,T)$ is predictable, the combined spot-futures market is arbitrage-free if and only if the futures and forward prices agree: for every underlying S and every $t \le T$,*

$$f_S(t,T) = F_S(t,T).$$

In the important special case of the futures analogue of the Black-Scholes model the bond price process – or interest-rates process – is deterministic, so predictable. We thus only consider the case of deterministic interest rates and a non-dividend paying stock as underlying.
Observe:

(i) Under deterministic or stochastic interest rates the spot-forward relationship is

$$F(t,T) = \frac{S(t)}{p(t,T)}$$

with $p(t,T)$ the price at date t of a zero-coupon bound.

(ii) consider the following sequence of investments in the period $[t,T]$ with subperiods $t, t+1, \dots, T$

at t: take a long position in $1/p(t,t+1)$ Futures contracts with maturity T

at $t+1$: close this position and invest the proceeds $\frac{1}{p(t,t+1)}\{f(t+1,T) - f(t,T)\}$ on a daily basis until date T with final wealth

$$\frac{1}{p(t,t+1)}\{f(t+1,T) - f(t,T)\}\frac{1}{p(t+1,t+2)\cdot \ldots \cdot p(T-1,T)}.$$

Also take a long position in $\frac{1}{p(t,t+1)p(t+1,t+2)}$ Futures contracts with maturity T.

at $t+2$: close/open positions as above.

at date T: we have the aggregate position

$$\frac{1}{p(t,t+1)\ldots p(T-1,T)}\{f(T;T) - f(t-1,T) + \ldots + f(t+1,T) - f(t,T)\}$$
$$= \frac{f(T,T) - f(t,T)}{p(t,t+1)\cdot \ldots \cdot p(T-1,T)}.$$

Lastly add a position of an investment of $f(t,T)$ Euros in a roll-over lending up to time T, which provides a wealth at T of

$$\frac{1}{p(t,t+1)\cdot \ldots \cdot p(T-1,T)}f(t,T).$$

By addition the portfolio value is

$$\frac{1}{p(t,t+1)\cdot\ldots\cdot p(T-1,T)} f(T,T)$$

and required an initial wealth of $f(t,T)$ since no payments are needed to enter a Futures contract.

In case of deterministic interest rates we find from the no-arbitrage condition

$$p(t,t+1)\cdot\ldots\cdot p(T-1,T) = p(t,t).$$

From $f(T,T) = S(T)$ the final value of the portfolio can thus be written as $\frac{S(T)}{p(t,T)}$ and required an investment of $f(t,T)$.

(iii) The position of buying at t $\frac{1}{p(t,T)}$ shares and keeping them until T requires an investment of $S(t)/p(t,T)$ and has a terminal value of $\frac{S(T)}{p(t,T)}$.

So (ii) and (iii) yield portfolios with same value at date T in all states of the world. By no-arbitrage (observe no in/out-flow of funds) they have the same value at any time t, in particular

$$f(t,T) = \frac{S(t)}{p(t,T)} = F(t,T).$$

1.4 Pricing Formulae for Options

1.4.1 Black-Scholes Formula

The Model

Recall the classical Black-Scholes model

$$\begin{aligned} dB(t) &= rB(t)dt, & B(0) &= 1, \\ dS(t) &= S(t)\left(bdt + \sigma dW(t)\right), & S(0) &= p \in (0,\infty), \end{aligned}$$

with constant coefficients $b \in I\!R,\ r,\sigma \in I\!R_+$. We write as usual $\tilde{S}(t) = S(t)/B(t)$ for the discounted stock price process (with the bank account being the natural numéraire), and get from Itô's formula

$$d\tilde{S}(t) = \tilde{S}(t)\left\{(b-r)dt + \sigma dW(t)\right\}.$$

Equivalent Martingale Measure

Because we use the Brownian filtration any pair of equivalent probability measures $I\!P \sim \mathbb{Q}$ on $\mathcal{F}_T$ is a Girsanov pair, i.e.

$$\left.\frac{d\mathbb{Q}}{dI\!P}\right|_{\mathcal{F}_t} = L(t)$$

with

$$L(t) = \exp\left\{-\int_0^t \gamma(s)dW(s) - \frac{1}{2}\int_0^t \gamma(s)^2 ds\right\},$$

and $(\gamma(t) : 0 \le t \le T)$ a measurable, adapted d-dimensional process with $\int_0^T \gamma(t)^2 dt < \infty \quad a.s.$. By Girsanov's theorem A.1.4 we have

$$dW(t) = d\tilde{W}(t) - \gamma(t)dt,$$

where $\tilde{W}$ is a $\mathbb{Q}$-Wiener process. Thus the $\mathbb{Q}$-dynamics for $\tilde{S}$ are

$$d\tilde{S}(t) = \tilde{S}(t)\left\{(b - r - \sigma\gamma(t))dt + \sigma d\tilde{W}(t)\right\}.$$

Since $\tilde{S}$ has to be a martingale under $\mathbb{Q}$ we must have

$$b - r - \sigma\gamma(t) = 0 \quad t \in [0, T],$$

and so we must choose

$$\gamma(t) \equiv \gamma = \frac{b - r}{\sigma},$$

(the 'market price of risk'). Indeed, this argument leads to a unique martingale measure, and we will make use of this fact later on. Using the product rule, we find the $\mathbb{Q}$-dynamics of S as

$$dS(t) = S(t)\left\{rdt + \sigma d\tilde{W}\right\}.$$

We see that the appreciation rate b is replaced by the interest rate r, hence the terminology risk-neutral (or yield-equating) martingale measure.
We also know that we have a unique martingale measure $\mathbb{P}^*$ (recall $\gamma = (b - r)/\sigma$ in Girsanov's transformation).

Pricing and Hedging Contingent Claims

Recall that a contingent claim X is a $\mathcal{F}_T$-measurable random variable such that $X/B(T) \in L^1(\Omega, \mathcal{F}_T, \mathbb{P}^*)$. (We write $\mathbb{E}^*$ for $\mathbb{E}_{\mathbb{P}^*}$ in this section.) By the risk-neutral valuation principle the price of a contingent claim X is given by

$$\Pi_X(t) = e^{\{-r(T-t)\}}\mathbb{E}^*\left[X|\,\mathcal{F}_t\right],$$

with $\mathbb{E}^*$ given via the Girsanov density

$$L(t) = \exp\left\{-\left(\frac{b-r}{\sigma}\right)W(t) - \frac{1}{2}\left(\frac{b-r}{\sigma}\right)^2 t\right\}.$$

Now consider a European call with strike K and maturity T on the stock S (so $\Phi(T) = (S(T) - K)^+$), we can evaluate the above expected value (which is easier than solving the Black-Scholes partial differential equation) and obtain:

Proposition 1.4.1 (Black-Scholes Formula)**.** *The Black-Scholes price process of a European call is given by*

$$C(t) = S(t)\Phi(d_1(S(t), T - t)) - Ke^{-r(T-t)}\Phi(d_2(S(t), T - t)). \tag{1.6}$$

The functions $d_1(s, t)$ and $d_2(s, t)$ are given by

$$\begin{aligned} d_1(s,t) &= \frac{\log(s/K) + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}, \\ d_2(s,t) &= d_1(s,t) - \sigma\sqrt{t} = \frac{\log(s/K) + (r - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}} \end{aligned}$$

We can also use an arbitrage approach to derive the Black-Scholes formula. For this consider a self-financing portfolio which has dynamics

$$dV_\varphi(t) = \varphi_0(t)dB(t) + \varphi_1(t)dS(t) = (\varphi_0(t)rB(t) + \varphi_1(t)\mu S(t))dt + \varphi_1(t)\sigma S(t)dW(t).$$

Assume that the portfolio value can be written as

$$V_\varphi(t) = V(t) = f(t, S(t))$$

for a suitable function $f \in C^{1,2}$. Then by Itô's formula

$$dV(t) = (f_t(t,S_t) + f_x(t,S_t)S_t\mu + \frac{1}{2}S_t^2\sigma^2 f_{xx}(t,S_t))dt + f_x(t,S_t)\sigma S_t dW_t.$$

Now we match the coefficients and find

$$\varphi_1(t) = f_x(t, S_t)$$

and

$$\varphi_0(t) = \frac{1}{rB(t)}(f_t(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 f_{xx}(t,S_t)).$$

Then looking at the total portfolio value we find that $f(t,x)$ must satisfy the Black-Scholes partial differential equation

$$f_t(t,x) + rxf_x(t,x) + \frac{1}{2}\sigma^2 x^2 f_{xx}(t,x) - rf(t,x) = 0 \tag{1.7}$$

and initial condition $f(T,x) = (x-K)^+$.

1.4.2 Options on Dividend-paying Stocks

We assume that the stock pays a dividend at some fixed rate κ and that the dividend payments are used in full for reinvestment. Consequently, a trading strategy $\varphi = (\varphi_0, \varphi_1)$ is self-financing if its wealth process

$$V_\varphi(t) = \varphi_0(t)B(t) + \varphi_1(t)S(t)$$

satisfies

$$dV_\varphi(t) = \varphi_0(t)dB(t) + \varphi_1(t)dS(t) + \kappa\varphi_1(t)S(t)dt,$$

or equivalently (using the stochastic dynamics of the stock),

$$dV_\varphi(t) = \varphi_0(t)dB(t) + \varphi_1(t)(\kappa + \mu)S(t)dt + \varphi_1(t)S(t)\sigma dW(t).$$

Consider now the auxiliary process

$$S^*(t) = e^{\kappa t}S(t).$$

From an application of Itô's lemma we see

$$dS^*(t) = \mu_\kappa S^*(t)dt + \sigma S^*(t)dW(t), \quad \text{with} \quad \mu_\kappa := \mu + \kappa.$$

In terms of this process we have

$$V_\varphi(t) = \varphi_0(t)B(t) + \varphi_1(t)e^{-\kappa t}S^*(t) \;\; \text{resp.} \;\; dV_\varphi(t) = \varphi_0(t)dB(t) + \varphi_1(t)e^{-\kappa t}dS^*(t).$$

For the discounted wealth $\tilde{V}_\varphi(t) = V_\varphi(t)/B(t)$ we find

$$d\tilde{V}_\varphi(t) = \varphi_1(t)e^{-\kappa t}d\tilde{S}(t) \;\; \text{with} \;\; \tilde{S}_\varphi(t) = S^*/B(t).$$

or

$$d\tilde{V}_\varphi(t) = \varphi_1(t)\sigma\tilde{S}(t)\left(dW(t) + \sigma^{-1}(\mu_\kappa - r)dt\right).$$

Thus we obtain a unique martingale measure $I\!P^*$ by using Girsanov's theorem with $\gamma = \sigma^{-1}(\mu_\kappa - r)$. The dynamics of $\tilde{V}_\varphi(t)$ and $\tilde{S}^*(t)$ are

$$d\tilde{V}_\varphi(t) = \sigma\varphi_1(t)\tilde{S}^*(t)d\tilde{W}(t) \;\; \text{and} \;\; d\tilde{S}^*(t) = \sigma\tilde{S}^*(t)d\tilde{W}(t)$$

with $\tilde{W}(t) = W(t) - (r - \mu_k)\sigma^{-1}t$. We can now simply repeat the argument used to obtain the Black-Scholes formula to prove

Proposition 1.4.2. *The arbitrage price at $t < T$ of a European call on a stock paying dividends at a constant rate κ during the option's lifetime is is given by the risk-neutral valuation formula*

$$C^{\kappa}(t) = B_t \mathbb{E}^* \left[B_T^{-1}(S_T - K)^+ \middle| \mathcal{F}_t \right] \tag{1.8}$$

or explicitly

$$C^{\kappa}(t) = \bar{S}(t)\Phi(d_1(\bar{S}(t), T-t)) - Ke^{-r(T-t)}\Phi(d_2(\bar{S}(t), T-t)). \tag{1.9}$$

where $\bar{S}(t) = S(t)e^{-\kappa(t-t)}$ and the functions $d_1(s,t)$ and $d_2(s,t)$ are as above.

Proof. The first equality is the risk-neutral valuation formula. For the second observe the

$$C^{\kappa}(t) = e^{-r(T-t)} \mathbb{E}^* \left[(S_T - K)^+ \middle| \mathcal{F}_t \right] = e^{-\kappa T} e^{-r(T-t)} \mathbb{E}^* \left[(S_T^* - e^{\kappa T} K)^+ \middle| \mathcal{F}_t \right].$$

The last expectation can now be evaluated similar to the corresponding expectation leading to the Black-Scholes equation.

1.4.3 Black's Futures Options Formula

We turn now to the problem of extending our option pricing theory from spot markets to futures markets. We assume that the stock-price dynamics S are given by geometric Brownian motion

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t),$$

and that interest rates are deterministic. We know that there exists a unique equivalent martingale measure, $\mathbb{P}^*$ (for the discounted stock price processes), with expectation $\mathbb{E}^*$. Write

$$f(t) := f_S(t, T^*)$$

for the futures price $f(t)$ corresponding to the spot price $S(t)$. Then risk-neutral valuation gives

$$f(t) = \mathbb{E}^*(S(T^*)|\mathcal{F}_t) \quad (t \in [0, T^*]),$$

while forward prices are given in terms of bond prices by

$$F(t) = S(t)/B(t, T^*) \quad (t \in [0, T^*]).$$

So by Proposition 1.3.1

$$f(t) = F(t) = S(t)e^{r(T^*-t)} \quad (t \in [0, T^*]).$$

So we can use the product rule to determine the dynamics of the futures price

$$df(t) = (\mu - r)f(t)dt + \sigma f(t)dW(t), \quad f(0) = S(0)e^{rT^*}.$$

In the following we study a general Futures market and assume

$$df(t) = \mu f(t)dt + \sigma f(t)dW(t).$$

Again, we say that a futures strategy is self-financing if

$$dV_\varphi^f(t) = \varphi_0(t)dB(t) + \varphi_1(t)df(t).$$

But observe that

$$V_\varphi^f(t) = \varphi_0(t)B(t),$$

since it costs nothing to enter a Futures position.
We call a probability measure $\mathbb{P}^* \sim \mathbb{P}$ a Futures martingale measure, if

$$\tilde{V}_\varphi^f(t) = \frac{V_\varphi^f(t)}{B(t)},$$

follows a (local) martingale.

Lemma 1.4.1. *$I\!\!P^* \sim I\!\!P$ is a Futures martingale measure if and only if f is a (local) martingale under $I\!\!P^*$.*

Proof. Using the product rule we see that $\tilde{V}_\varphi^f(t)$ satisfies for any self-financing φ

$$\begin{aligned} d\tilde{V}_\varphi^f(t) &= B(t)^{-1}\left(\varphi_0(t)dB(t) + \varphi_1(t)df(t)\right) - rB(t)^{-1}\tilde{V}_\varphi^f(t)dt \\ &= B(t)^{-1}\left(\varphi_0(t)dB(t) - r\tilde{V}_\varphi^f(t)dt\right) + B(t)^{-1}\varphi_1(t)df(t) \\ &= B(t)^{-1}\left(V_\varphi^f(t)B(t)^{-1}rB(t)dt - r\tilde{V}_\varphi^f(t)dt\right) + B(t)^{-1}\varphi_1(t)df(t). \blacksquare \end{aligned}$$

As usual we obtain from Girsanov's theorem

Proposition 1.4.3. *The unique martingale measure $I\!\!P^*$ on $(\Omega, I\!\!F)$ for the process f is given by*

$$\left.\frac{dI\!\!P^*}{dI\!\!P}\right|_{\mathcal{F}_t} = \exp\left\{-\frac{\mu}{\sigma}W(t) - \frac{1}{2}\frac{\mu^2}{\sigma^2}t\right\}.$$

Thus the $I\!\!P^*$-dynamics for the Futures price f are

$$df(t) = \sigma f(t)d\tilde{W}(t)$$

and the process

$$\tilde{W}(t) = W(t) + \mu\sigma^{-1}t$$

is a $I\!\!P^*$-Wiener process. Also

$$f(t) = f_0 \exp\left\{\sigma\tilde{W}(t) - \frac{1}{2}\sigma^2 t\right\}.$$

We turn now to the futures analogue of the Black-Scholes formula, due to Black (1976). We and use the same notation - strike K, expiry T as in the spot case, and write Φ for the standard normal distribution function.

Theorem 1.4.1. *The arbitrage price C of a European futures call option is*

$$C(t) = c(f(t), T-t),$$

where $c(f,t)$ is given by Black's futures options formula:

$$c(f,t) := e^{-rt}(f\Phi(\tilde{d}_1(f,t)) - K\Phi(\tilde{d}_2(f,t))),$$

where

$$\tilde{d}_{1,2}(f,t) := \frac{\log(f/K) \pm \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}}.$$

Proof. By risk-neutral valuation,

$$C(t) = B(t)I\!\!E^*\left[(f(T)-K)^+/B(T)|\mathcal{F}_t\right],$$

with $B(t) = e^{rt}$. For simplicity, we work with $t = 0$; the extension to the general case is immediate. Thus

$$\begin{aligned} C(0) &= I\!\!E^*\left[(f(T)-K)^+/B(T)\right] \\ &= I\!\!E^*\left[e^{-rT}f(T)\mathbf{1}_D\right] - I\!\!E^*\left[e^{-rT}K\mathbf{1}_D\right] \\ &= \mathbf{1}_1 - \mathbf{1}_2 \end{aligned}$$

say, where

$$D := \{f(T) > K\}.$$

Thus

$$\begin{aligned}\mathbf{1}_2 &= e^{-rT} K \mathbb{P}^*(f(T) > K) \\ &= e^{-rT} K \mathbb{P}^* \left(f(0) \exp\left\{ \sigma \tilde{W}(T) - \frac{1}{2}\sigma^2 T \right\} > K \right),\end{aligned}$$

where $\tilde{W}$ is a standard Brownian motion under $\mathbb{P}^*$. Now $\xi := -\tilde{W}(T)/\sqrt{T}$ is standard normal, with law Φ under $\mathbb{P}^*$, so

$$\begin{aligned}\mathbf{1}_2 &= e^{-rT} K \mathbb{P}^* \left(\xi < \frac{\log(f(0)/K) - \frac{1}{2}\sigma^2}{\sigma\sqrt{T}} \right) \\ &= e^{-rT} K \Phi \left(\frac{\log(f(0)/K) - \frac{1}{2}\sigma^2}{\sigma\sqrt{T}} \right) \\ &= e^{-rT} K \tilde{d}_2(f(0), T).\end{aligned}$$

To evaluate $\mathbf{1}_1$ define an auxiliary probability measure $\hat{\mathbb{P}}$ by setting

$$\left.\frac{d\hat{\mathbb{P}}}{d\mathbb{P}^*}\right|_{\mathcal{F}_t} = \exp\left\{ \sigma \tilde{W}(t) - \frac{1}{2}\sigma^2 t \right\},$$

and thus

$$\mathbf{1}_1 = \mathbb{E}^* \left[e^{-rT} f(T) \mathbf{1}_D \right] = e^{-rT} f(0) \hat{\mathbb{P}}(f(T) > K).$$

Since

$$\hat{W}(t) = \tilde{W}(t) - \sigma t$$

is a standard $\hat{\mathbb{P}}$-Wiener process we see

$$f(t) = f_0 \exp\left\{ \sigma \hat{W}(t) - \frac{1}{2}\sigma^2 t \right\}.$$

Thus

$$\begin{aligned}\mathbf{1}_1 &= e^{-rT} f(0) \hat{\mathbb{P}} \left(f(T) > K \right) \\ &= e^{-rT} f(0) \hat{\mathbb{P}} \left(f_0 \exp\left\{ \sigma \hat{W}(T) - \frac{1}{2}\sigma^2 T \right\} > K \right) \\ &= e^{-rT} f(0) \hat{\mathbb{P}} \left(-\sigma \hat{W}(T) < \log(f(0)/K) + \frac{1}{2}\sigma^2 T \right) \\ &= e^{-rT} f(0) \Phi \left(\tilde{d}_1(f(0), T) \right).\end{aligned}$$

■

Observe that the quantities $\tilde{d}_1$ and $\tilde{d}_2$ do not depend on the interest rate r. This is intuitively clear from the classical Black approach: one sets up a replicating risk-free portfolio consisting of a position in futures options and an offsetting position in the underlying futures contract. The portfolio requires no initial investment and therefore should not earn any interest.

Chapter 2

Data Analysis

2.1 Introduction

Our aim now is to discuss models for the distribution of of electricity prices (e.g. their returns). Recall the structure of the price processes

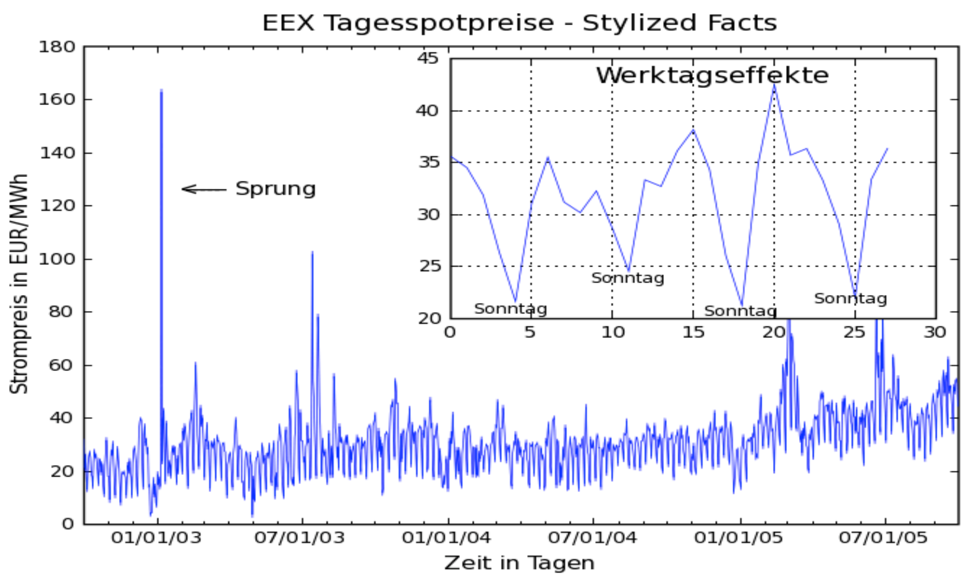


Figure 2.1: Spot prices

We start discussion possible choices for distributions and discuss their fit.

2.2 Distribution of electricity prices

2.2.1 Stable distributions

Stable distributions have been used to model financial assets for the following reasons

(i) stable laws are the only possible limit distributions for properly normalized and centered sums of independent identically distributed random variables. So they are a natural generalisation of normally distributed random variables.

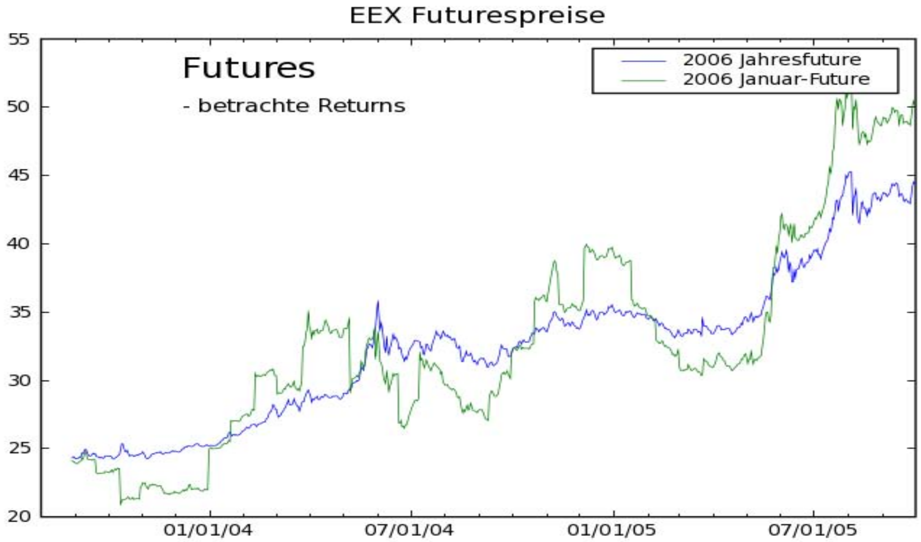


Figure 2.2: Futures prices

(ii) stable distributions are leptokurtic, that is they have heavier tails than the normal distribution.

Stable distributions require four parameters for a complete description

$$\alpha \in (0,2],\ \beta \in [-1,1],\ \sigma > 0,\ \mu \in \mathbb{R}.$$

α is the tail exponent and determines the rate which which the tails of the distribution tend to zero

$$\begin{cases} \lim_{x\to\infty} x^\alpha \mathbb{P}(X > x) = C_\alpha(1+\beta)\sigma^2, \\ \lim_{x\to\infty} x^\alpha \mathbb{P}(X < -x) = C_\alpha(1+\beta)\sigma^2, \end{cases}$$

When $\alpha > 1$ the mean of the distribution exists and equals μ. β is the skewness parameter; if it is positive (negative) the distribution is skewed to the right (left). σ is the scale parameter.
The drawback of these distributions is that their probability density functions and cumulative distribution functions do not have a closed form expression. However, the characteristic functions are well understood. We have for $X \sim S_\alpha(\sigma, \beta, \mu)$

$$\log\phi(t) = \begin{cases} -\sigma^\alpha |t|^\alpha \left\{1 - i\beta sign(t)\tan\frac{\pi\alpha}{2}\right\}, & \alpha \neq 1 \\ -\sigma|t|\left\{1 + i\beta sign(t)\frac{2}{\pi}\log|t|\right\}, & \alpha = 1 \end{cases}$$

2.2.2 Hyberbolic Distributions

Our concern here is the hyperbolic family, a four-parameter family with two type and two shape parameters. Recall that, for normal (Gaussian) distributions, the log-density is quadratic – that is, parabolic – and the tails are very thin. The hyperbolic family is specified by taking the log-density instead to be hyperbolic, and this leads to thicker tails as desired (but not as thick as for the stable family).
We need some background on Bessel functions. Recall the Bessel functions J_ν of the first kind, Watson (1944), §3.11, Y_ν of the second kind, Watson (1944), §3.53, and K_ν, Watson (1944), §3.7,

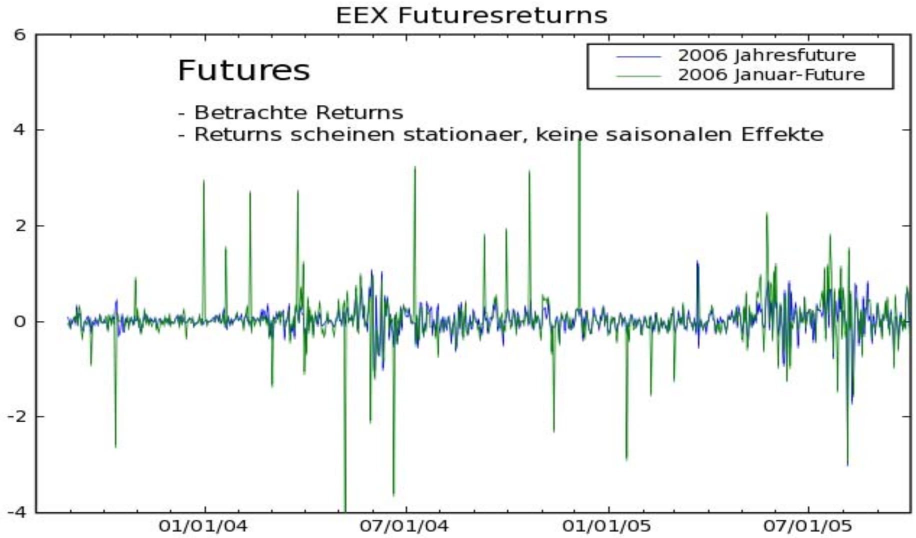


Figure 2.3: Futures returns

there called a Bessel function with imaginary argument or Macdonald function, nowadays usually called a Bessel function of the third kind. From the integral representation

$$K_\nu(x) = \frac{1}{2}\int_0^\infty u^{\nu-1}\exp\left\{-\frac{1}{2}x(u+1/u)\right\}du \quad (x>0) \tag{2.1}$$

one sees that

$$f(x) = \frac{(\psi/\chi)^{\frac{1}{2}\lambda}}{2K_\lambda(\sqrt{\chi\psi})}x^{\lambda-1}\exp\left\{-\frac{1}{2}(\psi x+\chi/x)\right\} \quad (x>0) \tag{2.2}$$

is a probability density function. The corresponding law is called the generalized inverse Gaussian $GIG_{\lambda,\psi,\chi}$; the inverse Gaussian is the case $\lambda = 1$: $IG_{\chi,\psi} = GIG_{1,\psi,\chi}$. These laws were introduced by Good (1953).

Now consider a Gaussian (normal) law $N(\mu+\beta\sigma^2,\sigma^2)$ where the parameter σ^2 is random and is sampled from $GIG_{1,\psi,\chi}$. The resulting law is a mean-variance mixture of normal laws, the mixing law being generalized inverse Gaussian. It is written $I\!\!E_{\sigma^2}N(\mu+\beta\sigma^2,\sigma^2)$; it has a density of the form

$$\frac{\sqrt{\alpha^2-\beta^2}}{2\alpha\delta K_1(\delta\sqrt{\alpha^2-\beta^2})}\exp\left\{-\alpha\sqrt{\delta^2+(x-\mu)^2}+\beta(x-\mu)\right\} \tag{2.3}$$

(Barndorff-Nielsen (1977)), where $\alpha^2 = \psi+\beta^2$ and $\delta^2 = \chi$. Just as the Gaussian law has log-density a quadratic – or parabolic – function, so this law has log-density a hyperbolic function. It is accordingly called a hyperbolic distribution. Various parametrizations are possible. Here μ is a location and δ a scale parameter, while $\alpha > 0$ and β $(0 \le |\beta| < \alpha)$ are shape parameters. One may pass from (α,β) to (ϕ,γ) via

$$\alpha = (\phi+\gamma)/2,\ \beta = (\phi-\gamma)/2,\ \text{so}\ \phi\gamma = \alpha^2-\beta^2,$$

and then to (ξ,χ) via

$$\xi = (1+\delta\sqrt{\phi\gamma})^{-\frac{1}{2}},\ \chi = \frac{\xi\beta}{\alpha} = \xi\frac{\phi-\gamma}{\phi+\gamma}.$$

This parameterization (in which ξ and χ correspond to the classical shape parameters of skewness and kurtosis) has the advantage of being affine invariant (invariant under changes of location and scale). The range of (ξ, χ) is the interior of a triangle

$$\nabla = \{(\xi, \chi) : 0 \leq |\chi| < \xi < 1\},$$

called the shape triangle.

Infinite Divisibility.

Recall (Feller (1971), XIII,7, Theorem 1) that a function ω is the Laplace transform of an infinitely divisible probability law on $I\!R_+$ iff $\omega = e^{-\psi}$, where $\psi(0) = 0$ and ψ has a completely monotone derivative (that is, the derivatives of ψ' alternate in sign). Grosswald (1976) showed that if

$$Q_\nu(x) := K_{\nu-1}(\sqrt{x})/(\sqrt{x}K_\nu(\sqrt{x})) \quad (\nu \geq 0, x > 0),$$

then Q_ν is completely monotone. Hence Barndorff-Nielsen and Halgreen (1977) showed that the generalized inverse Gaussian laws GIG are infinitely divisible. Now the GIG are the mixing laws giving rise to the hyperbolic laws as normal mean-variance mixtures. This transfers infinite divisibility (see e.g. Kelker (1971), Keilson and Steutel (1974), §§1,2), so the hyperbolic laws are infinite divisible.

Characteristic Functions.

The mixture representation transfers to characteristic functions on taking the Fourier transform. It gives the characteristic function of $hyp_{\zeta,\delta}$ as

$$\phi(u) = \phi(u; \zeta, \delta) = \frac{\zeta}{K_1(\zeta)} \frac{K_1\left(\sqrt{\zeta^2 + \delta^2 u^2}\right)}{\sqrt{\zeta^2 + \delta^2 u^2}}. \tag{2.4}$$

If $\phi(u)$ is the characteristic function of Z_1 in the corresponding Lévy process $Z = (Z_t)$, that of Z_t is $\phi_t = \phi^t$. The mixture representation of $hyp_{\zeta,\delta}$ gives

$$\phi_t(u) = \exp\{tk(u^2/2)\},$$

where $k(.)$ is the cumulant generating function of the law IG,

$$I\!E\left(e^{-sY}\right) = e^{k(s)},$$

where Y has law $IG_{\psi,\chi}$ (recall $\chi = \delta^2$), and Grosswald's result above is

$$Q_\nu(t) = \int_0^\infty q_\nu(x)dx/(x+t),$$

where

$$q_\nu = 2/(\pi^2 x(J_\nu^2(\sqrt{x}) + Y_\nu(\sqrt{x}))^2) > 0 \quad (x > 0)$$

(thus Q_ν is a Stieltjes transform, or iterated Laplace transform, Widder (1941), VIII). Using this and the Lévy-Khintchine formula Eberlein and Keller (1995) obtained the density $\nu(x)$ of the Lévy measure $\mu(dx)$ of Z as

$$\nu(x) = \frac{1}{\pi^2|x|} \int_0^\infty \frac{\exp\left\{-|x|\sqrt{2y + (\zeta/\delta)^2}\right\}}{y\left(J_1^2(\delta\sqrt{2y}) + Y_1^2(\delta\sqrt{2y})\right)} dy + \frac{\exp\{-|x|\zeta/\delta\}}{|x|}, \tag{2.5}$$

and then

$$\phi_t(u) = \exp\{tK(u^2/2)\}, \quad K(u^2/2) = \int_{-\infty}^{\infty} \left(e^{iux} - 1 - iux\right) \nu(x)dx.$$

Now (Watson (1944), §7.21)

$$\begin{aligned} J_\nu(x) &\sim \sqrt{2/\pi x}\cos\left(x - \frac{1}{2}\pi\nu - \frac{1}{4}\pi\right), \\ Y_\nu(x) &\sim \sqrt{2/\pi x}\sin\left(x - \frac{1}{2}\pi\nu - \frac{1}{4}\pi\right), \end{aligned} \quad (x \to \infty).$$

So the denominator in the integral in (2.5) is asymptotic to a multiple of $y^{\frac{1}{2}}$ as $y \to \infty$. The asymptotics of the integral as $x \downarrow 0$ are determined by that of the integral as $y \to \infty$, and (writing $\sqrt{2y + (\zeta/\delta)^2}$ as t, say) this can be read off from the Hardy-Littlewood-Karamata theorem for Laplace transforms (Feller (1971), XIII.5, Theorem 2, or Bingham, Goldie, and Teugels (1987), Theorem 1.7.1). We see that $\nu(x) \sim c/x^2$, $(x \downarrow 0)$ for c a constant. In particular the Lévy measure is infinite.

Tails and Shape.

The classic empirical studies of Bagnold (1941) and Bagnold and Barndorff-Nielsen (1979) reveal the characteristic pattern that, when log-density is plotted against log-size of particle, one obtains a unimodal curve approaching linear asymptotics at $\pm\infty$. Now the simplest such curve is the hyperbola, which contains four parameters: location of the mode, the slopes of the asymptotics, and curvature near the mode (the modal height is absorbed by the density normalization). This is the empirical basis for the hyperbolic laws in particle-size studies. Following Barndorff-Nielsen's suggested analogy, a similar pattern was sought, and found, in financial data, with log-density plotted against log-price. Studies by Eberlein and Keller (1995), Eberlein, Keller, and Prause (1998), Eberlein and Raible (1998), Bibby and Sørensen (1997), and other authors show that hyperbolic densities provide a good fit for a range of financial data, not only in the tails but throughout the distribution. The hyperbolic tails are log-linear: much fatter than normal tails but much thinner than stable ones.

Normal Inverse Gaussian Distribution

We start with the generalized hyperbolic distributions for log returns. For these distributions the densities are given by (to fix the notation):

$$\begin{aligned} d_{GH}(x; \lambda, \alpha, \beta, \delta, \mu) &= a(\lambda, \alpha, \beta, \delta, \mu) \\ &\quad \times(\delta^2 + (x-\mu)^2)^{(\lambda - \frac{1}{2})/2} \\ &\quad \times K_{\lambda - \frac{1}{2}}(\alpha\sqrt{\delta^2 + (x-\mu)^2}) \\ &\quad \times \exp\{\beta(x-\mu)\} \end{aligned} \tag{2.6}$$

where

$$a(\lambda, \alpha, \beta, \delta, \mu) = \frac{(\alpha^2 - \beta^2)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda - \frac{1}{2}}\delta^\lambda K_\lambda(\delta\sqrt{\alpha^2 - \beta^2})}$$

and K_ν denotes the modified Bessel function of the third kind

$$K_\nu(z) = \frac{1}{2}\int_0^\infty y^{\nu - 1}\exp\left\{-\frac{1}{2}z(y + y^{-1})\right\} dy$$

We will consider the Normal Inverse Gaussian Distribution (NIG), where the parameter $\lambda = -1/2$. So the density is

$$d_{NIG}(x) = \frac{\alpha}{\pi} \exp\left\{\delta\sqrt{\alpha^2-\beta^2} + \beta(x-\mu)\right\}$$

$$\times \frac{K_1\left(\alpha\delta\sqrt{1+\left(\frac{x-\mu}{\delta}\right)^2}\right)}{\sqrt{1+\left(\frac{x-\mu}{\delta}\right)^2}}.$$

2.3 Case Study: EEX

We estimate the moments (mean, variance, higher order) as usual

$$I\!E(X) = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$Var(X) = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

$$I\!E(X^k) = \mu^{(k)} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^k$$

For skewness S and kurtosis K we obtain

$$S = \frac{I\!E(X^3)}{Var(X)^{\frac{3}{2}}}$$

and

$$K = \frac{I\!E(X^4)}{Var(X)^2}$$

An application to the daily log returns gives

underlying	**mean**	**variance**	**kurtosis**	**skewness**
Strom Spot (EEX)	0.0003	0.0058	5.7	0.63
Strom Spot (PJM)*	0.0002	0.0006	14.1	0.36
Gas Spot (Zeebrugge)	0.0007	0.0005	6.6	-0.52
DaimlerChrysler	0,0002	$2.1 \cdot 10^{-5}$	1.9	0.05

**Pennsylvania, Jersey, Maryland*

Quelle: Energy and Power Risk Management *Eydeland/Wolyniec, Wiley&Sons*

We perform the standard tests for normality with the Jarque-Bera- and Kolmogorov-Smirnov -test

$$JB = n\left(\frac{skewness^2}{6} + \frac{(kurtosis-3)^2}{24}\right) \sim \chi^2(2)$$

$$KS = \sup|F_n(x) - F(x)| \quad \text{with}$$

$$F_n(x) = \frac{1}{n}\sum_{j=1}^{n} 1(\text{for} X_j < x)$$

Applied to the log returns we find

underlying	**JB**	**5% CI**	**KS**	**5% CI**
Strom Spot (EEX)	1570.8	<6.0	0.13	<0.041
Strom Spot (PJM)*	5054.4	<6.0	0.14	<0.044
Gas Spot (Zeebrugge)	1330.0	<6.0	0.11	<0.051
DaimlerChrysler	166.4	<6.0	0.05	<0.042

**Quelle:* Energy and Power Risk Management *Eydeland/Wolyniec, Wiley&Sons*

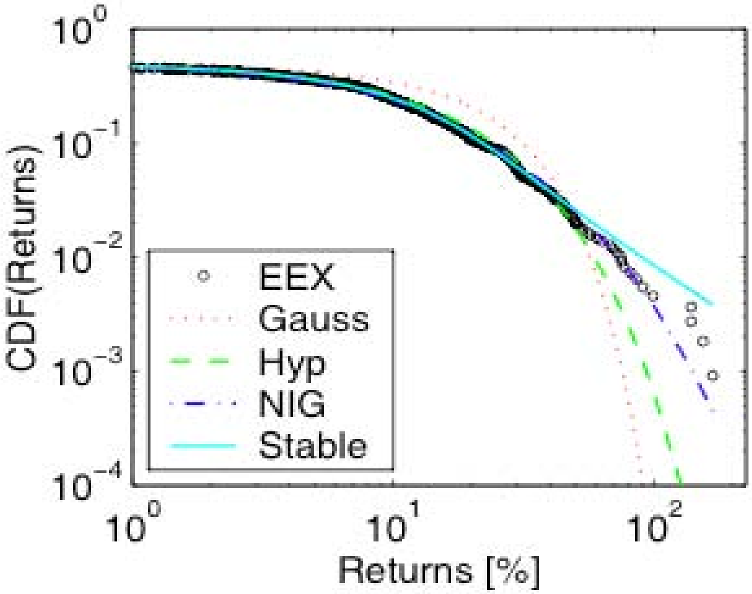


Figure 2.4: Tail behaviour of the data and the fitted distributions

A comparison of the values is given below

To Appear: Modelling and forecasting electricity loads and prices, *Rafal Weron*

Parameters	α	σ, δ	β	μ
Gaussian fit		11.4548		0.0083
Hyperbolic fit	0.2099	0.0851	-0.0001	0.0136
NIG fit	0.0469	3.2181	-0.0031	0.0083
α-stable fit	1.5104	2.9005	-0.2616	-0.4898
Test values	Anderson-Darling		Kolmogorov	
Gaussian fit	+INF		6.9894	
Hyperbolic fit	+INF		1.8669	
NIG fit	1.7890		0.9138	
α-stable fit	0.5419		0.6831	

Figure 2.5: Fit of distributions to de-seasonalized returns

Chapter 3

Stochastic Modelling of Spot Price Processes

3.1 Geometric Brownian Motion for Spot Prices

Although our empirical study has revealed that GBM might not provide a good fit to observed data, we will use it as a first modelling approach.
The standard form is

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

with (W_t) a standard BM.
Recall the spot-forward relationship (1.2) and consider the variation

$$F(t,T) = S_t e^{(r+u-y)(T-t)}$$

where y is the convenience yield (benefit of ownership) and u represents the storage costs.
In case we set the parameter $\mu = r + u - y$ we also obtain

$$F(t,T) = \mathbb{E}(S_T | S_t)$$

since $S_T = S_0 \exp\{(r + u - y - \frac{1}{2}\sigma^2)T + \sigma W_T\}$ and $\mathbb{E}(\exp(\sigma W_T)) = e^{\frac{1}{2}\sigma^2 T}$.
So we assume that

$$dS_t = (r + u - y)S_t dt + \sigma S_t dW_t.$$

The assumption that u and y are constants is often seen as a crude approximation.
In (Schwartz 1997) the standard BM with mean reversion is used to describe the evolution of the convenience yield. Thus

$$\begin{aligned} dS_t &= (r - \delta_t)S_t dt + \sigma_1 S_t dW_t^{(1)} \\ d\delta_t &= [\kappa(\alpha - \delta_t) - \lambda]dt + \sigma_2 dW_t^{(2)} \end{aligned}$$

with $dW_t^{(1)} dW_t^{(2)} = \rho dt$
where

- δ denotes the random convenience yield
- κ, α, σ_2 are respectively the strength of mean reversion, the long-term value and the instantaneous volatility of convenience yield
- λ is a constant associated with the market price of risk associated with the unhedgable convenience yield.

Now the forward price is given by the equality

$$F(t,T) = S_t \cdot \exp[-\delta \frac{1-e^{-\kappa(T-t)}}{\kappa} + A(t,T)]$$

where

$$\begin{aligned} A(t,T) &= \left(r - \alpha + \frac{\lambda}{\kappa} + \frac{1}{2}\frac{\sigma_2^2}{\kappa^2} - \frac{\sigma_1\sigma_2\rho}{\kappa}\right)(T-t) \\ &+ \frac{1}{4}\sigma_2^2 \frac{1-e^{2\kappa(T-t)}}{\kappa^3} \\ &+ \left[\left(\alpha - \frac{\lambda}{\kappa}\right)\kappa + \sigma_1\sigma_2\rho - \frac{\sigma_2^2}{\kappa}\right]\frac{1-e^{-\kappa(T-t)}}{\kappa^2}. \end{aligned}$$

In the GBM case we can use the standard formulae for call and put options and obtain

$$\begin{aligned} C_{BS} &= C_{BS}(t,S;T,X,r,\sigma) &= S\Phi(d_1) - Xe^{-r(T-t)}\Phi(d_2) \\ P_{BS} &= P_{BS}(t,S;T,X,r,\sigma) &= Xe^{-r(T-t)}\Phi(-d_2) - S\Phi(-d_1) \end{aligned}$$

with

$$d_1 = \frac{\log(S/X) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = d_2 + \sigma\sqrt{T-t}.$$

Introducing storage costs and convenience yield this gives

$$C = C_{BS}(t, Se^{(u-y)(T-t)}; T, X, r, \sigma)$$

and

$$P = P_{BS}(t, Se^{(u-y)(T-t)}; T, X, r, \sigma),$$

which now in terms of forward (futures) prices can be written

$$\begin{aligned} C &= C_{BS}(t, F(t,T)e^{-r(T-t)}; T, X, r, \sigma) \\ P &= P_{BS}(t, R(t,T)e^{-r(T-t)}; T, X, r, \sigma). \end{aligned}$$

The benefit of this representation is that it does not require the knowledge of the convenience yield for option valuation. A useful generalization of these valuation formulas is to the case of non constant volatility, $\sigma = \sigma(t)$.
Then the pricing formulas are still valid when σ is replaced by

$$\bar{\sigma} = \sqrt{\frac{1}{T-t}\int_t^T \sigma^2(u)du}.$$

We can use the above formulas to obtain implied volatilities, which are often used as a forward-looking measure of price uncertainty. Thus we need to solve

$$C_{t,T,X}^{\text{market}} = C(t, S_t; T, X, r, \sigma_I)$$

for σ_I. Often straddles, i.e. combinations of put and calls with the same strike, are used to find σ_I.
Example 3.1 Assume that the current price of natural gas is \$ 2.50. The ATM straddle, strike \$ 2.50, expiring 6 month from now is quoted at \$ 0.42. The risk free rate is 5 %. Thus we need to solve

$$0.42 = C(0, 2.50; 0.5, 2.50, 0.05, \sigma_I) + P(0, 2.50; 0.5, 2.50, 0.05, \sigma_I)$$

using a standard numerical algorithm we find $\sigma_I = 0.3$.
This method does not take into account the term-structure of volatility. Since different times to expiry imply different volatilities we need to use time dependent volatility.
Example 3.2
Option 1: $t = 0$, $S_0 = \$\ 2.50$, $T_1 = 0.5$, $X_1 = \$\ 2.50$, $r = 0.05$; $C_1 = \$\ 0.24$.
Option 2: $t = 0$, $S_0 = \$\ 2.50$, $T_2 = 1$, $X_2 = \$\ 2.70$, $r = 0.05$; $C_2 = \$\ 0.37$.
This leads to $\sigma_I(T_1) = 0.3$, $\sigma_I(T_2) = 0.4$.
We now use the formula with time dependent volatility and need to match

$$\int_0^{T_1} \sigma^2(s)ds = T_1\sigma_I^2(T_1)$$

and

$$\int_0^{T_2} \sigma^2(s)ds = T_2\sigma_I^2(T_2).$$

Using the data

$$\int_0^{0.5} \sigma^2(s)ds = 0.5 \cdot (0.3)^2$$

and

$$\int_0^{1} \sigma^2(s)ds = (0.4)^2.$$

Among the infinite many functions satisfying these equations we can choose a piecewise constant function

$$\begin{aligned} \sigma(t) &= 0.30 \quad \text{for} \quad 0 \le t \le 0.5 \quad \text{and} \\ \sigma(t) &= 0.48 \quad \text{for} \quad 0.5 \le t \le 1. \end{aligned}$$

In case we are given implied volatility for every T we have

$$\int_0^{T} \sigma^2(t)dt = T\sigma_I^2(T)$$

or

$$\sigma^2(t) = \frac{\partial}{\partial}[t\sigma_I^2(t)].$$

Since volatility is positive $t\sigma_I^2(t)$ must be an increasing function. Unfortunately, there are empirical term structures which cannot be fitted.

3.2 Modifications of GBM: Mean Reversion

Recall that a price process is said to be mean reverting toward a certain level, called a long-term mean, if it exhibits the following property: the further it moves away from this level, the higher the probability that in the future it will move back to it. GBM does not have the mean reverting property.
Suggested modifications are

$$\frac{dS_t}{S_t} = \kappa(S_\infty - S_t)dt + \sigma dW_t$$

i.e. the spot price shows the mean-reverting property or

$$\frac{dS_t}{S_t} = \kappa(\theta - \log S_t)dt + \sigma dW_t,$$

here $\theta - \frac{1}{2\kappa}\sigma^2$ is the long-term mean of logarithms of spot prices (returns) and κ is the speed of mean reversion.
Write $Z_t = \log S_t$ and use Itô's lemma to find

$$dZ_t = \kappa(\theta - \frac{1}{2\kappa}\sigma^2 - Z_t)dt + \sigma dW_t.$$

Introducing $Y_t = e^{\kappa t} Z_t$ and again by Itô's lemma obtain

$$dY_t = \kappa e^{\kappa t} Z_t dt + e^{\kappa t} dZ_t = \kappa(\theta - \frac{1}{2\kappa}\sigma^2)e^{\kappa t}dt + \sigma e^{kt} dW_t$$

and we obtain the conditional distribution

$$Y_T|Y_t \sim \mathcal{N}\left[Y_t + (\theta - \frac{1}{2\kappa}\sigma^2)(e^{\kappa T} - e^{kt}), \sigma\sqrt{\frac{e^{2\kappa T} - e^{2\kappa t}}{2\kappa}}\right]$$

and

$$Z_T|Z_t \sim \mathcal{N}\left[e^{-\kappa(T-t)}Z_t + (\theta - \frac{1}{2\kappa}\sigma^2)(1 - e^{-\kappa(T-t)}), \sigma\sqrt{\frac{1 - e^{-2\kappa(T-t)}}{2\kappa}}\right].$$

In order to use the model for pricing we need to modify the drift of Z_t to make risk-neutral pricing possible. We do this by calibrating the model to the forward curve.
Assume that t_0 is the current time and that the long-term log-of-price parameter θ depends deterministically on t, i.e. $\theta = \theta(t)$. Also, assume that κ is constant and has already been determined.
Now $Y_t = e^{\kappa Z} Z_t$ is again normally distributed with

$$\text{expectation} \quad I\!E(Y_t) = \int\limits_{t_0}^{t} \kappa e^{\kappa s}[\theta(s) - \frac{1}{2\kappa}\sigma^2]ds$$

and

$$\text{variance} \quad Var(Y_t) = \sigma^2 \frac{e^{2\kappa t} - e^{2\kappa t_0}}{2\kappa}.$$

We can solve the expression for the expectation for θ

$$\theta(t) = \frac{1}{\kappa} e^{-\kappa t} \frac{\partial}{\partial t} I\!E(Y_t) + \frac{1}{2\kappa}\sigma^2.$$

Thus

$$\begin{aligned}\theta(t) &= \frac{1}{\kappa} e^{-\kappa t} \frac{\partial}{\partial t} I\!E(e^{\kappa t} Z_t) + \frac{1}{2\kappa}\sigma^2 \\ &= I\!E(Z_t) + \frac{1}{\kappa}\frac{\partial}{\partial t} I\!E(Z_t) + \frac{1}{2\kappa}\sigma^2.\end{aligned}$$

In the risk-neutral world $F(t_0, t) = I\!E(S_t)$ where $F(t_0, t)$ is the forward curve at t_0 and t the time parameter. The terminal price is lognormally distributed because the distribution of its log, Z_t, is normally distributed.
So

$$I\!E(Z_t) = I\!E(\log S_t) = \log(F(t_0, t)) - \frac{1}{2} var(Z_t).$$

We know that

$$var(Z_t) = \sigma^2 \frac{1 - e^{-2\kappa(t-t_0)}}{2\kappa}.$$

Hence

$$\begin{aligned}\theta(t) &= \frac{1}{\kappa} e^{-\kappa t} \frac{\partial}{\partial t} \mathbb{E}(e^{\kappa t} Z_t) + \frac{1}{2\kappa}\sigma^2 \\ &= \log F(t_0,t) + \frac{1}{\kappa}\frac{1}{F(t_0,t)}\frac{\partial}{\partial t}F(t_0,t) + \frac{\sigma^2}{4\kappa}(1 - e^{-2\kappa(t-t_0)}).\end{aligned}$$

Now we can compute the price of a European call as

$$C(t_0, S_0; T, X, \theta, \kappa, r, \sigma) = e^{-r(T-t_0)} \mathbb{E}(\max(S_T - X, \sigma)),$$

where the expectation is computed with $\theta(t)$ as defined above. The terminal price S_t is lognormally distributed with expectation $F(t_0, T)$ by construction of θ.
The standard deviation of the logarithm of S_T is

$$\tilde{\sigma}_T = \sigma \sqrt{\frac{1 - e^{-2\kappa(T-t_0)}}{2\kappa}}.$$

Therefore the expectation can be expressed in a Black-Scholes type framework with

$$C(t_0, S_0; T, X, \theta, \kappa, \sigma) = C_{BS}(t_0, e^{-r(T-t_0)} F(t_0, T); T, X, r, \tilde{\sigma}_t).$$

3.3 Jump-Diffusion Processes

Using Jump-Diffusion Processes (JDPs) has the advantage that spikes (large jumps) and fat-tails of the return distribution can be modelled. JDPs are combinations of a diffusion process (BM) and a jump process, mostly a Poisson process q_t. Recall that

$$\begin{aligned}\mathbb{P}(dq_t = 0) &= 1 - \lambda dt \quad \text{"no jump} \\ \mathbb{P}(dq_t = 1) &= \lambda dt \quad \text{"one jump"} \\ \mathbb{P}(dq_t > 1) &= 0\boldsymbol{o}(dt) \quad \text{"more than one jump".}\end{aligned}$$

Furthermore, we assume that increments of the Poisson process on any two non-overlapping intervals are independent. The parameter λ is called the intensity of the Poisson process. We have

$$\mathbb{E}(q_t) = \lambda t \quad , \quad Var(q_t) = \lambda t.$$

If $\tau_1, \tau_2, \ldots, \tau_n, \ldots$ are the arrival times of jumps, then the random variable $X_i = \tau_{i+1} - \tau_i$, the length of the intervals between jumps, are independent and have exponential distribution with parameter λ, that is

$$\mathbb{P}(X_i < x) = \begin{cases} 1 - e^{-\lambda x} & \text{for} \quad x \geq 0 \\ 0 & \text{for} \quad x < 0. \end{cases}$$

For a pure Poisson process all jumps have magnitude one. Since such a constant magnitude is not realistic, jumps of a random magnitude are usually introduced in pricing models. Thus a compound Poisson process is used for a jump component.
The typical representation is

$$\frac{dS_t}{S_t} = (\mu - \lambda\kappa)dt + \sigma dW_t + (Y_t - 1)dq_t$$

or after the usual $Z_t = \log S_t$

$$dZ_t = (\mu - \lambda\kappa - \frac{1}{2}\sigma^2)dt + \sigma dW_t + \log(Y_t)dq_t,$$

with	S_t	the spot price
	W_t	standard BM
	μ	expected instantaneous rate of relative change in spot prices
	σ	volatility (on intervals not containing jumps)
	q_t	Poisson process
	λ	intensity of the Poisson process
	$Y_t - 1$	a random variable representing the magnitude of jumps in price returns, $Y_t \geq 0$
	κ	the expected jump magnitude, $\mathbb{E}(Y_t - 1) = \kappa$.

The drift adjustment is made to ensure that the total rate of relative price changes (expected diffusion and expected jump rate) is equal to μ.
A typical choice for the random sizes of jumps is

$$\log(Y_t) \sim \mathcal{N}(\gamma, \delta)$$

with $\gamma = \log(1+\kappa) - \delta^2/2$. (Observe Y_t is lognormally distributed.) We can now derive the distribution of S_t or $Z_t = \log S_t$. Conditionally on exactly n jumps in a time interval $(t, t+\triangle t)$ we have

$$\mathbb{P}(\triangle Z_t = x | \sharp \quad \text{jumps} = n) = \mathcal{N}((\mu - \lambda\kappa - \frac{1}{2}\sigma^2)\triangle t + n\gamma, \sqrt{\sigma^2 \triangle t + n\delta^2}).$$

We know that for a Poisson process the probability of n jumps in the interval $(t, t+\triangle t)$ is

$$\mathbb{P}(dq_t = n) = \frac{e^{-\lambda \triangle t}(\lambda \triangle t)^n}{n!}.$$

Therefore we find for the unconditional distribution

$$\mathbb{P}(\triangle Z_t \leq x) = \sum_{n=0}^{\infty} \frac{e^{-\lambda \triangle t}(\lambda \triangle t)^n}{n!} \Phi(x; (\mu - \lambda\kappa - \frac{1}{2}\sigma^2)\triangle t + n\gamma, \sqrt{\sigma^2 \triangle t + n\delta^2}).$$

By independence this argument allows to compute the distribution of $Z_T | Z_t$ (replace $\triangle t$ by $\tau = T - t$). In order to price options in this model we replace the drift rate μ by r (to use the risk-neutral valuation approach. In addition, the market is incomplete here and a further argument is needed). Then, as usual,

$$F(t, T) = \mathbb{E}(S_T) = e^{r(T-t)} S_t$$

and the price of European call with strike X is $(\tau = T - t)$

$$C_{JDP} = \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} C_{BS}(t, S; T, X, r - \lambda\kappa + \frac{n\lambda}{\tau}, \sqrt{\sigma^2 + \frac{n\delta^2}{\tau}})$$

and for the put

$$P_{JDP} = \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} P_{BS}(t, S; T, X, r - \lambda\kappa + \frac{n\lambda}{\tau}, \sqrt{\sigma^2 + \frac{n\delta^2}{\tau}}).$$

In order to include spikes, we need to force to return the process quickly to its pre-jumps level. Hence we add mean reversion

$$\frac{dS_t}{S_t} = \tilde{\kappa}(\theta - \lambda\kappa - \log S_t)dt + \sigma dW_t + (Y_t - 1)dq_t$$

with $\tilde{\kappa}$ the strength of mean reversion and

$$\theta - \lambda\kappa - \frac{1}{2\kappa}\sigma^2$$

the long-term mean of the logarithm of spot prices.

The equivalent form is

$$dZ_t = \tilde{\kappa}(\theta - \lambda\kappa - \frac{1}{2\kappa}\sigma^2 - Z_t)dt + \sigma dW_t + \log(Y_t)dq_t.$$

The drawback is that we now have at least six parameters for calibration: $\tilde{\kappa}, \theta, \lambda, \sigma, \kappa, \delta$. Additionally, if we want to calibrate the model we might need to introduce more parameters (or functions).

3.4 Modelling based on Non-Gaussian Ornstein-Uhlenbeck processes

3.4.1 The Normal Inverse Gaussian Distribution

The normal inverse Gaussian distribution is a 4 parameter family of distributions belonging to the class of generalized hyperbolic distributions. We shall denote it by $NIG(\mu, \alpha, \beta, \delta)$. The density function of the NIG distribution (compare chapter 2) is explicitly given as

$$f(x;\mu,\alpha,\beta,\delta) = \frac{\alpha\delta}{\pi}\exp\{\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)\}\frac{K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})}{\sqrt{\delta^2+(x-\mu)^2}}\,. \tag{3.1}$$

Here, $\mu \in I\!R$ is the location of the density, $\beta \in I\!R$ is the skewness parameter, $\alpha \geq |\beta|$ measures the heaviness of the tails and finally, $\delta \geq 0$ is the scale parameter. The function K_1 is the modified Bessel function of the third kind and index 1.
Let $IG(\delta, \gamma)$ denote the inverse Gaussian distribution with density function

$$g(z;\delta,\gamma) = \frac{\delta}{\sqrt{2\pi}}\,e^{\delta\gamma}z^{-3/2}\exp\{-\frac{1}{2}(\delta^2 z^{-1}+\gamma^2 z)\}, \qquad z > 0\,. \tag{3.2}$$

The $NIG(\mu, \alpha, \beta, \delta)$ distribution is a normal variance-mean mixture [1]. In fact, it occurs as the marginal distribution of X for a pair of random variables (Z, X) where Z follows the $IG(\delta, \sqrt{\alpha^2-\beta^2})$ distribution while conditional on Z the distribution of X is normal: $X \sim N(\mu+\beta Z, Z)$. This is the reason why we refer to the distribution (3.1) as the normal inverse Gaussian distribution.
We give now some properties of the normal inverse Gaussian distributions and of random variables that are NIG distributed.
It follows immediately from (3.1) that the moment generating function of the normal inverse Gaussian distribution is

$$\begin{aligned} m(u;\mu,\alpha,\beta,\delta) &= I\!E\, e^{ux} = \int_{-\infty}^{\infty} e^{ux} f(x;\mu,\alpha,\beta,\delta)\,dx \\ &= \int_{-\infty}^{\infty} e^{\left[\delta\sqrt{\alpha^2-\beta^2}+\beta x+ux-\beta u\right]}\,\frac{\alpha\delta}{\pi}\,\frac{K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})}{\sqrt{\delta^2+(x-\mu)^2}}\,dx \\ &= \int_{-\infty}^{\infty} e^{\left[\delta\sqrt{\alpha^2-\beta^2}+(\beta+u)(x-\mu)+\mu u\right]}\,\frac{\alpha\delta}{\pi}\,\frac{K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})}{\sqrt{\delta^2+(x-\mu)^2}}\,dx \\ &= e^{\left[\delta\sqrt{\alpha^2-\beta^2}-\delta\sqrt{\alpha^2-(\beta+u)^2}+u\mu\right]}\int_{-\infty}^{\infty} f(x;\mu,\alpha,\beta+u,\delta)\,dx\,. \end{aligned}$$

[1] A random variable X is said to be of variance-mean mixture type if X can be interpreted in law as $X \stackrel{d}{=} \beta Z + \varepsilon$, where ε and $Z(>0)$ are independent random variables with $\varepsilon \sim N(\mu, Z)$.

Thus,

$$m_{NIG}(u;\mu,\alpha,\beta,\delta) = \exp\left\{\delta\left[\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+u)^2}\right] + \mu u\right\}. \tag{3.3}$$

Hence, all moments of $NIG(\mu,\alpha,\beta,\delta)$ have simple explicit expressions and, in particular, the mean and variance are

$$\begin{aligned} I\!\!E\,[X] &= \mu + \delta\,\frac{\beta/\alpha}{(1-(\beta/\alpha)^2)^{1/2}} \\ Var\,[X] &= \frac{\delta}{\alpha\big(1-(\beta/\alpha)^2\big)^{3/2}} \end{aligned}$$

The moment generating function of the corresponding inverse Gaussian distribution can be calculated analogously:

$$\begin{aligned} m_{IG}(u;\delta,\sqrt{\alpha^2-\beta^2}) &= I\!\!E\, e^{uz} = \int_{-\infty}^{\infty} e^{uz} g(z;\delta,\sqrt{\alpha^2-\beta^2})\,dz \\ &= e^{\left[\delta\sqrt{\alpha^2-\beta^2}-\delta\sqrt{\alpha^2-\beta^2+2u}\right]}\,\frac{\delta}{\sqrt{2\pi}}\,e^{\delta\sqrt{\alpha^2-\beta^2+2u}} \\ &\quad\times \int_{-\infty}^{\infty} z^{-3/2}\exp\Big\{-\frac{1}{2}\big(\delta^2 z^{-1} + (\alpha^2-\beta^2+2u)z\big)\Big\}\,dz \\ &= \exp\left\{\delta\left[\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-\beta^2+2u}\right]\right\}. \end{aligned} \tag{3.4}$$

We define the normal inverse Gaussian Lévy process as the homogeneous Lévy process (i. e. Lévy process with stationary increments) $L = \{L_t,\, t \geq 0\}$ for which the moment generating function of L_t is

$$m_t(u;\mu,\alpha,\beta,\delta) = I\!\!E e^{uL_t} = m(\mu,\alpha,\beta,\delta)^t \tag{3.5}$$

where $m(\mu,\alpha,\beta,\delta)$ is given by (3.3). The moment generating function m_t of L_t is thus expressible as

$$m_t(u;\mu,\alpha,\beta,\delta) = m(u;t\mu,\alpha,\beta,t\delta)\,. \tag{3.6}$$

Representation by subordination

As a direct consequence of the mixture representation of the normal inverse Gaussian distribution we find that the normal inverse Gaussian Lévy process L_t may be described, via random time change of a Brownian motion, as

$$L_t = \mu t + W_{Z_t}\,, \tag{3.7}$$

where W_t is the Brownian motion with drift β and diffusion coefficient 1 and where Z_t, stochastically independent of W_t, is the inverse Gaussian Lévy process with parameters δ and $\sqrt{\alpha^2-\beta^2}$. The latter process is defined as the homogeneous Lévy process for which the density is given by (3.2). The variate Z_t has the interpretation of being the first passage time to level δt of a Brownian motion with drift $\sqrt{\alpha^2-\beta^2}$ and diffusion coefficient 1. In other words, (3.7) represents the normal inverse Gaussian Lévy process as a subordination of Brownian motion by the inverse Gaussian Lévy process.

3.4.2 Ornstein-Uhlenbeck Processes

An important role in the financial applications play processes driven by Lévy process. They allow to construct a large family of mean-reverting jump processes with linear dynamics on which various properties such as positiveness or the choice of a marginal distribution can be imposed.

In this section we will consider the so-called *Ornstein-Uhlenbeck-type processes* or *OU processes* for short. They offer the possibility of capturing important distributional deviations from Gaussianity and for flexible modelling of dependence structures.
For any $t > 0$ and $\lambda > 0$ we can use the representation of a random variable X in the class L as follows:

$$\begin{aligned} X &= \int_0^\infty e^{-\lambda s}\, dL_{\lambda s} \\ &= \int_t^\infty e^{-\lambda s}\, dL_{\lambda s} + \int_0^t e^{-\lambda s}\, dL_{\lambda s} \\ &= e^{-\lambda t} X_0 + U_t\,, \end{aligned}$$

where

$$X_0 = \int_0^\infty e^{-\lambda s}\, dL_{\lambda(s+t)}$$

and

$$U_t = e^{-\lambda t} \int_0^t e^{\lambda s}\, dL_{\lambda(t-s)}$$

X_0 and U_t being independent. Note that $X_0 \overset{d}{=} X$ and

$$U_t \overset{d}{=} \int_0^t e^{-\lambda(t-s)}\, dL_{\lambda s}\,.$$

Definition 3.4.1. *A stochastic process $(X_t)_{t\geq 0}$, such that $X_t \overset{d}{=} X$, is said to be of* ***Ornstein-Uhlenbeck type*** *if it satisfies a stochastic differential equation of the form*

$$dX_t = -\lambda X_t\, dt + dL_{\lambda t}\,, \tag{3.8}$$

for any $\lambda > 0$. The process L_t is a homogeneous Lévy process, termed the ***Background Driving Lévy Process*** *(BDLP) corresponding to the process X_t.*

The process $(X_t)_{t\geq 0}$ is stationary on the positive half-line, i. e. there exists a distribution D, called the *stationary distribution* or the *marginal distribution*, such that X_t follows D for every t if the initial X_0 is chosen according to D. If X_t is an OU process with marginal law D, then we say that X_t is a D-OU process.

Remark 3.4.1. *Positive OU processes present a particular interest in the context of stochastic volatility modelling.*

The stationary process $(X_t)_{t\geq 0}$ can be extended to a stationary process on the whole real line. To do this we introduce an independent copy of the process L_t but modify it to be RCLL, thus obtaining a process $\bar{L}_t$.
For $t < 0$ define L_t by $L_t = \bar{L}_{-t}$, and for $t \in I\!R$ let

$$X_t = e^{-\lambda t} \int_{-\infty}^t e^{\lambda s}\, dL_{\lambda s}\,.$$

Then $(L_t)_{t\in I\!R}$ is a (homogeneous, RCLL) Lévy process; and $(X_t)_{t\in I\!R}$ is a strictly stationary process of Ornstein-Uhlenbeck type.
From the above discussion it follows, in particular, that there exists a $(X_t)_{t\in I\!R}$ stationary OU process such that $X_t \sim NIG(\mu, \alpha, \beta, \delta)$ for every $t \in I\!R$. We refer to this process as the *normal inverse Gaussian OU process*, or the *NIG OU process* for short.

3.4.3 The Spot Price Model with Lévy Noise

Summarizing antecedent results we propose in the last section of the chapter the model of (deseasonalized) spot prices as the exponential of a non-Gaussian Ornstein-Uhlenbeck process.
Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, I\!P)$ be a complete filtered probability space which satisfies the *usual conditions*, i.e. the σ-algebra $\mathcal{F}$ is $I\!P$-complete and the $\mathcal{F}_t$, $t \geq 0$ must contain all the sets in $\mathcal{F}$ of $I\!P$-probability zero, and be right-continuous ($\mathcal{F}_t = \mathcal{F}_{t+}, t \geq 0$).
Introduce a Lévy process L_t with Lévy-Khintchine representation

$$L_t = \chi t + \sigma W_t + \int\limits_{|z|<1} z\tilde{N}\big((0,t], dz\big) + \int\limits_{|z|\geq 1} zN\big((0,t], dz\big)\,, \tag{3.9}$$

where W_t is a standard Brownian motion, χ a constant, $\sigma > 0$ a constant, $N(dt, dz)$ a homogeneous Poisson random measure associated to the Lévy process L_t and $\tilde{N}(dt, dz) = N(dt, dz) - dt\,\mathcal{L}(dz)$ its compensated (Poisson) measure. The σ-finite measure $\mathcal{L}(dz)$ on the Borel sets of $I\!R$ is called the *Lévy measure*, and satisfies the conditions

$$\mathcal{L}(\{0\}) = 0 \qquad \text{and} \qquad \int\limits_{I\!R} \min(1, z^2)\,\mathcal{L}(dz) = \int\limits_{I\!R} (1 \wedge z^2)\,\mathcal{L}(dz) < \infty\,.$$

Alternatively, the Lévy-Khintchine formula can be written as

$$I\!E \exp\{iuL_t\} = \exp\{t\psi(u)\},$$

with $\psi(u)$ being the *cumulant characteristic function* or *characteristic exponent*, which satisfies

$$\psi(u) = i\chi u - \frac{\sigma^2}{2}u^2 + \int\limits_{I\!R} \{e^{iuz} - 1 - iuz\mathbf{1}_{\{|z|<1\}}\}\mathcal{L}(dz)\,. \tag{3.10}$$

We say that our infinitely divisible distribution has a triplet of *Lévy characteristics* (or *Lévy triplet* for short) $\big[\chi, \sigma^2, \mathcal{L}(dz)\big]$. If the Lévy measure is of the form $\mathcal{L}(dz) = l(z)dz$, we call $l(z)$ the *Lévy density*. The Lévy density has the same mathematical requirements as a probability density, except that it does not need to be integrable and must have zero mass at the origin.
From the Lévy-Khintchine formula, we see that, in general, a Lévy process consists of three independent parts: a linear deterministic part, a Brownian part and a pure jump part. The Lévy measure $\mathcal{L}(dz)$ dictates how the jumps occur. Jumps of sizes in the set A occur according to a Poisson process with intensity parameter $\int\limits_A \mathcal{L}(dz)$.
Assume S_t is the spot price at time t, which we model as the stochastic process

$$S_t = \Lambda(t)\, e^{X_t}\,, \tag{3.11}$$

where $\Lambda(t) : [0, T] \longrightarrow I\!R$ is continuous deterministic function of time modelling the seasonality. T is assumed to be a fixed finite planning horizon. The non-Gaussian Ornstein-Uhlenbeck process X_t has dynamics

$$dX_t = a(m - X_t)\,dt + dL_t\,, \tag{3.12}$$

and initial state $X_0 = x$. The speed of mean-reversion is given by $a \geq 0$, while $m > 0$ indicates a long-term mean of the process.
To solve (3.12) we, first, find the solution to the homogeneous differential equation. It is $\check{X}_t = xe^{-at}$, whereas the solution $\hat{X}_t$, derived by the method of variation of constants, results from

$$\begin{aligned} d(CX_t) = -ae^{-at}C(t) + C'(t)e^{-at} &= -ae^{-at}C(t) + am\,dt + dL_t \\ \Leftrightarrow\; C'(t) &= e^{at}(am\,dt + dL_t)\,. \end{aligned}$$

Integrating the last equality leads to

$$C(t) = am \int_0^t e^{as}\, ds + \int_0^t e^{as}\, dL_s = m(e^{at} - 1) + \int_0^t e^{as}\, dL_s \,.$$

$$\Longrightarrow \qquad \hat{X}_t = C(t)e^{-at} = m(1 - e^{-at}) + \int_0^t e^{-a(t-s)}\, dL_s \,.$$

Thus, the solution X_t to the stochastic differential equation (3.12) is the sum of $\check{X}_t$ and $\hat{X}_t$:

$$X_t = xe^{-at} + m(1 - e^{-at}) + \int_0^t e^{-a(t-s)}\, dL_s \,. \tag{3.13}$$

Concluding, we remark that in modelling considerations we shall pay particular attention to the process X_t, being the NIG OU process, i. e. the BDLP of X_t will be the NIG Lévy process[2] L_t. The representation of L_t, as the sum of three independent homogeneous Lévy processes, is given in Theorem **??**.

In particular, when $L_t = \chi t + \sigma W_t$, the process S_t reduces to the classical mean-reversion model of Schwartz. Choosing $a = 0$ in (3.12), we obtain $X_t = L_t$ and the spot dynamics becomes the exponential of a Lévy process, generalizing the classical geometric Brownian motion model. Hence, our geometric spot price model (3.11) generalizes geometric Brownian motion and Schwartz' mean-reversion dynamics.

We also remark in passing that starting the dynamics of the spot price directly in exponential form rather than as the solution to a stochastic differential equation is advantageous when fitting to data.

[2]A process L_t is called NIG Lévy process if L_1 is distributed according to the NIG distribution.

Chapter 4

Forward Price Processes

4.1 Heath-Jarrow-Morton (HJM) type models

Recall the Black Futures model in §1.4.3. There a futures with fixed maturity was considered. We extend that now to include the whole term structure for futures in analogy to the HJM-type models for interest rates. So

$$dF(t,T) = \mu(t,T,F(t,T))dt + \sum_j \sigma_j(t,T,F(t,T))dW_t^j, \quad 0 \le t \in T,$$

where W_t^j are independent BMs. As usual under the risk neutral measure the drift term disappears (unlike in the interest market Forwards are traded).
So we consider only

$$dF(t,T) = \sum_j \sigma_j(t,T,F(t,T))dW_t^j.$$

We now consider as an Example the two-factor forward curve dynamics model introduced by (Schwartz and Smith 2000). Here

$$\log(F(t,T)) = e^{-\kappa(T-t)}X_t + \xi_t + A(T-t),$$

where the variables X_t and ξ_t are linked to the spot with

$$\log(S_t) = X_t + \xi_t$$

and

$$\begin{aligned} dX_t &= (-\kappa X_t - \lambda_x)dt + \sigma_x dW_t^x \\ d\xi_t &= (\mu_\xi - \lambda_\xi)dt + \sigma_\xi dW_t^\xi \\ & \mathbb{E}(dW_t^X dW_t^\xi) = \rho_{x,\xi}dt \end{aligned}$$

with

$$A(\tau) = \mu_\xi \tau - (1-e^{-\kappa\tau})\frac{\lambda_x}{\kappa} + \frac{1}{2}\left[(1-e^{-2\kappa\tau})\frac{\sigma_x^2}{2\kappa} + \sigma_\xi^2\tau + 2(1-e^{-\kappa\tau})\frac{\rho_{x\xi}\sigma_x\sigma_\xi}{\kappa}\right].$$

Parameters $(\kappa, \sigma_x, \mu_\xi, \sigma_\xi, \rho_{x,\xi})$ have their usual meaning and λ_x, λ_ξ are risk premia introduced for proper risk-adjustment.

ξ_t characterizes the long term behavior of spot prices
X_t represents the short-term deviation of spot prices from long-term levels.

A related approach is to use

$$\frac{dF(t,T)}{F(t,T)} = \sum_j \sigma_j(t,T)dW_\tau^j$$

to obtain a log-normal structure.
We find that

$$F(t,T) = F(0,T)\cdot \exp\left\{\sum_j \left[\frac{1}{2}\int_0^t \sigma_j^2(u,T)du + \int_0^t \sigma_j(u,T)dW_u^j\right]\right\}.$$

A widely used model of this type is the (Schwartz 1997) model with

$$\begin{aligned}\sigma_1(t,T) &= \sigma_1 - \rho\sigma_2\frac{1-e^{-\kappa(T-t)}}{\kappa}\\ \sigma_2(t,T) &= \sigma_2\sqrt{1-\rho^2}\frac{1-e^{-\kappa(T-t)}}{\kappa}.\end{aligned}$$

It is obvious that as soon as σ_i are chosen a correlation structure between different contracts of the curve is introduced. E.g. in case of $j=2$ the correlation of $\log F(t,T_1)$ and $\log F(t,T_2)$ is

$$\rho(t,T_1,T_2) = \frac{\sigma_1(t,T_1)\sigma_1(t,T_2)+\sigma_2(t,T_1)\cdot\sigma_2(t,T_2)}{\sqrt{\sigma_1^2(t,T_1)+\sigma_2^2(t,T_1)}\sqrt{\sigma_1^2(t,T_2)+\sigma_2^2(t,T_2)}}.$$

4.2 Market Models

Due to the lack of sufficient historical data some modelling approaches try to enhance the available data set by adding to it forward-looking market data. This leads to market models, where we only consider tradable contracts. the dynamics are described by

$$dF_{t,k} = \sum_j \sigma_j(t,j,F_{t,k})dW_t^j,\quad 0\le t\le T,\quad k=1,\dots,K$$

where $F_{t,k}\equiv F_{t,T_k}$ is the k-th tradable forward contract. In case the volatilities are assumed to be linear functions of the forward prices we have

$$\frac{dF_{t,k}}{F_{t,k}} = \sum_{j=1}^J \eta_{j,k}(t)dW_t^j,\quad k=1,\dots K\quad (W_t^1,\dots,W_t^J)\ \text{uncorrelated.}$$

If the number of common factors J equals the number of tradable contracts K, then we can represent the dynamics of the forward curve by using K-correlated contract-specific factors

$$\frac{dF_{t,k}}{F_{t,k}} = v_k(t)d\tilde{W}_{t,k}\quad k=1,\dots K$$

where we have the correlation condition

$$\mathrm{Corr}(\tilde{W}_{t,k},\tilde{W}_{t,j}) = \rho(t,k,j).$$

We now apply the market model approach to a multi-commodity case. Assume we have N commodities (Gas, Oil, Coal, Electricity,...). For the i-th commodity the evolution of monthly forward prices $F_{t,k}^i$, with k being the index of the forward, is given by

$$dF_{t,k}^i = \sigma_k^i(t,T,F_{t,k})d\tilde{W}_{t,k}^i,\quad t\ge 0, i=1,\dots,N\quad k=1,\dots K,$$

where $\tilde{W}_{t,k}^i$ are correlated BMs. Typically the volatility is $\sigma_k^i(t,T)F_{t,k}^i$, so that we obtain a log-normal distribution. Observe that these equations also imply the evolution of the spot prices via $S_t^i = F_{t,t}^i$. As initial conditions we use the current forward curve for the commodity i, $i=1,\dots,N$; $F_{0,k}^i$.

Thus we have $I\!E(F^i_{t,k}) = F^i_{0,k}$ for every $t \geq 0$. We can also calibrate the model to any option prices we observe

$$C^i_M = C_B(F^i_{0,k}; F_k, X^i_k, r_k, \sigma^i_N),$$

where C^i_M is the quoted option price of the i-th commodity, $F^i_{0,k}$ is the current price of the forward with settlement date T_k, X^i_k is the strike, r_k is the risk-free rate, and σ^i_M is the volatility of the monthly forward prices. The option value is then computed using Black's formula. Finally we can try to match correlations between logreturns for different forward contracts of the same commodity, as well as the correlation between different commodities.
Thus we need to have

$$\frac{I\!E(d\log(F^i_t, T_1) \cdot d\log(F^j_t, T_2))}{\sqrt{Var(d\log(F^i_{t,T_1}))}\sqrt{Var(d\log(F^j_{t,T_2}))}} = \rho^{i,j}(t, T_1, T_2).$$

For the case of log-normal specifications

$$\frac{dF^i_{t,T_k}}{F^i_{t,T_k}} = \sigma^i(t, T_k)d\tilde{W}_{t,k}, \quad t \geq 0, i = 1, \ldots, N, \quad k = 1, \ldots K.$$

1. Matching current forward prices $I\!E(F^i_{t,T_k}) = F^i_{0,T_k}$ as above.
2. **Option prices.** We need to find $\sigma^i(t, T_k)$ such that

   $$\frac{1}{T_k - t}\int_t^{T_k} [\sigma^i(S, T_k)]^2 ds = \sigma^i_M(t, T_k)^2,$$

   where $\sigma^i_M(t, T_k)$ is the implied volatility (from Black's formula).

   Therefore

   $$[\sigma^i(t, T_K)]^2 = -\frac{\partial}{\partial t}\{T_k - t[\sigma^i_M(t, T_k)]^2\}.$$

3. The correlation structure can be matched as above.

Appendix A

Continuous-time Financial Market Models

A.1 The Stock Price Process and its Stochastic Calculus

A.1.1 Continuous-time Stochastic Processes

A *stochastic process* $X = (X(t))_{t\geq 0}$ is a family of random variables defined on $(\Omega, \mathcal{F}, I\!P, I\!F)$. We say X is *adapted* if $X(t) \in \mathcal{F}_t$ (i.e. $X(t)$ is $\mathcal{F}_t$-measurable) for each t: thus $X(t)$ is known when $\mathcal{F}_t$ is known, at time t.
The martingale property in continuous time is just that suggested by the discrete-time case:

Definition A.1.1. *A stochastic process $X = (X(t))_{0\leq t<\infty}$ is a martingale relative to $(I\!F, I\!P)$ if*

(i) X is adapted, and $I\!E\,|X(t)| < \infty$ for all $\leq t < \infty$;

(ii) $I\!E[X(t)|\mathcal{F}_s] = X(s)$ $I\!P - a.s.$ $(0 \leq s \leq t)$,

and similarly for sub- and supermartingales.

There are regularisation results, under which one can take $X(t)$ RCLL in t (basically $t \to I\!EX(t)$ has to be right-continuous). Then the analogues of the results for discrete-time martingales hold true.
Interpretation. Martingales model fair games. Submartingales model favourable games. Supermartingales model unfavourable games.
Brownian motion originates in work of the botanist Robert Brown in 1828. It was introduced into finance by Louis Bachelier in 1900, and developed in physics by Albert Einstein in 1905.

Definition A.1.2. *A stochastic process $X = (X(t))_{t\geq 0}$ is a standard (one-dimensional) Brownian motion, BM or $BM(I\!R)$, on some probability space $(\Omega, \mathcal{F}, I\!P)$, if*

(i) $X(0) = 0$ a.s.,

(ii) X has independent increments: $X(t+u) - X(t)$ is independent of $\sigma(X(s) : s \leq t)$ for $u \geq 0$,

(iii) X has stationary increments: the law of $X(t+u) - X(t)$ depends only on u,

(iv) X has Gaussian increments: $X(t+u) - X(t)$ is normally distributed with mean 0 and variance u, $X(t+u) - X(t) \sim N(0, u)$,

(v) X has continuous paths: $X(t)$ is a continuous function of t, i.e. $t \to X(t, \omega)$ is continuous in t for all $\omega \in \Omega$.

We shall henceforth denote standard Brownian motion $BM(\mathbb{R})$ by $W = (W(t))$ (W for Wiener), though $B = (B(t))$ (B for Brown) is also common. Standard Brownian motion $BM(\mathbb{R}^d)$ in d dimensions is defined by $W(t) := (W_1(t), \ldots, W_d(t))$, where $W_1, \ldots, W_d$ are independent standard Brownian motions in one dimension (independent copies of $BM(\mathbb{R})$).
We have Wiener's theorem:

Theorem A.1.1 (Wiener)**.** *Brownian motion exists.*

For further background, see any measure-theoretic text on stochastic processes. A treatment starting directly from our main reference of measure-theoretic results, Williams Williams (1991), is Rogers and Williams Rogers and Williams (1994), Chapter 1. The classic is Doob's book, Doob (1953), VIII.2. Excellent modern texts include **?** (see particularly Karatzas and Shreve (1991), §2.2-4 for construction).

A.1.2 Stochastic Analysis

Stochastic integration was introduced by K. Itô in 1944, hence its name Itô calculus. It gives a meaning to

$$\int_0^t X dY = \int_0^t X(s,\omega) dY(s,\omega),$$

for suitable stochastic processes X and Y, the integrand and the integrator. We shall confine our attention here mainly to the basic case with integrator Brownian motion: $Y = W$. Much greater generality is possible: for Y a continuous martingale, see Karatzas and Shreve (1991) or Revuz and Yor (1991); for a systematic general treatment, see Protter (2004).
Suppose that b is adapted and locally integrable (so $\int_0^t b(s)ds$ is defined as an ordinary integral), and σ is adapted and measurable with $\int_0^t \mathbb{E}\left(\sigma(u)^2\right) du < \infty$ for all t (so $\int_0^t \sigma(s)dW(s)$ is defined as a stochastic integral). Then

$$X(t) := x_0 + \int_0^t b(s)ds + \int_0^t \sigma(s)dW(s)$$

defines a stochastic process X with $X(0) = x_0$. It is customary, and convenient, to express such an equation symbolically in differential form, in terms of the stochastic differential equation

$$dX(t) = b(t)dt + \sigma(t)dW(t), \quad X(0) = x_0. \tag{A.1}$$

Now suppose $f : \mathbb{R} \to \mathbb{R}$ is of class C^2. The question arises of giving a meaning to the stochastic differential $df(X(t))$ of the process $f(X(t))$, and finding it. Given a partition $\mathcal{P}$ of $[0,t]$, i.e. $0 = t_0 < t_1 < \ldots < t_n = t$, we can use Taylor's formula to obtain

$$\begin{aligned} f(X(t)) - f(X(0)) &= \sum_{k=0}^{n-1} f(X(t_{k+1})) - f(X(t_k)) \\ &= \sum_{k=0}^{n-1} f'(X(t_k))\Delta X(t_k) \\ &+ \frac{1}{2}\sum_{k=0}^{n-1} f''(X(t_k) + \theta_k \Delta X(t_k))(\Delta X(t_k))^2 \end{aligned}$$

with $0 < \theta_k < 1$. We know that $\sum(\Delta X(t_k))^2 \to \langle X \rangle (t)$ in probability (so, taking a subsequence, with probability one), and with a little more effort one can prove

$$\sum_{k=0}^{n-1} f''(X(t_k) + \theta_k \Delta X(t_k))(\Delta X(t_k))^2 \to \int_0^t f''(X(u)) d\langle X \rangle (u).$$

The first sum is easily recognized as an approximating sequence of a stochastic integral; indeed, we find

$$\sum_{k=0}^{n-1} f'(X(t_k))\Delta X(t_k) \to \int_0^t f'(X(u))dX(u).$$

So we have

Theorem A.1.2 (Basic Itô formula)**.** *If X has stochastic differential given by A.1 and $f \in C^2$, then $f(X)$ has stochastic differential*

$$df(X(t)) = f'(X(t))dX(t) + \frac{1}{2}f''(X(t))d\langle X\rangle(t),$$

or writing out the integrals,

$$f(X(t)) = f(x_0) + \int_0^t f'(X(u))dX(u) + \frac{1}{2}\int_0^t f''(X(u))d\langle X\rangle(u).$$

More generally, suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a function, continuously differentiable once in its first argument (which will denote time), and twice in its second argument (space): $f \in C^{1,2}$. By the Taylor expansion of a smooth function of several variables we get for t close to t_0 (we use subscripts to denote partial derivatives: $f_t := \partial f/\partial t$, $f_{tx} := \partial^2 f/\partial t\partial x$):

$$\begin{aligned} f(t, X(t)) &= f(t_0, X(t_0)) \\ &\quad + (t-t_0)f_t(t_0, X(t_0)) + (X(t) - X(t_0))f_x(t_0, X(t_0)) \\ &\quad + \frac{1}{2}(t-t_0)^2 f_{tt}(t_0, X(t_0)) + \frac{1}{2}(X(t)-X(t_0))^2 f_{xx}(t_0, X(t_0)) \\ &\quad + (t-t_0)(X(t)-X(t_0))f_{tx}(t_0, X(t_0)) + \ldots, \end{aligned}$$

which may be written symbolically as

$$df = f_t dt + f_x dX + \frac{1}{2}f_{tt}(dt)^2 + f_{tx}dtdX + \frac{1}{2}f_{xx}(dX)^2 + \ldots.$$

In this, we substitute $dX(t) = b(t)dt + \sigma(t)dW(t)$ from above, to obtain

$$\begin{aligned} df &= f_t dt + f_x(bdt + \sigma dW) \\ &\quad + \frac{1}{2}f_{tt}(dt)^2 + f_{tx}dt(bdt + \sigma dW) + \frac{1}{2}f_{xx}(bdt + \sigma dW)^2 + \ldots \end{aligned}$$

Now using the formal multiplication rules $dt \cdot dt = 0, dt \cdot dW = 0, dW \cdot dW = dt$ (which are just shorthand for the corresponding properties of the quadratic variations, we expand

$$(bdt + \sigma dW)^2 = \sigma^2 dt + 2b\sigma dtdW + b^2(dt)^2 = \sigma^2 dt + \text{higher-order terms}$$

to get finally

$$df = \left(f_t + bf_x + \frac{1}{2}\sigma^2 f_{xx}\right)dt + \sigma f_x dW + \text{higher-order terms}.$$

As above the higher-order terms are irrelevant, and summarising, we obtain *Itô's lemma*, the analogue for the Itô or stochastic calculus of the chain rule for ordinary (Newton-Leibniz) calculus:

Theorem A.1.3 (Itô's Lemma)**.** *If $X(t)$ has stochastic differential given by A.1, then $f = f(t, X(t))$ has stochastic differential*

$$df = \left(f_t + bf_x + \frac{1}{2}\sigma^2 f_{xx}\right)dt + \sigma f_x dW.$$

That is, writing f_0 *for* $f(0, x_0)$*, the initial value of* f*,*

$$f = f_0 + \int_0^t (f_t + bf_x + \frac{1}{2}\sigma^2 f_{xx})dt + \int_0^t \sigma f_x dW.$$

We will make good use of:

Corollary A.1.1. $\mathbb{E}\left(f(t, X(t))\right) = f_0 + \int_0^t \mathbb{E}\left(f_t + bf_x + \frac{1}{2}\sigma^2 f_{xx}\right) dt.$

Proof. $\int_0^t \sigma f_2 dW$ is a stochastic integral, so a martingale, so its expectation is constant (= 0, as it starts at 0). ∎

Geometric Brownian Motion

Now that we have both Brownian motion W and Itô's Lemma to hand, we can introduce the most important stochastic process for us, a relative of Brownian motion - *geometric* (or *exponential*, or *economic*) Brownian motion.
Suppose we wish to model the time evolution of a stock price $S(t)$ (as we will, in the Black-Scholes theory). Consider how S will change in some small time-interval from the present time t to a time $t + dt$ in the near future. Writing $dS(t)$ for the change $S(t + dt) - S(t)$ in S, the *return* on S in this interval is $dS(t)/S(t)$. It is economically reasonable to expect this return to decompose into two components, a *systematic* part and a *random* part. The systematic part could plausibly be modelled by μdt, where μ is some parameter representing the mean rate of return of the stock. The random part could plausibly be modelled by $\sigma dW(t)$, where $dW(t)$ represents the noise term driving the stock price dynamics, and σ is a second parameter describing how much effect this noise has - how much the stock price fluctuates. Thus σ governs how volatile the price is, and is called the *volatility* of the stock. The role of the driving noise term is to represent the random buffeting effect of the multiplicity of factors at work in the economic environment in which the stock price is determined by supply and demand.
Putting this together, we have the stochastic differential equation

$$dS(t) = S(t)(\mu dt + \sigma dW(t)), \quad S(0) > 0, \tag{A.2}$$

due to Itô in 1944. This corrects Bachelier's earlier attempt of 1900 (he did not have the factor $S(t)$ on the right - missing the interpretation in terms of returns, and leading to negative stock prices!) Incidentally, Bachelier's work served as Itô's motivation in introducing Itô calculus. The mathematical importance of Itô's work was recognised early, and led on to the work of Doob (1953), Meyer (1976) and many others (see the memorial volume Ikeda, Watanabe, M., and Kunita (1996) in honour of Itô's eightieth birthday in 1995). The economic importance of geometric Brownian motion was recognised by Paul A. Samuelson in his work from 1965 on (Samuelson (1965)), for which Samuelson received the Nobel Prize in Economics in 1970, and by Robert Merton (see Merton (1990) for a full bibliography), in work for which he was similarly honoured in 1997.
The differential equation (A.2) above has the unique solution

$$S(t) = S(0) \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma dW(t)\right\}.$$

For, writing

$$f(t, x) := \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma x\right\},$$

we have

$$f_t = \left(\mu - \frac{1}{2}\sigma^2\right) f, \quad f_x = \sigma f, \quad f_{xx} = \sigma^2 f,$$

and with $x = W(t)$, one has

$$dx = dW(t), \quad (dx)^2 = dt.$$

Thus Itô's lemma gives

$$\begin{aligned} df(t,W(t)) &= f_t dt + f_x dW(t) + \frac{1}{2} f_{xx}(dW(t))^2 \\ &= f\left(\left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW(t) + \frac{1}{2}\sigma^2 dt\right) \\ &= f(\mu dt + \sigma dW(t)), \end{aligned}$$

so $f(t,W(t))$ is a solution of the stochastic differential equation, and the initial condition $f(0,W(0)) = S(0)$ as $W(0) = 0$, giving existence.

A.1.3 Girsanov's Theorem

Consider first independent $N(0,1)$ random variables $Z_1,\dots,Z_n$ on a probability space $(\Omega, \mathcal{F}, I\!\!P)$. Given a vector $\gamma = (\gamma_1,\dots,\gamma_n)$, consider a new probability measure $\tilde{I\!\!P}$ on $(\Omega,\mathcal{F})$ defined by

$$\tilde{I\!\!P}(d\omega) = \exp\left\{\sum_{i=1}^n \gamma_i Z_i(\omega) - \frac{1}{2}\sum_{i=1}^n \gamma_i^2\right\} I\!\!P(d\omega).$$

As $\exp\{.\} > 0$ and integrates to 1, as $\int \exp\{\gamma_i Z_i\} dI\!\!P = \exp\{\frac{1}{2}\gamma_i^2\}$, this is a probability measure. It is also equivalent to $I\!\!P$ (has the same null sets), again as the exponential term is positive. Also

$$\begin{aligned} &\tilde{I\!\!P}(Z_i \in dz_i, \quad i = 1,\dots,n) \\ = \; &\exp\left\{\sum_{i=1}^n \gamma_i Z_i - \frac{1}{2}\sum_{i=1}^n \gamma_i^2\right\} I\!\!P(Z_i \in dz_i, \quad i=1,\dots,n) \\ = \; &(2\pi)^{-\frac{n}{2}} \exp\left\{\sum_{i=1}^n \gamma_i z_i - \frac{1}{2}\sum_{i=1}^n \gamma_i^2 - \frac{1}{2}\sum_{i=1}^n z_i^2\right\} \prod_{i=1}^n dz_i \\ = \; &(2\pi)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^n (z_i - \gamma_i)^2\right\} dz_1 \dots dz_n. \end{aligned}$$

This says that if the Z_i are independent $N(0,1)$ under $I\!\!P$, they are independent $N(\gamma_i, 1)$ under $\tilde{I\!\!P}$. Thus the effect of the *change of measure* $I\!\!P \to \tilde{I\!\!P}$, from the original measure $I\!\!P$ to the *equivalent* measure $\tilde{I\!\!P}$, is to *change the mean*, from $0 = (0,\dots,0)$ to $\gamma = (\gamma_1,\dots,\gamma_n)$.
This result extends to infinitely many dimensions - i.e., from random vectors to stochastic processes, indeed with random rather than deterministic means. Let $W = (W_1,\dots W_d)$ be a d-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, I\!\!P, I\!\!F)$ with the filtration $I\!\!F$ satisfying the usual conditions. Let $(\gamma(t) : 0 \le t \le T)$ be a measurable, adapted d-dimensional process with $\int_0^T \gamma_i(t)^2 dt < \infty \quad a.s.,\ i = 1,\dots,d$, and define the process $(L(t) : 0 \le t \le T)$ by

$$L(t) = \exp\left\{-\int_0^t \gamma(s)' dW(s) - \frac{1}{2}\int_0^t \|\gamma(s)\|^2 ds\right\}. \tag{A.3}$$

Then L is continuous, and, being the stochastic exponential of $-\int_0^t \gamma(s)' dW(s)$, is a local martingale. Given sufficient integrability on the process γ, L will in fact be a (continuous) martingale. For this, *Novikov's condition* suffices:

$$I\!\!E\left(\exp\left\{\frac{1}{2}\int_0^T \|\gamma(s)\|^2 ds\right\}\right) < \infty. \tag{A.4}$$

We are now in the position to state a version of Girsanov's theorem, which will be one of our main tools in studying continuous-time financial market models.

Theorem A.1.4 (Girsanov)**.** *Let γ be as above and satisfy Novikov's condition; let L be the corresponding continuous martingale. Define the processes $\tilde{W}_i$, $i = 1, \dots, d$ by*

$$\tilde{W}_i(t) := W_i(t) + \int_0^t \gamma_i(s) ds, \quad (0 \le t \le T), \quad i = 1, \dots, d.$$

Then under the equivalent probability measure $\tilde{I\!P}$ (defined on $(\Omega, \mathcal{F}_T)$) with Radon-Nikodým derivative

$$\frac{d\tilde{I\!P}}{dI\!P} = L(T),$$

the process $\tilde{W} = (\tilde{W}_1, \dots, \tilde{W}_d)$ is d-dimensional Brownian motion.

In particular, for $\gamma(t)$ constant $(= \gamma)$, change of measure by introducing the Radon-Nikodým derivative $\exp\left\{-\gamma W(t) - \frac{1}{2}\gamma^2 t\right\}$ corresponds to a change of drift from c to $c - \gamma$. If $I\!F = (\mathcal{F}_t)$ is the Brownian filtration (basically $\mathcal{F}_t = \sigma(W(s), 0 \le s \le t)$ slightly enlarged to satisfy the usual conditions) any pair of equivalent probability measures $\mathbb{Q} \sim I\!P$ on $\mathcal{F} = \mathcal{F}_T$ is a Girsanov pair, i.e.

$$\left.\frac{d\tilde{\mathbb{Q}}}{dI\!P}\right|_{\mathcal{F}_t} = L(t)$$

with L defined as above. Girsanov's theorem (or the Cameron-Martin-Girsanov theorem) is formulated in varying degrees of generality, discussed and proved, e.g. in Karatzas and Shreve (1991), §3.5, Protter (2004), III.6, Revuz and Yor (1991), VIII, Dothan (1990), §5.4 (discrete time), §11.6 (continuous time).

A.2 Financial Market Models

A.2.1 The Financial Market Model

We start with a general model of a frictionless (compare Chapter 1) security market where investors are allowed to trade continuously up to some fixed finite planning horizon T. Uncertainty in the financial market is modelled by a probability space $(\Omega, \mathcal{F}, I\!P)$ and a filtration $I\!F = (\mathcal{F}_t)_{0 \le t \le T}$ satisfying the usual conditions of right-continuity and completeness. We assume that the σ-field $\mathcal{F}_0$ is trivial, i.e. for every $A \in \mathcal{F}_0$ either $I\!P(A) = 0$ or $I\!P(A) = 1$, and that $\mathcal{F}_T = \mathcal{F}$.
There are $d + 1$ primary traded assets, whose price processes are given by stochastic processes $S_0, \dots, S_d$. We assume that the processes $S_0, \dots, S_d$ represent the prices of some traded assets (stocks, bonds, or options).
We have not emphasised so far that there was an implicit numéraire behind the prices $S_0, \dots, S_d$; it is the numéraire relevant for domestic transactions at time t. The formal definition of a numéraire is very much as in the discrete setting.

Definition A.2.1. *A numéraire is a price process $X(t)$ almost surely strictly positive for each $t \in [0, T]$.*

We assume now that $S_0(t)$ is a non-dividend paying asset, which is (almost surely) strictly positive and use S_0 as numéraire. 'Historically' (see Harrison and Pliska (1981)) the money market account $B(t)$, given by $B(t) = e^{r(t)}$ with a positive deterministic process $r(t)$ and $r(0) = 0$, was used as a numéraire, and the reader may think of $S_0(t)$ as being $B(t)$.
Our principal task will be the pricing and hedging of contingent claims, which we model as $\mathcal{F}_T$-measurable random variables. This implies that the contingent claims specify a stochastic cash-flow at time T and that they may depend on the whole path of the underlying in $[0, T]$ - because $\mathcal{F}_T$ contains all that information. We will often have to impose further integrability conditions on the contingent claims under consideration. The fundamental concept in (arbitrage) pricing and

hedging contingent claims is the interplay of self-financing replicating portfolios and risk-neutral probabilities. Although the current setting is on a much higher level of sophistication, the key ideas remain the same.
We call an $\mathbb{R}^{d+1}$-valued predictable process

$$\varphi(t) = (\varphi_0(t), \ldots, \varphi_d(t)), \quad t \in [0,T]$$

with $\int_0^T \mathbb{E}(\varphi_0(t))dt < \infty, \sum_{i=0}^d \int_0^T \mathbb{E}(\varphi_i^2(t))dt < \infty$ a trading strategy (or dynamic portfolio process). Here $\varphi_i(t)$ denotes the number of shares of asset i held in the portfolio at time t - to be determined on the basis of information available *before* time t; i.e. the investor selects his time t portfolio after observing the prices $S(t-)$. The components $\varphi_i(t)$ may assume negative as well as positive values, reflecting the fact that we allow short sales and assume that the assets are perfectly divisible.

Definition A.2.2. *(i) The value of the portfolio φ at time t is given by the scalar product*

$$V_\varphi(t) := \varphi(t) \cdot S(t) = \sum_{i=0}^d \varphi_i(t) S_i(t), \quad t \in [0,T].$$

The process $V_\varphi(t)$ is called the value process, or wealth process, of the trading strategy φ.
(ii) The gains process $G_\varphi(t)$ is defined by

$$G_\varphi(t) := \int_0^t \varphi(u) dS(u) = \sum_{i=0}^d \int_0^t \varphi_i(u) dS_i(u).$$

(iii) A trading strategy φ is called self-financing if the wealth process $V_\varphi(t)$ satisfies

$$V_\varphi(t) = V_\varphi(0) + G_\varphi(t) \quad \textit{for all}\, t \in [0,T].$$

Remark A.2.1. *(i) The financial implications of the above equations are that all changes in the wealth of the portfolio are due to capital gains, as opposed to withdrawals of cash or injections of new funds.*
(ii) The definition of a trading strategy includes regularity assumptions in order to ensure the existence of stochastic integrals.

Using the special numéraire $S_0(t)$ we consider the discounted price process

$$\tilde{S}(t) := \frac{S(t)}{S_0(t)} = (1, \tilde{S}_1(t), \ldots \tilde{S}_d(t))$$

with $\tilde{S}_i(t) = S_i(t)/S_0(t),\ i = 1, 2, \ldots, d$. Furthermore, the discounted wealth process $\tilde{V}_\varphi(t)$ is given by

$$\tilde{V}_\varphi(t) := \frac{V_\varphi(t)}{S_0(t)} = \varphi_0(t) + \sum_{i=1}^d \varphi_i(t) \tilde{S}_i(t)$$

and the discounted gains process $\tilde{G}_\varphi(t)$ is

$$\tilde{G}_\varphi(t) := \sum_{i=1}^d \int_0^t \varphi_i(t) d\tilde{S}_i(t).$$

Observe that $\tilde{G}_\varphi(t)$ does not depend on the numéraire component φ_0.
It is convenient to reformulate the self-financing condition in terms of the discounted processes:

Proposition A.2.1. *Let φ be a trading strategy. Then φ if self-financing if and only if*

$$\tilde{V}_\varphi(t) = \tilde{V}_\varphi(0) + \tilde{G}_\varphi(t).$$

Of course, $V_\varphi(t) \geq 0$ if and only if $\tilde{V}_\varphi(t) \geq 0$.

The *proof* follows by the numéraire invariance theorem using S_0 as numéraire. ■

Remark A.2.2. *The above result shows that a self-financing strategy is completely determined by its initial value and the components $\varphi_1, \ldots, \varphi_d$. In other words, any set of predictable processes $\varphi_1, \ldots, \varphi_d$ such that the stochastic integrals $\int \varphi_i d\tilde{S}_i$, $i = 1, \ldots, d$ exist can be uniquely extended to a self-financing strategy φ with specified initial value $\tilde{V}_\varphi(0) = v$ by setting the cash holding as*

$$\varphi_0(t) = v + \sum_{i=1}^{d} \int_0^t \varphi_i(u) d\tilde{S}_i(u) - \sum_{i=1}^{d} \varphi_i(t)\tilde{S}_i(t), \; t \in [0, T].$$

A.2.2 Equivalent Martingale Measures

We develop a relative pricing theory for contingent claims. Again the underlying concept is the link between the no-arbitrage condition and certain probability measures. We begin with:

Definition A.2.3. *A self-financing trading strategy φ is called an arbitrage opportunity if the wealth process V_φ satisfies the following set of conditions:*

$$V_\varphi(0) = 0, \quad I\!P(V_\varphi(T) \geq 0) = 1, \quad \textit{and} \quad I\!P(V_\varphi(T) > 0) > 0.$$

Arbitrage opportunities represent the limitless creation of wealth through risk-free profit and thus should not be present in a well-functioning market.
The main tool in investigating arbitrage opportunities is the concept of equivalent martingale measures:

Definition A.2.4. *We say that a probability measure $\boldsymbol{Q}$ defined on $(\Omega, \mathcal{F})$ is an equivalent martingale measure if:*

(i) $\boldsymbol{Q}$ is equivalent to $I\!P$,
(ii) the discounted price process $\tilde{S}$ is a $\boldsymbol{Q}$ martingale.

We denote the set of martingale measures by $\mathcal{P}$.

A useful criterion in determining whether a given equivalent measure is indeed a martingale measure is the observation that the growth rates relative to the numéraire of all given primary assets under the measure in question must coincide. For example, in the case $S_0(t) = B(t)$ we have:

Lemma A.2.1. *Assume $S_0(t) = B(t) = e^{r(t)}$, then $\boldsymbol{Q} \sim I\!P$ is a martingale measure if and only if every asset price process S_i has price dynamics under $\boldsymbol{Q}$ of the form*

$$dS_i(t) = r(t)S_i(t)dt + dM_i(t),$$

where M_i is a $\boldsymbol{Q}$-martingale.

The *proof* is an application of Itô's formula.
In order to proceed we have to impose further restrictions on the set of trading strategies.

Definition A.2.5. *A self-financing trading strategy φ is called tame (relative to the numéraire S_0) if*

$$\tilde{V}_\varphi(t) \geq 0 \quad \textit{for all}\, t \in [0, T].$$

We use the notation Φ for the set of tame trading strategies.

We next analyse the value process under equivalent martingale measures for such strategies.

Proposition A.2.2. *For $\varphi \in \Phi$ $\tilde{V}_\varphi(t)$ is a martingale under each $\mathbb{Q} \in \mathcal{P}$.*

This observation is the key to our first central result:

Theorem A.2.1. *Assume $\mathcal{P} \neq \emptyset$. Then the market model contains no arbitrage opportunities in Φ.*

Proof. For any $\varphi \in \Phi$ and under any $\mathbb{Q} \in \mathcal{P}$ $\tilde{V}_\varphi(t)$ is a martingale. That is,

$$\mathbb{E}_{\mathbb{Q}}\left(\tilde{V}_\varphi(t)|\mathcal{F}_u\right) = \tilde{V}_\varphi(u), \quad \text{for all} \quad u \leq t \leq T.$$

For $\varphi \in \Phi$ to be an arbitrage opportunity we must have $\tilde{V}_\varphi(0) = V_\varphi(0) = 0$. Now

$$\mathbb{E}_{\mathbb{Q}}\left(\tilde{V}_\varphi(t)\right) = 0, \quad \text{for all} \quad 0 \leq t \leq T.$$

Now φ is tame, so $\tilde{V}_\varphi(t) \geq 0,\ 0 \leq t \leq T$, implying $\mathbb{E}_{\mathbb{Q}}\left(\tilde{V}_\varphi(t)\right) = 0,\ 0 \leq t \leq T$, and in particular $\mathbb{E}_{\mathbb{Q}}\left(\tilde{V}_\varphi(T)\right) = 0$. But an arbitrage opportunity φ also has to satisfy $\mathbb{P}\left(V_\varphi(T) \geq 0\right) = 1$, and since $\mathbb{Q} \sim \mathbb{P}$, this means $\mathbb{Q}\left(V_\varphi(T) \geq 0\right) = 1$. Both together yield

$$\mathbb{Q}\left(V_\varphi(T) > 0\right) = \mathbb{P}\left(V_\varphi(T) > 0\right) = 0,$$

and hence the result follows. ■

A.2.3 Risk-neutral Pricing

We now assume that there exists an equivalent martingale measure $\mathbb{P}^*$ which implies that there are no arbitrage opportunities with respect to Φ in the financial market model. Until further notice we use $\mathbb{P}^*$ as our reference measure, and when using the term martingale we always assume that the underlying probability measure is $\mathbb{P}^*$. In particular, we restrict our attention to contingent claims X such that $X/S_0(T) \in L^1(\mathcal{F}, \mathbb{P}^*)$.
We now define a further subclass of trading strategies:

Definition A.2.6. *A self-financing trading strategy φ is called ($\mathbb{P}^*$-) admissible if the relative gains process*

$$\tilde{G}_\varphi(t) = \int_0^t \varphi(u) d\tilde{S}(u)$$

is a ($\mathbb{P}^*$-) martingale. The class of all ($\mathbb{P}^*$-) admissible trading strategies is denoted $\Phi(\mathbb{P}^*)$.

By definition $\tilde{S}$ is a martingale, and $\tilde{G}$ is the stochastic integral with respect to $\tilde{S}$. We see that any sufficiently integrable processes $\varphi_1, \ldots, \varphi_d$ give rise to $\mathbb{P}^*$-admissible trading strategies.
We can repeat the above argument to obtain

Theorem A.2.2. *The financial market model $\mathcal{M}$ contains no arbitrage opportunities in $\Phi(\mathbb{P}^*)$.*

Under the assumption that no arbitrage opportunities exist, the question of pricing and hedging a contingent claim reduces to the existence of replicating self-financing trading strategies. Formally:

Definition A.2.7. *(i) A contingent claim X is called attainable if there exists at least one admissible trading strategy such that*

$$V_\varphi(T) = X.$$

We call such a trading strategy φ a replicating strategy for X.
(ii) The financial market model $\mathcal{M}$ is said to be complete if any contingent claim is attainable.

Again we emphasise that this depends on the class of trading strategies. On the other hand, it does not depend on the numéraire: it is an easy exercise in the continuous asset-price process case to show that if a contingent claim is attainable in a given numéraire it is also attainable in any other numéraire and the replicating strategies are the same.
If a contingent claim X is attainable, X can be replicated by a portfolio $\varphi \in \Phi(I\!P^*)$. This means that holding the portfolio and holding the contingent claim are equivalent from a financial point of view. In the absence of arbitrage the (arbitrage) price process $\Pi_X(t)$ of the contingent claim must therefore satisfy

$$\Pi_X(t) = V_\varphi(t).$$

Of course the questions arise of what will happen if X can be replicated by more than one portfolio, and what the relation of the price process to the equivalent martingale measure(s) is. The following central theorem is the key to answering these questions:

Theorem A.2.3 (Risk-Neutral Valuation Formula)**.** *The arbitrage price process of any attainable claim is given by the risk-neutral valuation formula*

$$\Pi_X(t) = S_0(t) I\!E_{I\!P^*}\left[\left. \frac{X}{S_0(T)} \right| \mathcal{F}_t \right]. \tag{A.5}$$

The uniqueness question is immediate from the above theorem:

Corollary A.2.1. *For any two replicating portfolios $\varphi, \psi \in \Phi(I\!P^*)$ we have*

$$V_\varphi(t) = V_\psi(t).$$

Proof of Theorem A.2.3 Since X is attainable, there exists a replicating strategy $\varphi \in \Phi(I\!P^*)$ such that $V_\varphi(T) = X$ and $\Pi_X(t) = V_\varphi(t)$. Since $\varphi \in \Phi(I\!P^*)$ the discounted value process $\tilde{V}_\varphi(t)$ is a martingale, and hence

$$\begin{aligned} \Pi_X(t) &= V_\varphi(t) = S_0(t)\tilde{V}_\varphi(t) \\ &= S_0(t) I\!E_{I\!P^*}\left[\left. \tilde{V}_\varphi(T) \right| \mathcal{F}_t \right] = S_0(t) I\!E_{I\!P^*}\left[\left. \frac{V_\varphi(T)}{S_0(T)} \right| \mathcal{F}_t \right] \\ &= S_0(t) I\!E_{I\!P^*}\left[\left. \frac{X}{S_0(T)} \right| \mathcal{F}_t \right]. \end{aligned}$$

■

Bibliography

Bagnold, R.A., 1941, *The physics of blown sand and desert dunes.* (Matthew, London).

Bagnold, R.A., and O.E. Barndorff-Nielsen, 1979, The pattern of natural size distributions, *Sedimentology* 27, 199–207.

Barndorff-Nielsen, O.E., 1977, Exponentially decreasing distributions for the logarithm of particle size, *Proc. Roy. Soc. London A* 353, 401–419.

Barndorff-Nielsen, O.E., and O. Halgreen, 1977, Infinite diversibility of the hyperbolic and generalized inverse Gaussian distributions, *Z. Wahrschein.* 38, 309–312.

Benth, Fred Espen, Lars O. Dahl, and Kenneth H. Karlsen, 2004, Quasi Monte-Carlo evaluation of sensitivities of options in commodity and energy markets, *International Journal of theoretical and Applied Finance* 7 .

Bibby, B.M., and M. Sørensen, 1997, A hyperbolic diffusion model for stock prices, *Finance and Stochastics* 1, 25–41.

Bingham, N.H., C.M. Goldie, and J.L. Teugels, 1987, *Regular Variation.* (Cambridge University Press, Cambridge).

Black, F., 1976, The pricing of commodity contracts, *J. Financial Economics* 31, 167–179.

Burger, Markus, Bernhard Klar, Alfred Müller, and Gero Schindlmayr, 2004, A spot market model for pricing derivatives in electricity markets, *Quantitative Finance* 4, 109–122.

Clewlow, Les, and Chris Strickland, 1999, Valuing Energy Options in a One Factor Model Fitted to Forward Prices, *Working Paper* Online: http://ssrn.com/abstract=160608.

de Jong, Cyriel, and Ronald Huisman, 2002, Option Formulas for Mean-Reverting Power Prices with Spikes, *Working Paper* Online: http://www.energyforum.net/downloads/reports/Mean-reverting_with_spikes.pdf.

Doob, J. L., 1953, *Stochastic processes.* (Wiley, New York).

Dothan, M. U., 1990, *Prices in financial markets.* (Oxford University Press, Oxford).

Eberlein, E., and U. Keller, 1995, Hyperbolic distributions in finance, *Bernoulli* 1, 281–299.

Eberlein, E., U. Keller, and K. Prause, 1998, New insights into smile, mispricing and Value-at-Risk: The hyperbolic model, *J. Business* 71, 371–406.

Eberlein, E., and S. Raible, 1998, Term structure models driven by general Lévy processes, *Mathematical Finance* 9, 31–53.

Feller, W., 1971, *An introduction to probability theory and its applications, Volume 2.* (Wiley & Sons, Chichester) 2nd edn.

Geman, H., 2005, *Commodities and commodity derivatives.* (John Wiley & Sons).

Geman, Helyette, and Alexander Eydeland, 1999, Some Fundamentals of Electricity Derivatives, in *Energy Modelling & The Management of Uncertainty* (Risk Books,) Online Version 7/98: http://www.ise.ufl.edu/rmfe/seminar/eydeland/Eydeland_geman.pdf.

Good, I.J., 1953, The population frequency of species and the estimation of population parameters, *Biometrika* 1953, 237–240.

Grosswald, E., 1976, The Student t-distribution function of any degree of freedom is infinitely divisible, *Z. Wahrschein.* 36, 103–109.

Harrison, J.M., and S.R. Pliska, 1981, Martingales and stochastic integrals in the theory of continuous trading, *Stochastic Processes and their Applications* 11, 215–260.

Ikeda, N., S. Watanabe, Fukushima M., and H. Kunita (eds.), 1996, *Itô stochastic calculus and probability theory.* (Springer, Tokyo Berlin New York) Festschrift for Kiyosi Itô's eightieth birthday, 1995.

Karatzas, I., and S. Shreve, 1991, *Brownian Motion and Stochastic Calculus.* (Springer-Verlag, Berlin Heidelberg New York) 2nd edn. 1rst edition 1998.

Keilson, J., and F.W. Steutel, 1974, Mixtures of distributions, moment inequalities and measures of exponentiality and normality, *Annals of Probability* 2, 112–130.

Kelker, D., 1971, Infinite divisibility and variance mixtures of the normal distribution, *Annals of Mathematical Statistics* 42, 802–808.

Knittel, Christopher R., and Michael R. Roberts, 2001, An empirical examination of deregulated electricity prices, *Working Paper* Online: http://www.ucei.berkeley.edu/PDFDown.html, PWP-087.

Merton, R.C., 1990, *Continuous-time finance.* (Blackwell, Oxford).

Meyer, P.-A., 1976, Un cours sur les intégrales stochastiques, in *Séminaire de Probabilités X* no. 511 in Lecture Notes in Mathematics pp. 245–400. Springer, Berlin Heidelberg New York.

Protter, P., 2004, *Stochastic integration and differential equations.* (Springer, New York) 2nd edn. 1rst edition, 1992.

Revuz, D., and M. Yor, 1991, *Continuous martingales and Brownian motion.* (Springer, New York).

Rogers, L.C.G., and D. Williams, 1994, *Diffusions, Markov processes and martingales, Volume 1: Foundation.* (Wiley, New York) 2nd edn. 1st ed. D. Williams, 1970.

Samuelson, P.A., 1965, Rational theory of warrant pricing, *Industrial Management Review* 6, 13–39.

Schwartz, E., 1997, The stochastic behaviour of commodity prices: Implications for valuation and hedging, *Journal of Finance* 52, 923–973.

Schwartz, Eduardo S, and James E Smith, 2000, Short-Term Variations and Long-Term Dynamics in Commodity Prices, *Management Science* 46, 893–911 Online: http://faculty.fuqua.duke.edu/~jes9/bio/Linked_Paper_List.htm.

Watson, G.N., 1944, *A treatise on the theory of Bessel functions.* (Cambridge University Press, Cambridge) 2nd edn. 1st ed. 1922.

Widder, 1941, *The Laplace transform.* (Princeton University Press, Princeton).

Williams, D., 1991, *Probability with martingales.* (Cambridge University Press, Cambridge).