Energy Derivatives

Lecture Notes
LSE

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THIS IS A PRELIMINARY VERSION
THERE WILL BE UPDATES DURING THE COURSE

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Short Description.

Content.
Within the last few years the markets for commodities, in particular energy-related commodities, has changed substantially. New regulations and products have resulted in a spectacular growth in spot and derivative trading. In particular, electricity markets have changed fundamentally over the last couple of years. Due to deregulation energy companies are now allowed to trade not only the commodity electricity, but also various derivatives on electricity on several Energy Exchanges (such as the EEX).

Specific topics

1. Basic Principles of Commodity Markets, models for forwards and futures.
2. Stylized facts of electricity markets; statistical analysis of spot and futures markets.
3. Spot and Forward Market Models for Electricity, mathematical models based on Lévy processes (including a short intro to such processes).
4. Special derivatives for the Electricity markets.

Literature.

- Eydeland, A. Wolyniec, K.: *Energy and Power Risk Management*, Wiley 2003
- Geman, H.: *Commodities and Commodity Derivatives*, Wiley 2005.

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Contents

Chapter 1

Fundamentals

1.1 Markets and Price Processes

Since the deregulation of electricity markets in the end of the 1990s, power can be traded at exchanges like the Nordpool or the European Energy Exchange (EEX). All exchanges have established spot and futures markets.
The spot market usually is organised as an auction, which manages the distribution of power in the near future, i. e. one day ahead. Empirical studies, such as Knittel and Roberts (2001) using hourly prices in the California power market, show that spot prices exhibit seasonalities on different time scales, a strong mean-reversion and are very volatile and spiky in nature. Because of inherent properties of electricity as an almost non-storable commodity such a price behaviour has to be expected, see Geman (2005).
Due to the volatile behaviour of the spot market and to ensure that power plants can be deployed optimally, **power forwards and futures** are traded. Power exchanges established the trade of forwards and futures early on and by now large volumes are traded. A power forward contract is characterized by a fixed delivery price per MWh, a delivery period and the total amount of energy to deliver. Especially the length of the delivery period and the exact time of delivery determine the value and statistical characteristics of the contract vitally. One can observe, that contracts with a long delivery period show less volatile prices than those with short delivery. These facts give rise to a a term structure of volatility in most power forward markets, which has to be modelled accurately in order to be able to price options on futures. Figure 1.1 gives an example of such a term structure for futures traded at the EEX. Additionally, seasonalities can be observed in the forward curve within a year. Monthly contracts during winter months show higher prices than comparable contracts during the summer (cp. Figure 1.2).
Aside from spot and forward markets, valuing options is an issue for market participants. While some research has been done on the valuation of options on spot power, hardly any results can be found on options on forwards and futures. Both types impose different problems for the valuation.
Spot options fail most of the arbitrage and replication arguments, since power is almost non-storable. Some authors take the position to find a realistic model to describe the prices of spot prices and then value options via risk-neutral expectations (cp. de Jong and Huisman (2002), Benth, Dahl, and Karlsen (2004), Burger, Klar, Müller, and Schindlmayr (2004)). Other ideas explicitly take care of the special situation in the electricity production and use power plants to replicate certain contingent claims (cp. Geman and Eydeland (1999)).
Forward and futures options are heavily influenced by the length of their delivery period and their time to maturity. In Clewlow and Strickland (1999), for example, a one-factor model is presented, that tries to fit the term structure of volatility, but that does not incorporate a delivery period, since it is constructed for oil and gas markets.
As an example let us have a look at the EEX spot market. Here we have the following structure

- the EEX spot market is a day-ahead auction for single hours of the following day

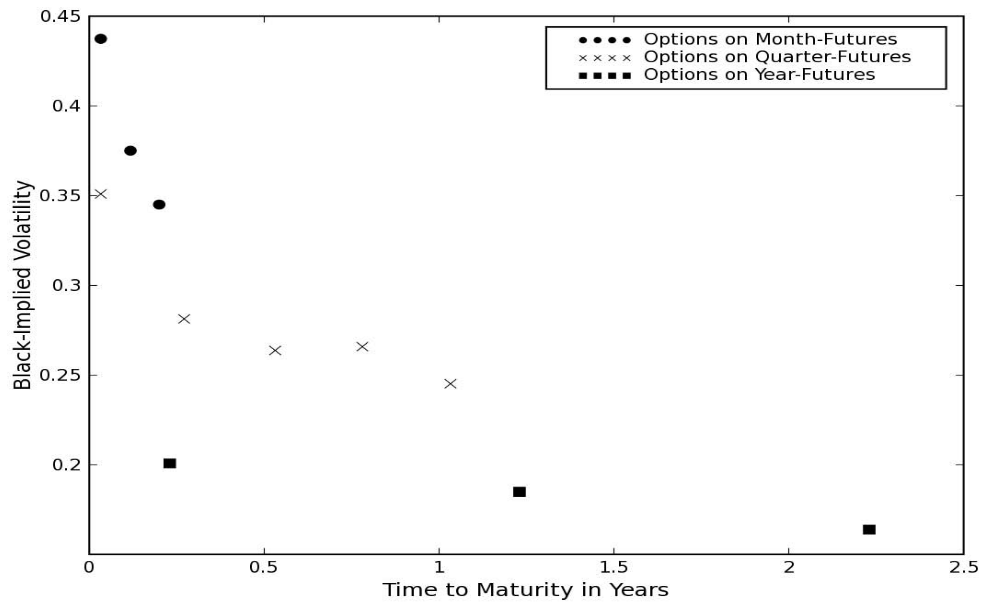


Figure 1.1: Implied volatilities of futures with different maturities and delivery periods, Sep. 14

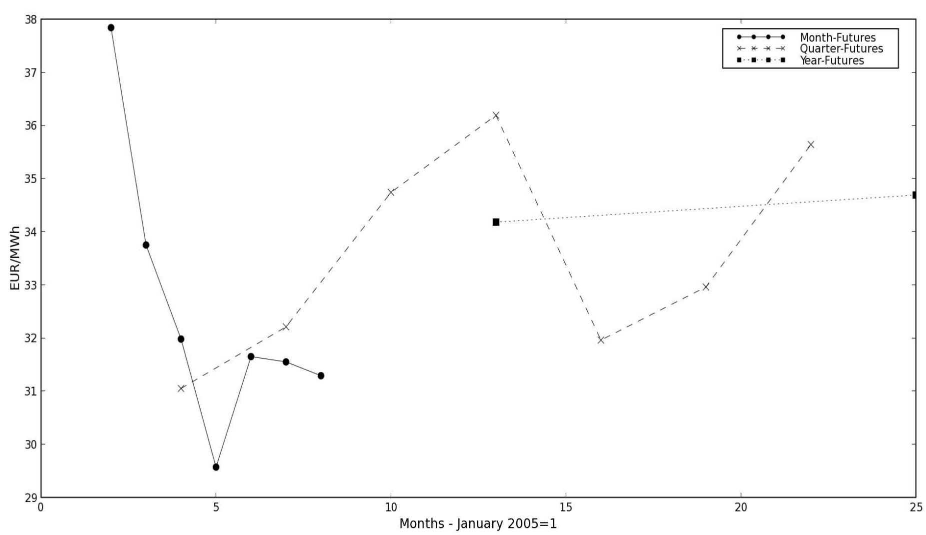


Figure 1.2: Forward prices of futures with different maturities and delivery periods, Feb. 18

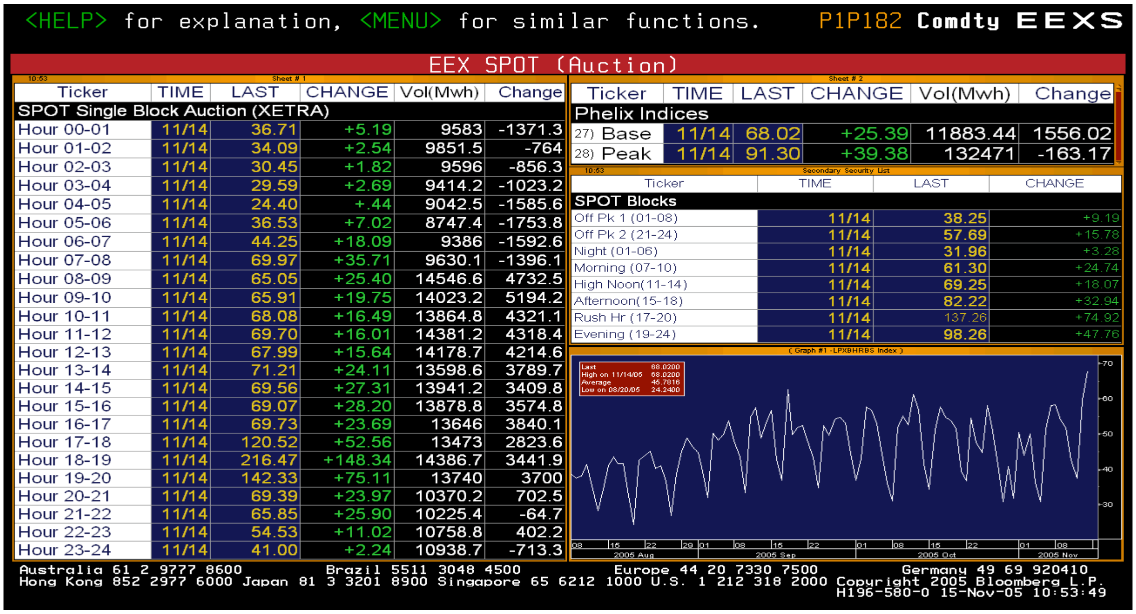
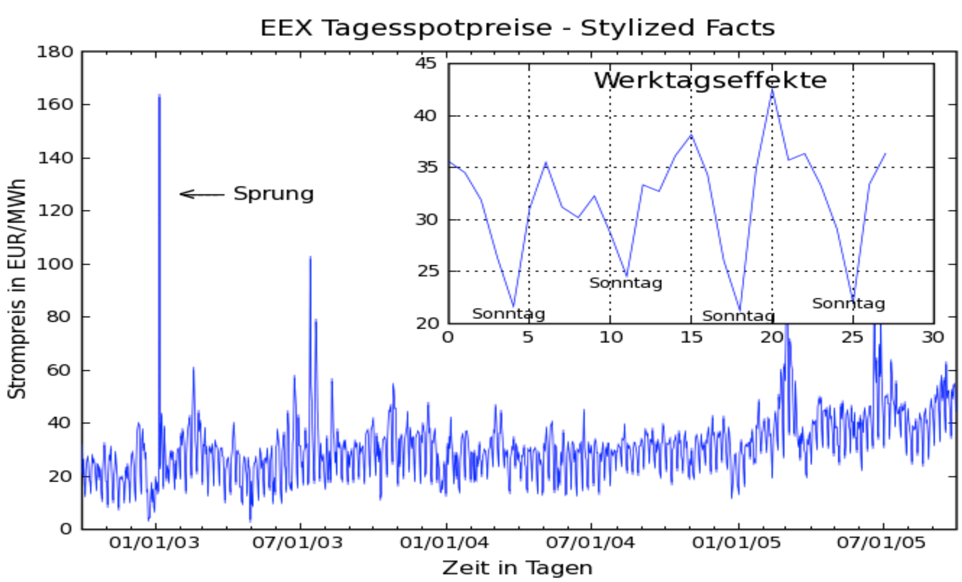
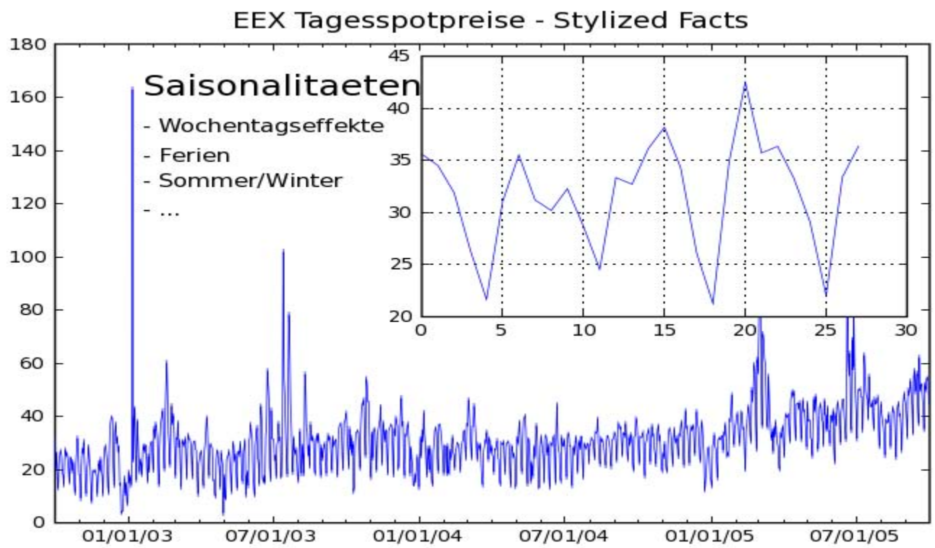


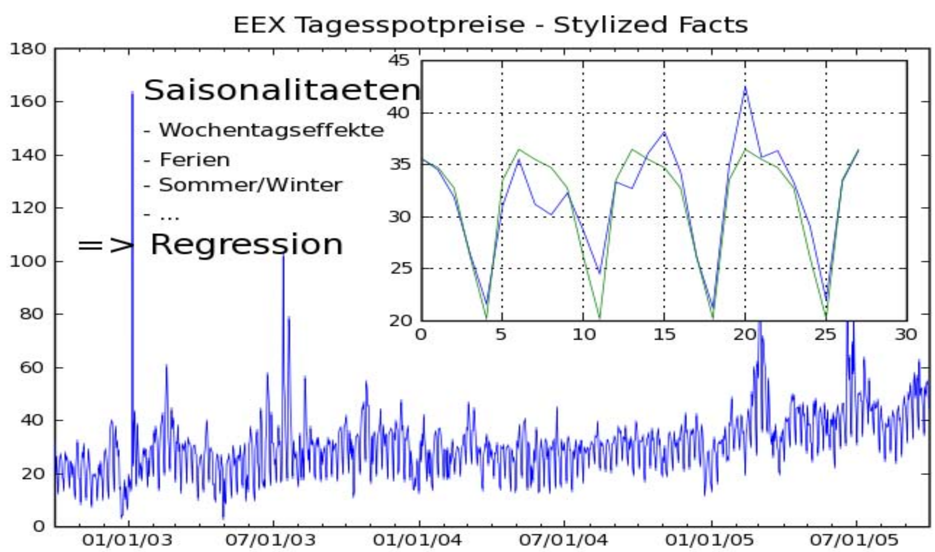
Figure 1.3: Bloomberg screen for energy spot prices

- participants submit their price offer/bit curves, the EEX system prices are equilibrium prices that clear the market.
- EEX day prices are the average of the 24-single hours.
- on fridays the hours for the whole weekend are auctioned.
- similar structures can be found on other power exchanges(Nord Pool, APX, etc.).

The following are examples of price processe



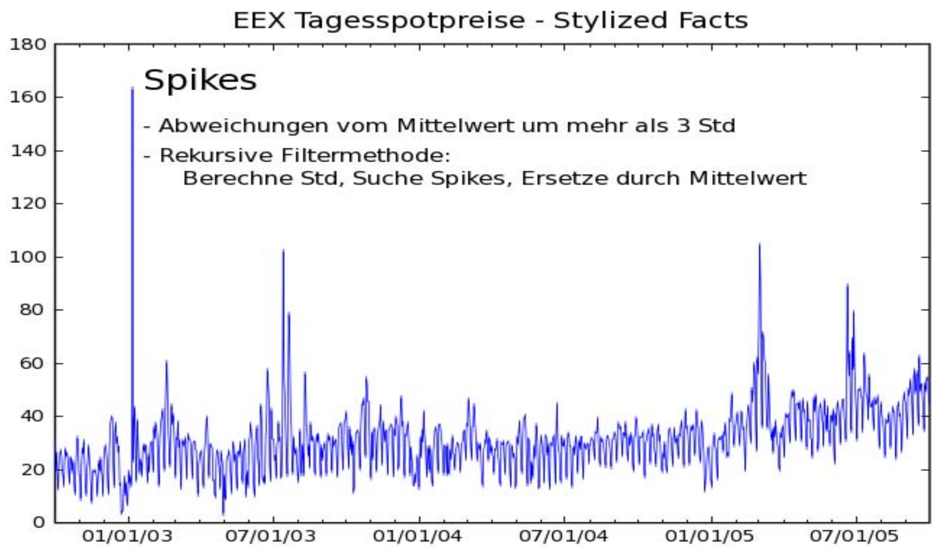


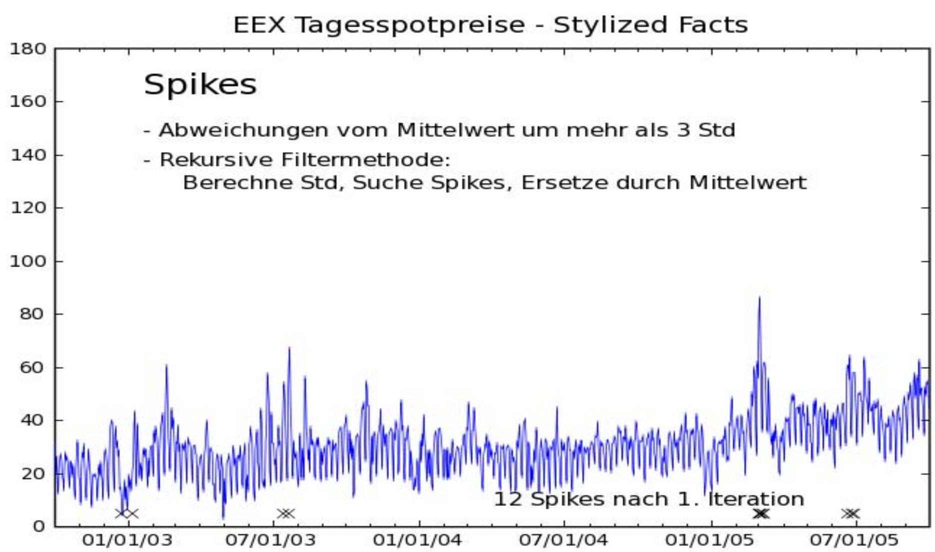


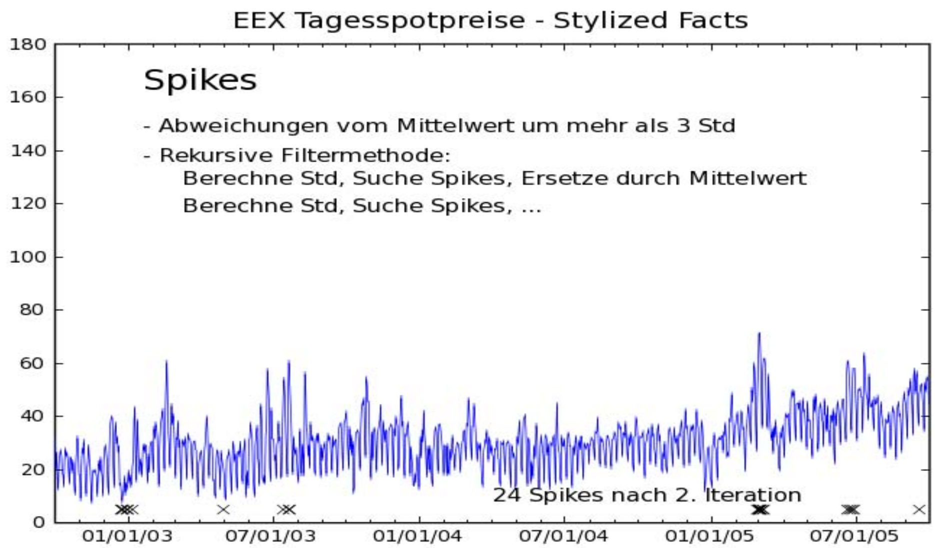
To analyse seasonalities one can preform a regression analysis. This can be done by standard methods assuming a model for the mean, e. g.

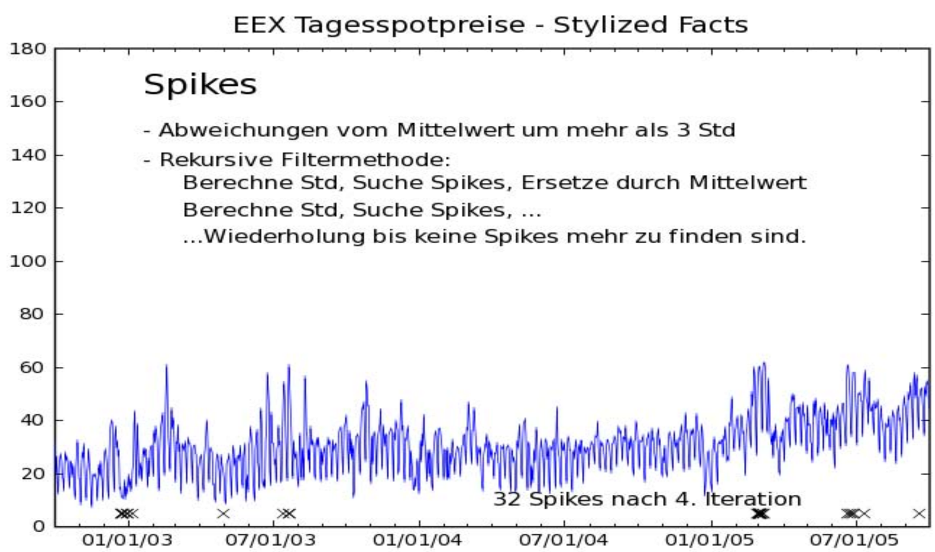
$$\begin{aligned} S_t &= \beta_1 \cdot 1(\text{if } t \in \text{Mondays}) + \ldots + \beta_7 \cdot 1(\text{if } t \in \text{Sundays}) \\ &\quad + \text{ other calendar day effects} \\ &\quad +\beta_8 \cdot t \text{ for long term linear trend} \\ &\quad +\beta_9 \sin(\frac{2\pi}{365}(t-c)) \text{ for summer/winter seasonality} \\ &\quad + \ldots \end{aligned}$$

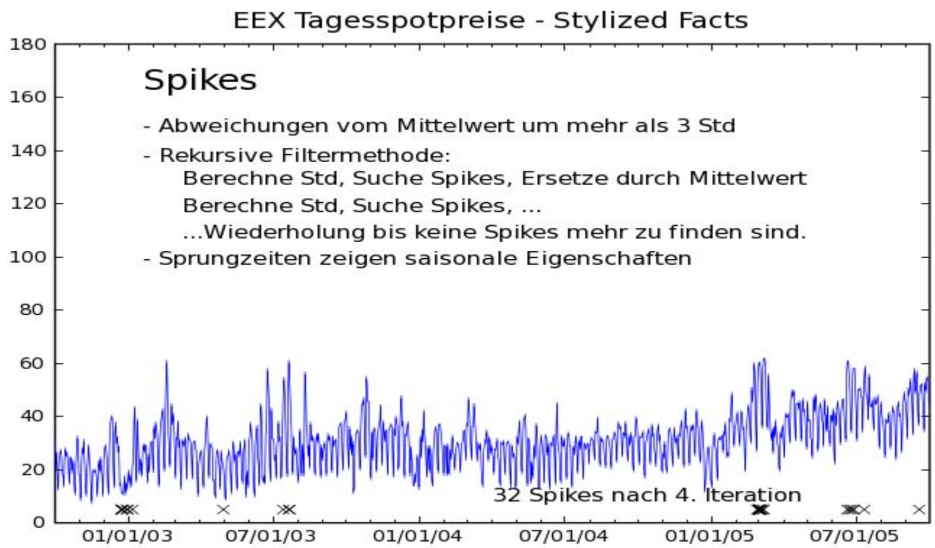
The unknown parameters $\beta_1, \ldots, \beta_p$ can be estimated easily by Least-Squares-Methods.
We also observe spikes


EEX Tagesspotpreise - Stylized Facts
Spikes
- Abweichungen vom Mittelwert um mehr als 3 Std
- Rekursive Filtermethode:
Berechne Std, Suche Spikes, Ersetze durch Mittelwert
180
160
140
120
100
80
60
40
20
0
01/01/03
07/01/03
01/01/04
07/01/04
01/01/05
07/01/05


EEX Tagesspotpreise - Stylized Facts
Spikes
- Abweichungen vom Mittelwert um mehr als 3 Std
- Rekursive Filtermethode:
Berechne Std, Suche Spikes, Ersetze durch Mittelwert
12 Spikes nach 1. Iteration
180
160
140
120
100
80
60
40
20
0
01/01/03
07/01/03
01/01/04
07/01/04
01/01/05
07/01/05


EEX Tagesspotpreise - Stylized Facts
Spikes
- Abweichungen vom Mittelwert um mehr als 3 Std
- Rekursive Filtermethode:
Berechne Std, Suche Spikes, Ersetze durch Mittelwert
Berechne Std, Suche Spikes, ...
24 Spikes nach 2. Iteration
180
160
140
120
100
80
60
40
20
0
01/01/03
07/01/03
01/01/04
07/01/04
01/01/05
07/01/05

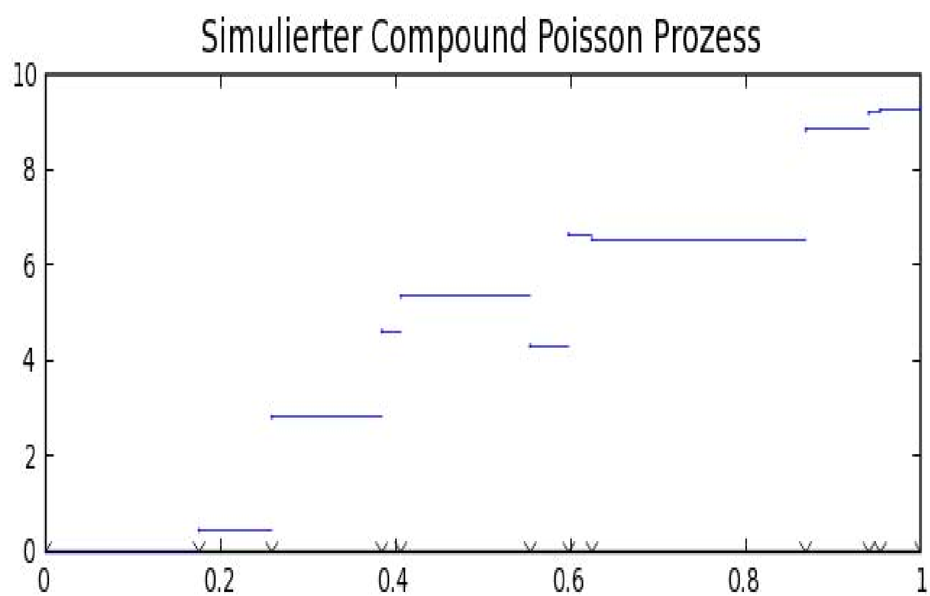

EEX Tagesspotpreise - Stylized Facts
Spikes
- Abweichungen vom Mittelwert um mehr als 3 Std
- Rekursive Filtermethode:
Berechne Std, Suche Spikes, Ersetze durch Mittelwert
Berechne Std, Suche Spikes, ...
...Wiederholung bis keine Spikes mehr zu finden sind.
32 Spikes nach 4. Iteration
180
160
140
120
100
80
60
40
20
0
01/01/03
07/01/03
01/01/04
07/01/04
01/01/05
07/01/05

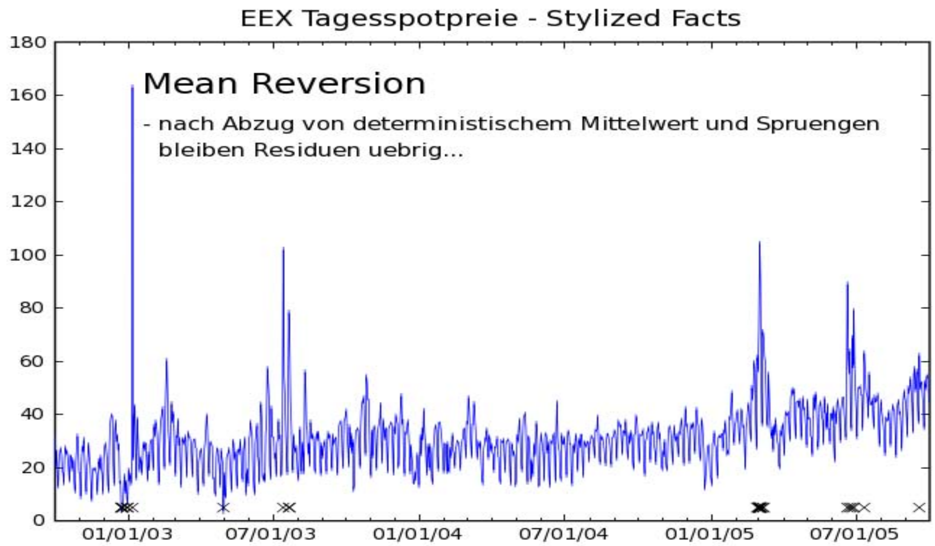


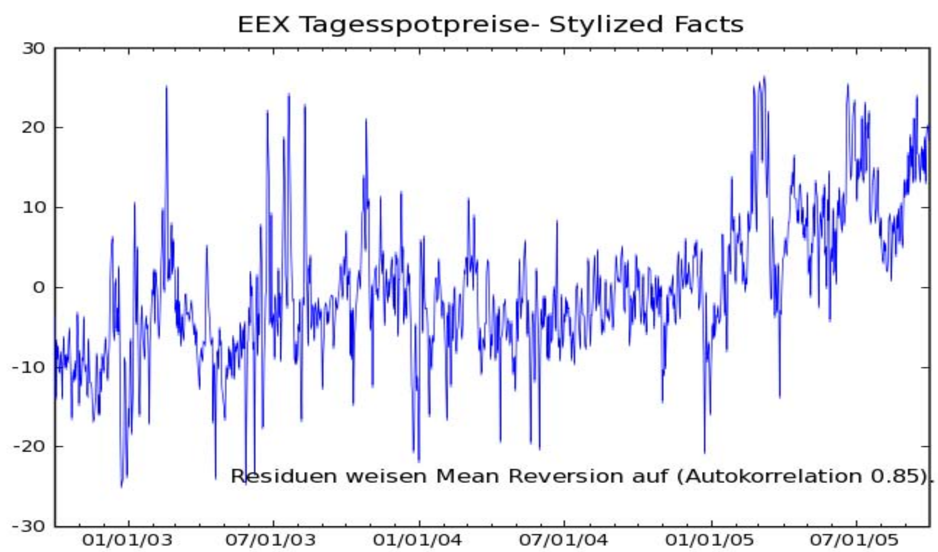
Spikes are often modelled by jumps. The process of jumps is often modelled by a compound poisson process

$$CP_t := \sum_{i=1}^{N_t} J_i$$

N_t is a Poisson process with intensity λ, which randomly jumps by 1 unit, so it counts how many jumps occurred up to time t. J_i is the random jump size of the ith jump.







1.2 Basic Products and Structures

We mostly have been dealing so far with derivatives based on underlying assets – stock – existing, and available for trading, now. It frequently happens, however, that the underlying assets relevant in a particular market will instead be available at some time in the future, and need not even exist now. Obvious examples include crop commodities – wheat, sugar, coffee etc. – which might not yet be planted, or be still growing, and so whose eventual price remains uncertain – for instance, because of the uncertainty of future weather. The principal factors determining yield of crops such as cereals, for instance, are rainfall and hours of sunshine during the growing season. Oil, gas, coals are another example of a commodity widely traded in the future, and here the uncertainty is more a result of political factors, shipping costs etc. Our focus here will be on electricity later on. Financial assets, such as currencies, bonds and stock indexes, may also be traded in the future,

on exchanges such as the London International Financial Futures and Options Exchange (LIFFE) and the Tokyo International Financial Futures Exchange (TIFFE), and we shall restrict attention to financial futures for simplicity.
We thus have the existence of two parallel markets in some asset – the *spot market*, for assets traded in the present, and the *futures market*, for assets to be realized in the future. We may also consider the combined spot-futures market.
We now want briefly look at the most important products.

1.2.1 Forwards

A *forward contract* is an agreement to buy or sell an asset S at a certain future date T for a certain price K. The agent who agrees to buy the underlying asset is said to have a *long* position, the other agent assumes a *short* position. The settlement date is called *delivery date* and the specified price is referred to as *delivery price*. The *forward price* $F(t,T)$ is the delivery price which would make the contract have zero value at time t. At the time the contract is set up, $t = 0$, the forward price therefore equals the delivery price, hence $F(0,T) = K$. The forward prices $F(t,T)$ need not (and will not) necessarily be equal to the delivery price K during the life-time of the contract.
The payoff from a long position in a forward contract on one unit of an asset with price $S(T)$ at the maturity of the contract is

$$S(T) - K.$$

Compared with a call option with the same maturity and strike price K we see that the investor now faces a downside risk, too. He has the obligation to buy the asset for price K.

1.2.2 Futures Markets

Futures prices, like spot prices, are determined on the floor of the exchange by supply and demand, and are quoted in the financial press. *Futures contracts*, however – contracts on assets traded in the futures markets – have various special characteristics. Parties to futures contracts are subject to a daily settlement procedure known as *marking to market*. The initial deposit, paid when the contract is entered into, is adjusted daily by *margin payments* reflecting the daily movement in futures prices. The underlying asset and price are specified in the contract, as is the delivery date. Futures contracts are highly liquid – and indeed, are intended more for trading than for delivery. Being assets, futures contracts may be the subject of *futures options*.
We shall as before write $t = 0$ for the time when a contract, or option, is written, t for the present time, T for the expiry time of the option, and T^* for the delivery time specified in the futures (or forward) contract. We will have $T^* \geq T$, and in general $T^* > T$; beyond this, T^* will not affect the pricing of options with expiry T.

Swaps

A *swap* is an agreement whereby two parties undertake to exchange, at known dates in the future, various financial assets (or cash flows) according to a prearranged formula that depends on the value of one or more underlying assets. Examples are currency swaps (exchange currencies) and interest-rate swaps (exchange of fixed for floating set of interest payments).

1.3 Basic Pricing Relations

1.3.1 Storage, Inventory and Convenience Yield

The theory of storage aims to explain the differences between spot and Futures (Forward) prices by analyzing why agents hold inventories. Inventories allow to meet unexpected demand, avoid the cost of frequent revisions in the production schedule and eliminate manufacturing disruption. This motivates the concept of convenience yield as a benefit, that accrues to the owner of the

physical commodity but not to the holder of a forward contract. Thus the convenience yield is comparable to the dividend yield for stocks. A modern view is to view storage (inventory) as a timing option, that allows to put the commodity to the market when prices are high and hold it when the prices are low.
We will model the convenience yield y

- expressed as a rate, meaning that the benefit in a monetary amount for the holder of the commodity will be equal to $S(t)ydt$ over the interval $(t, t + dt)$, if $S(t)$ is the spot price at time t;
- defined as the difference between the positive gain attached to the physical commodity minus the cost of storage. Hence the convenience yield may be positive or negative depending on the period, commodity and cost of storage.

In recent literature the convenience yield is often modelled as a random variable, which allows to explain various shapes of forward curves over time.

1.3.2 Futures Prices and Expectation of Future Spot Prices

The rational expectation hypothesis (REH) (mainly used in the context of interest rates) states that the current futures price $f(t, T)$ for a commodity (interest rate) with delivery a time $T > t$ is the best estimator for the price $S(T)$ of the commodity. In mathematical terms

$$f(t, T) = \mathbb{E}[S(T)|\mathcal{F}_t]. \tag{1.1}$$

where $\mathcal{F}_t$ represents the information available at time t. The REH has been statistically tested in many studies for a wide range of commodities (resulting most of the time in a rejection).
When equality in (1.1) does not hold futures prices are biased estimators of future spot prices. If

- $>$ holds, then $f(t, T)$ is an up-ward biased estimate, then risk-aversion among market participants is such that buyers are willing to pay more than the expected spot price in order to secure access to the commodity at time T (political unrest);
- $<$ holds, then $f(t, T)$ is an down-ward biased estimate, this may reflect a perception of excess supply in the future.

No general theory for the bias has been developed. It may depend on the specific commodity, the actual forecast of the future spot price by market participant, and on the risk aversion of the participants.

1.3.3 Spot-Forward Relationship in Commodity Markets

Under the no-arbitrage assumption we have

$$F(t, T) = S(t)e^{(r-y)(T-t)} \tag{1.2}$$

where r is the interest rate at time t for maturity T and y is the convenience yield. We start by proving this relationship for stocks as underlying

Non-dividend paying stocks

Consider the portfolio

	t	T
buy stock	$-S(t)$	delivery
borrow to finance	$S(t)$	$-S(t)e^{r(T-t)}$
sell forward on S		F(t,T)

All quantities are known at t, the time t cash-flow is zero, so the cash-flow at T needs to be zero so we have

$$F(t, T) = S(t)e^{r(T-t)} \tag{1.3}$$

dividend paying stocks

Assume a continuous dividend κ, so we have a dividend rate of $\kappa S(t)dt$, which is immediately reinvested in the stock. We thus have a growth rate of $e^{g(T-t)}$ over the period of the quantity of stocks detained. Thus we only have to buy $e^{-g(T-t)}$ shares of stock S at time time. Replace in the above portfolio and obtain

$$F(t,T) = S(t)e^{(r-g)(T-t)} \tag{1.4}$$

storable commodity

Here the convenience yield plays the role of the dividend an we obtain (**??**).
In case of a linear rate this relationship is of the form

$$F(t,T) = S(t)\left\{1 + (r-y)(T-t)\right\}.$$

With the decomposition $y = y_1 - c$ with y_1 the *benefit from the physical commodity* and c the *storage cost* we have

$$F(t,T) = S(t)\left\{1 + r(T-t) + c(T-t) - y_1(T-t)\right\}.$$

Observe that (**??**) implies

(i) spot and forward are redundant (one can replace the other) and form a linear relationship (unlike options)

(ii) with two forward prices we can derive the value of $S(t)$ and y

(iii) knowledge of $S(t)$ and y allows us to construct the whole forward curve

(iv) for $r - y < 0$ we have backwardation; for $y - r > 0$ we have contango.

1.3.4 Futures Pricing Relations

The value $V_f(t)$ of a futures contract entered in at 0 at time t is

$$V_f(t) = e^{-r(T-t)}(f(t,T) - f(0,T)) \tag{1.5}$$

Despite their fundamental differences, futures prices $f_S(t,T)$ – on a stock S at time t with expiry T – and the corresponding forward prices $F_S(t,T)$, are closely linked. We use the notation $p(t,T)$ for the bond price process.

Proposition 1.3.1. *If the bond price process $p(t,T)$ is predictable, the combined spot-futures market is arbitrage-free if and only if the futures and forward prices agree: for every underlying S and every $t \leq T$,*

$$f_S(t,T) = F_S(t,T).$$

In the important special case of the futures analogue of the Black-Scholes model, to which we turn below, the bond price process – or interest-rates process – is deterministic, so predictable.

1.4 Pricing Formulae for Options

1.4.1 Black-Scholes Formula

The Model

We concentrate on the classical Black-Scholes model

$$\begin{aligned} dB(t) &= rB(t)dt, & B(0) &= 1, \\ dS(t) &= S(t)\left(bdt + \sigma dW(t)\right), & S(0) &= p \in (0,\infty), \end{aligned}$$

with constant coefficients $b \in I\!R,\ r, \sigma \in I\!R_+$. We write as usual $\tilde{S}(t) = S(t)/B(t)$ for the discounted stock price process (with the bank account being the natural numéraire), and get from Itô's formula

$$d\tilde{S}(t) = \tilde{S}(t)\,\{(b-r)dt + \sigma dW(t)\}.$$

Equivalent Martingale Measure

Because we use the Brownian filtration any pair of equivalent probability measures $I\!P \sim \boldsymbol{Q}$ on $\mathcal{F}_T$ is a Girsanov pair, i.e.

$$\left.\frac{d\boldsymbol{Q}}{dI\!P}\right|_{\mathcal{F}_t} = L(t)$$

with

$$L(t) = \exp\left\{ -\int_0^t \gamma(s)dW(s) - \frac{1}{2}\int_0^t \gamma(s)^2 ds \right\},$$

and $(\gamma(t) : 0 \le t \le T)$ a measurable, adapted d-dimensional process with $\int_0^T \gamma(t)^2 dt < \infty \quad a.s.$. By Girsanov's theorem A.1.4 we have

$$dW(t) = d\tilde{W}(t) - \gamma(t)dt,$$

where $\tilde{W}$ is a $\boldsymbol{Q}$-Wiener process. Thus the $\boldsymbol{Q}$-dynamics for $\tilde{S}$ are

$$d\tilde{S}(t) = \tilde{S}(t)\left\{(b - r - \sigma\gamma(t))dt + \sigma d\tilde{W}(t)\right\}.$$

Since $\tilde{S}$ has to be a martingale under $\boldsymbol{Q}$ we must have

$$b - r - \sigma\gamma(t) = 0 \quad t \in [0, T],$$

and so we must choose

$$\gamma(t) \equiv \gamma = \frac{b-r}{\sigma},$$

(the 'market price of risk'). Indeed, this argument leads to a unique martingale measure, and we will make use of this fact later on. Using the product rule, we find the $\boldsymbol{Q}$-dynamics of S as

$$dS(t) = S(t)\left\{rdt + \sigma d\tilde{W}\right\}.$$

We see that the appreciation rate b is replaced by the interest rate r, hence the terminology risk-neutral (or yield-equating) martingale measure.
We also know that we have a unique martingale measure $I\!P^*$ (recall $\gamma = (b-r)/\sigma$ in Girsanov's transformation).

Pricing and Hedging Contingent Claims

Recall that a contingent claim X is a $\mathcal{F}_T$-measurable random variable such that $X/B(T) \in L^1(\Omega, \mathcal{F}_T, I\!P^*)$. (We write $I\!E^*$ for $I\!E_{I\!P^*}$ in this section.) By the risk-neutral valuation principle the price of a contingent claim X is given by

$$\Pi_X(t) = e^{\{-r(T-t)\}} I\!E^*\left[X|\,\mathcal{F}_t\right],$$

with $I\!E^*$ given via the Girsanov density

$$L(t) = \exp\left\{ -\left(\frac{b-r}{\sigma}\right) W(t) - \frac{1}{2}\left(\frac{b-r}{\sigma}\right)^2 t \right\}.$$

Now consider a European call with strike K and maturity T on the stock S (so $\Phi(T) = (S(T) - K)^+$), we can evaluate the above expected value (which is easier than solving the Black-Scholes partial differential equation) and obtain:

Proposition 1.4.1 (Black-Scholes Formula)**.** *The Black-Scholes price process of a European call is given by*

$$C(t) = S(t)\Phi(d_1(S(t), T-t)) - Ke^{-r(T-t)}\Phi(d_2(S(t), T-t)). \tag{1.6}$$

The functions $d_1(s,t)$ and $d_2(s,t)$ are given by

$$\begin{aligned} d_1(s,t) &= \frac{\log(s/K) + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}, \\ d_2(s,t) &= d_1(s,t) - \sigma\sqrt{t} = \frac{\log(s/K) + (r - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}} \end{aligned}$$

Proposition 1.4.2. *The replicating strategy in the classical Black-Scholes model is given by*

$$\begin{aligned} \varphi_0 &= \frac{F(t,S(t)) - F_s(t,S(t))S(t)}{B(t)}, \\ \varphi_1 &= F_s(t,S(t)). \end{aligned}$$

We can also use an arbitrage approach to derive the Black-Scholes formula. For this consider a self-financing portfolio which has dynamics

$$dV_\varphi(t) = \varphi_0(t)dB(t) + \varphi_1(t)dS(t) = (\varphi_0(t)rB(t) + \varphi_1(t)\mu S(t))dt + \varphi_1(t)\sigma S(t)dW(t).$$

Assume that the portfolio value can be written as

$$V_\varphi(t) = V(t) = f(t, S(t))$$

for a suitable function $f \in C^{1,2}$. Then by Itô's formula

$$dV(t) = (f_t(t,S_t) + f_x(t,S_t)S_t\mu + \frac{1}{2}S_t^2\sigma^2 f_{xx}(t,S_t))dt + f_x(t,S_t)\sigma S_t dW_t.$$

Now we match the coefficients and find

$$\varphi_1(t) = f_x(t, S_t)$$

and

$$\varphi_0(t) = \frac{1}{rB(t)}(f_t(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 f_{xx}(t,S_t)).$$

Then looking at the total portfolio value we find that $f(t,x)$ must satisfy the Black-Scholes partial differential equation

$$f_t(t,x) + rxf_x(t,x) + \frac{1}{2}\sigma^2 x^2 f_{xx}(t,x) - rf(t,x) = 0 \tag{1.7}$$

and initial condition $f(T,x) = (x-K)^+$.

1.4.2 Options on Dividend-paying Stocks

1.4.3 Black's Futures Options Formula

We turn now to the problem of extending our option pricing theory from spot markets to futures markets. We assume that the stock-price dynamics S are given by geometric Brownian motion

$$dS(t) = bS(t)dt + \sigma S(t)dW(t),$$

and that interest rates are deterministic. We know that there exists a unique equivalent martingale measure, $I\!P^*$ (for the discounted stock price processes), with expectation $I\!E^*$. Write

$$f(t) := f_S(t, T^*)$$

for the futures price $f(t)$ corresponding to the spot price $S(t)$. Then risk-neutral valuation gives

$$f(t) = I\!E^*(S(T^*)|\mathcal{F}_t) \quad (t \in [0, T^*]),$$

while forward prices are given in terms of bond prices by

$$F(t) = S(t)/B(t, T^*) \quad (t \in [0, T^*]).$$

So by Proposition 1.3.1

$$f(t) = F(t) = S(t)e^{r(T^*-t)} \quad (t \in [0, T^*]).$$

So we can use the product rule (**??**) to determine the dynamics of the futures price

$$df(t) = (b - r)f(t)dt + \sigma f(t)dW(t), \quad f(0) = S(0)e^{rT^*}.$$

We know that the unique equivalent martingale measure in this setting is given by means of a Girsanov density

$$L(t) = \exp\{-\frac{b-r}{\sigma}W(t) - \frac{1}{2}(\frac{b-r}{\sigma})^2 t\},$$

so the $I\!P^*$-dynamics of the futures price are $df(t) = \sigma f(t)d\tilde{W}(t)$ with $\tilde{W}$ a $I\!P^*$-Brownian motion, so f is a $I\!P^*$-martingale. Observe that our derivation depends critically on the fact that interest rates are deterministic; for a more extended treatment, we refer to Musiela and Rutkowski (1997), §6.1.

We turn briefly now to the futures analogue of the Black-Scholes formula, due to Black (1976). We and use the same notation - strike K, expiry T as in the spot case, and write N for the standard normal distribution function.

Theorem 1.4.1. *The arbitrage price C of a European futures call option is*

$$C(t) = c(f(t), T - t),$$

where $c(f, t)$ is given by Black's futures options formula:

$$c(f, t) := e^{-rt}(fN(d_1(f, t)) - KN(d_2(f, t))),$$

where

$$d_{1,2}(f, t) := \frac{\log(f/K) \pm \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}}.$$

Proof. By risk-neutral valuation,

$$C(t) = B(t)I\!E^*\left[(f(T) - K)^+/B(T)|\mathcal{F}_t\right],$$

with $B(t) = e^{rt}$. For simplicity, we work with $t = 0$; the extension to the general case is immediate. Thus

$$\begin{aligned} C(0) &= I\!E^*\left[(f(T) - K)^+/B(T)\right] \\ &= I\!E^*\left[e^{-rT}f(T)\mathbf{1}_D\right] - I\!E^*\left[e^{-rT}K\mathbf{1}_D\right] \\ &= \mathbf{1}_1 - \mathbf{1}_2 \end{aligned}$$

say, where

$$D := \{f(T) > K\}.$$

Thus

$$\begin{aligned}\mathbf{1}_2 &= e^{-rT} K \mathbb{P}^*(f(T) > K) \\ &= e^{-rT} K \mathbb{P}^* \left(f(0) \exp\left\{ \sigma \tilde{W}(T) - \frac{1}{2}\sigma^2 T \right\} > K \right),\end{aligned}$$

where $\tilde{W}$ is a standard Brownian motion under $\mathbb{P}^*$. Now $\xi := -\tilde{W}(T)/\sqrt{T}$ is standard normal, with law N under $\mathbb{P}^*$, so

$$\begin{aligned}\mathbf{1}_2 &= e^{-rT} K \mathbb{P}^* \left(\xi < \frac{\log(f(0)/K) - \frac{1}{2}\sigma^2}{\sigma\sqrt{T}} \right) \\ &= e^{-rT} K N \left(\frac{\log(f(0)/K) - \frac{1}{2}\sigma^2}{\sigma\sqrt{T}} \right) \\ &= e^{-rT} K d_2(f(0), T).\end{aligned}$$

A similar calculation, also proceeding as in the spot-market case, gives

$$\mathbf{1}_1 = e^{-rT} f(0) d_1(f(0), T).$$

■

Observe that the quantities d_1 and d_2 do not depend on the interest rate r. This is intuitively clear from the classical Black approach: one sets up a replicating risk-free portfolio consisting of a position in futures options and an offsetting position in the underlying futures contract. The portfolio requires no initial investment and therefore should not earn any interest.

Appendix A

Continuous-time Financial Market Models

A.1 The Stock Price Process and its Stochastic Calculus

A.1.1 Continuous-time Stochastic Processes

A *stochastic process* $X = (X(t))_{t\geq 0}$ is a family of random variables defined on $(\Omega, \mathcal{F}, I\!P, I\!F)$. We say X is *adapted* if $X(t) \in \mathcal{F}_t$ (i.e. $X(t)$ is $\mathcal{F}_t$-measurable) for each t: thus $X(t)$ is known when $\mathcal{F}_t$ is known, at time t.
The martingale property in continuous time is just that suggested by the discrete-time case:

Definition A.1.1. *A stochastic process $X = (X(t))_{0\leq t<\infty}$ is a martingale relative to $(I\!F, I\!P)$ if*

(i) X is adapted, and $I\!E\,|X(t)| < \infty$ for all $\leq t < \infty$;

(ii) $I\!E[X(t)|\mathcal{F}_s] = X(s)$ $I\!P-a.s.$ $(0 \leq s \leq t)$,

and similarly for sub- and supermartingales.

There are regularisation results, under which one can take $X(t)$ RCLL in t (basically $t \to I\!EX(t)$ has to be right-continuous). Then the analogues of the results for discrete-time martingales hold true.
Interpretation. Martingales model fair games. Submartingales model favourable games. Supermartingales model unfavourable games.
Brownian motion originates in work of the botanist Robert Brown in 1828. It was introduced into finance by Louis Bachelier in 1900, and developed in physics by Albert Einstein in 1905.

Definition A.1.2. *A stochastic process $X = (X(t))_{t\geq 0}$ is a standard (one-dimensional) Brownian motion, BM or $BM(I\!R)$, on some probability space $(\Omega, \mathcal{F}, I\!P)$, if*

(i) $X(0) = 0$ a.s.,

(ii) X has independent increments: $X(t+u) - X(t)$ is independent of $\sigma(X(s) : s \leq t)$ for $u \geq 0$,

(iii) X has stationary increments: the law of $X(t+u) - X(t)$ depends only on u,

(iv) X has Gaussian increments: $X(t+u) - X(t)$ is normally distributed with mean 0 and variance u, $X(t+u) - X(t) \sim N(0, u)$,

(v) X has continuous paths: $X(t)$ is a continuous function of t, i.e. $t \to X(t, \omega)$ is continuous in t for all $\omega \in \Omega$.

We shall henceforth denote standard Brownian motion $BM(\mathbb{R})$ by $W = (W(t))$ (W for Wiener), though $B = (B(t))$ (B for Brown) is also common. Standard Brownian motion $BM(\mathbb{R}^d)$ in d dimensions is defined by $W(t) := (W_1(t), \ldots, W_d(t))$, where $W_1, \ldots, W_d$ are independent standard Brownian motions in one dimension (independent copies of $BM(\mathbb{R})$).
We have Wiener's theorem:

Theorem A.1.1 (Wiener)**.** *Brownian motion exists.*

For further background, see any measure-theoretic text on stochastic processes. A treatment starting directly from our main reference of measure-theoretic results, Williams Williams (1991), is Rogers and Williams Rogers and Williams (1994), Chapter 1. The classic is Doob's book, Doob (1953), VIII.2. Excellent modern texts include **?** (see particularly Karatzas and Shreve (1991), §2.2-4 for construction).

A.1.2 Stochastic Analysis

Stochastic integration was introduced by K. Itô in 1944, hence its name Itô calculus. It gives a meaning to

$$\int_0^t X dY = \int_0^t X(s,\omega) dY(s,\omega),$$

for suitable stochastic processes X and Y, the integrand and the integrator. We shall confine our attention here mainly to the basic case with integrator Brownian motion: $Y = W$. Much greater generality is possible: for Y a continuous martingale, see Karatzas and Shreve (1991) or Revuz and Yor (1991); for a systematic general treatment, see Protter (2004).
The first thing to note is that stochastic integrals with respect to Brownian motion, if they exist, must be quite different from the measure-theoretic integral. For, the Lebesgue-Stieltjes integrals described there have as integrators the difference of two monotone (increasing) functions, which are locally of bounded variation. But we know that Brownian motion is of infinite (unbounded) variation on every interval. So Lebesgue-Stieltjes and Itô integrals must be fundamentally different.
In view of the above, it is quite surprising that Itô integrals can be defined at all. But if we take for granted Itô's fundamental insight that they can be, it is obvious how to begin and clear enough how to proceed. We begin with the simplest possible integrands X, and extend successively in much the same way that we extended the measure-theoretic integral.

Indicators.

If $X(t,\omega) = \mathbf{1}_{[a,b]}(t)$, there is exactly one plausible way to define $\int X dW$:

$$\int_0^t X(s,\omega) dW(s,\omega) := \begin{cases} 0 & \text{if } t \le a, \\ W(t) - W(a) & \text{if } a \le t \le b, \\ W(b) - W(a) & \text{if } t \ge b. \end{cases}$$

Simple Functions.

Extend by linearity: if X is a linear combination of indicators, $X = \sum_{i=1}^n c_i \mathbf{1}_{[a_i,b_i]}$, we should define

$$\int_0^t X dW := \sum_{i=1}^n c_i \int_0^t \mathbf{1}_{[a_i,b_i]} dW.$$

Already one wonders how to extend this from constants c_i to suitable random variables, and one seeks to simplify the obvious but clumsy three-line expressions above.

We begin again, this time calling a stochastic process X *simple* if there is a partition $0 = t_0 < t_1 < \ldots < t_n = T < \infty$ and uniformly bounded $\mathcal{F}_{t_n}$-measurable random variables ξ_k ($|\xi_k| \le C$ for all $k = 0, \ldots, n$ and ω, for some C) and if $X(t, \omega)$ can be written in the form

$$X(t,\omega) = \xi_0(\omega)\mathbf{1}_{\{0\}}(t) + \sum_{i=0}^{n} \xi_i(\omega)\mathbf{1}_{(t_i,t_{i+1}]}(t) \quad (0 \le t \le T, \omega \in \Omega).$$

Then if $t_k \le t < t_{k+1}$,

$$\begin{aligned} I_t(X) := \int_0^t X dW &= \sum_{i=0}^{k-1} \xi_i (W(t_{i+1}) - W(t_i)) + \xi_k (W(t) - W(t_k)) \\ &= \sum_{i=0}^{n} \xi_i (W(t \wedge t_{i+1}) - W(t \wedge t_i)). \end{aligned}$$

Note that by definition $I_0(X) = 0 \quad I\!\!P - a.s.$. We collect some properties of the stochastic integral defined so far:

Lemma A.1.1. *(i)* $I_t(aX + bY) = aI_t(X) + bI_t(Y)$.
(ii) $I\!\!E(I_t(X)|\mathcal{F}_s) = I_s(X) \quad I\!\!P - a.s. \quad (0 \le s < t < \infty)$*, hence* $I_t(X)$ *is a continuous martingale.*

The stochastic integral for simple integrands is essentially a martingale transform, and the above is essentially the proof that martingale transforms are martingales.
We pause to note a property of square-integrable martingales which we shall need below. Call $M(t) - M(s)$ the increment of M over $(s, t]$. Then for a martingale M, the product of the increments over disjoint intervals has zero mean. For, if $s < t \le u < v$,

$$\begin{aligned} & I\!\!E\left[(M(v) - M(u))(M(t) - M(s))\right] \\ = \;& I\!\!E\left[I\!\!E((M(v) - M(u))(M(t) - M(s))|\mathcal{F}_u)\right] \\ = \;& I\!\!E\left[(M(t) - M(s))I\!\!E((M(v) - M(u))|\mathcal{F}_u)\right], \end{aligned}$$

taking out what is known (as $s, t \le u$). The inner expectation is zero by the martingale property, so the left-hand side is zero, as required.
We now can add further properties of the stochastic integral for simple functions.

Lemma A.1.2. *(i) We have the Itô isometry*

$$I\!\!E\left((I_t(X))^2\right) = I\!\!E\left(\int_0^t X(s)^2 ds\right).$$

(ii)$I\!\!E\left((I_t(X) - I_s(X))^2|\mathcal{F}_s\right) = I\!\!E\left(\int_s^t X(u)^2 du\right) \quad I\!\!P - a.s.$

The Itô isometry above suggests that $\int_0^t X dW$ should be defined only for processes with

$$\int_0^t I\!\!E\left(X(u)^2\right) du < \infty \quad \text{for all} \quad t.$$

We then can transfer convergence on a suitable L^2-space of stochastic processes to a suitable L^2-space of martingales. This gives us an L^2-theory of stochastic integration, for which Hilbert-space methods are available.
For the financial applications we have in mind, there is a fixed time-interval - $[0, T]$ say - on which we work (e.g., an option is written at time $t = 0$, with expiry time $t = T$). Then the above becomes

$$\int_0^T I\!\!E(X(u)^2) du < \infty.$$

Approximation.

We seek a class of integrands suitably approximable by simple integrands. It turns out that:

(i) The suitable class of integrands is the class of $(\mathcal{B}([0,\infty)) \otimes \mathcal{F})$-measurable, $(\mathcal{F}_t)$- adapted processes X with $\int_0^t \mathbb{E}\left(X(u)^2\right) du < \infty$ for all $t > 0$.
(ii) Each such X may be approximated by a sequence of simple integrands X_n so that the stochastic integral $I_t(X) = \int_0^t X dW$ may be defined as the limit of $I_t(X_n) = \int_0^t X_n dW$.
(iii) The properties from both lemmas above remain true for the stochastic integral $\int_0^t X dW$ defined by (i) and (ii).

Example.

We calculate $\int W(u)dW(u)$. We start by approximating the integrand by a sequence of simple functions.

$$X_n(u) = \begin{cases} W(0) = 0 & \text{if } \ 0 \le u \le t/n, \\ W(t/n) & \text{if } \ t/n < u \le 2t/n, \\ \vdots & \vdots \\ W\left(\frac{(n-1)t}{n}\right) & \text{if } \ (n-1)t/n < u \le t. \end{cases}$$

By definition,

$$\int_0^t W(u)dW(u) = \lim_{n\to\infty} \sum_{k=0}^{n-1} W\left(\frac{kt}{n}\right)\left(W\left(\frac{(k+1)t}{n}\right) - W\left(\frac{kt}{n}\right)\right).$$

Rearranging terms, we obtain for the sum on the right

$$\begin{aligned} &\sum_{k=0}^{n-1} W\left(\frac{kt}{n}\right)\left(W\left(\frac{(k+1)t}{n}\right) - W\left(\frac{kt}{n}\right)\right) \\ = \ &\frac{1}{2}W(t)^2 - \frac{1}{2}\left[\sum_{k=0}^{n-1}\left(W\left(\frac{(k+1)t}{n}\right) - W\left(\frac{kt}{n}\right)\right)^2\right]. \end{aligned}$$

Since the second term approximates the quadratic variation of W and hence tends to t for $n \to \infty$, we find

$$\int_0^t W(u)dW(u) = \frac{1}{2}W(t)^2 - \frac{1}{2}t. \tag{A.1}$$

Note the contrast with ordinary (Newton-Leibniz) calculus! Itô calculus requires the second term on the right – the Itô correction term – which arises from the quadratic variation of W.

One can construct a closely analogous theory for stochastic integrals with the Brownian integrator W above replaced by a square-integrable martingale integrator M. The properties above hold, with (i) in Lemma replaced by

$$\mathbb{E}\left[\left(\int_0^t X(u)dM(u)\right)^2\right] = \mathbb{E}\left[\int_0^t X(u)^2 d\langle M\rangle(u)\right].$$

Quadratic Variation, Quadratic Covariation.

We shall need to extend quadratic variation and quadratic covariation to stochastic integrals. The quadratic variation of $I_t(X) = \int_0^t X(u)dW(u)$ is $\int_0^t X(u)^2 du$. This is proved in the same way as the case $X \equiv 1$, that W has quadratic variation process t. More generally, if $Z(t) = \int_0^t X(u)dM(u)$ for a continuous martingale integrator M, then $\langle Z \rangle (t) = \int_0^t X^2(u)d\langle M \rangle (u)$. Similarly (or by polarisation), if $Z_i(t) = \int_0^t X_i(u)dM_i(u)$ $(i = 1, 2)$, $\langle Z_1, Z_2 \rangle (t) = \int_0^t X_1(u)X_2(u)d\langle M_1, M_2 \rangle (u)$.

A.1.3 Itô's Lemma

Suppose that b is adapted and locally integrable (so $\int_0^t b(s)ds$ is defined as an ordinary integral), and σ is adapted and measurable with $\int_0^t \mathbb{E}\left(\sigma(u)^2\right) du < \infty$ for all t (so $\int_0^t \sigma(s)dW(s)$ is defined as a stochastic integral). Then

$$X(t) := x_0 + \int_0^t b(s)ds + \int_0^t \sigma(s)dW(s)$$

defines a stochastic process X with $X(0) = x_0$. It is customary, and convenient, to express such an equation symbolically in differential form, in terms of the stochastic differential equation

$$dX(t) = b(t)dt + \sigma(t)dW(t), \quad X(0) = x_0. \tag{A.2}$$

Now suppose $f : \mathbb{R} \to \mathbb{R}$ is of class C^2. The question arises of giving a meaning to the stochastic differential $df(X(t))$ of the process $f(X(t))$, and finding it. Given a partition $\mathcal{P}$ of $[0, t]$, i.e. $0 = t_0 < t_1 < \ldots < t_n = t$, we can use Taylor's formula to obtain

$$\begin{aligned} f(X(t)) - f(X(0)) &= \sum_{k=0}^{n-1} f(X(t_{k+1})) - f(X(t_k)) \\ &= \sum_{k=0}^{n-1} f'(X(t_k))\Delta X(t_k) \\ &+ \frac{1}{2}\sum_{k=0}^{n-1} f''(X(t_k) + \theta_k \Delta X(t_k))(\Delta X(t_k))^2 \end{aligned}$$

with $0 < \theta_k < 1$. We know that $\sum(\Delta X(t_k))^2 \to \langle X \rangle (t)$ in probability (so, taking a subsequence, with probability one), and with a little more effort one can prove

$$\sum_{k=0}^{n-1} f''(X(t_k) + \theta_k \Delta X(t_k))(\Delta X(t_k))^2 \to \int_0^t f''(X(u))d\langle X \rangle (u).$$

The first sum is easily recognized as an approximating sequence of a stochastic integral; indeed, we find

$$\sum_{k=0}^{n-1} f'(X(t_k))\Delta X(t_k) \to \int_0^t f'(X(u))dX(u).$$

So we have

Theorem A.1.2 (Basic Itô formula)**.** *If X has stochastic differential given by A.2 and $f \in C^2$, then $f(X)$ has stochastic differential*

$$df(X(t)) = f'(X(t))dX(t) + \frac{1}{2}f''(X(t))d\langle X \rangle (t),$$

or writing out the integrals,

$$f(X(t)) = f(x_0) + \int_0^t f'(X(u))dX(u) + \frac{1}{2}\int_0^t f''(X(u))d\langle X \rangle (u).$$

More generally, suppose that $f : I\!R^2 \to I\!R$ is a function, continuously differentiable once in its first argument (which will denote time), and twice in its second argument (space): $f \in C^{1,2}$. By the Taylor expansion of a smooth function of several variables we get for t close to t_0 (we use subscripts to denote partial derivatives: $f_t := \partial f/\partial t$, $f_{tx} := \partial^2 f/\partial t \partial x$):

$$\begin{aligned} f(t, X(t)) &= f(t_0, X(t_0)) \\ &+(t-t_0)f_t(t_0, X(t_0)) + (X(t) - X(t_0))f_x(t_0, X(t_0)) \\ &+\frac{1}{2}(t-t_0)^2 f_{tt}(t_0, X(t_0)) + \frac{1}{2}(X(t) - X(t_0))^2 f_{xx}(t_0, X(t_0)) \\ &+(t-t_0)(X(t) - X(t_0))f_{tx}(t_0, X(t_0)) + \ldots, \end{aligned}$$

which may be written symbolically as

$$df = f_t dt + f_x dX + \frac{1}{2} f_{tt}(dt)^2 + f_{tx} dt dX + \frac{1}{2} f_{xx}(dX)^2 + \ldots.$$

In this, we substitute $dX(t) = b(t)dt + \sigma(t)dW(t)$ from above, to obtain

$$\begin{aligned} df &= f_t dt + f_x(bdt + \sigma dW) \\ &+\frac{1}{2} f_{tt}(dt)^2 + f_{tx} dt(bdt + \sigma dW) + \frac{1}{2} f_{xx}(bdt + \sigma dW)^2 + \ldots \end{aligned}$$

Now using the formal multiplication rules $dt \cdot dt = 0, dt \cdot dW = 0, dW \cdot dW = dt$ (which are just shorthand for the corresponding properties of the quadratic variations, we expand

$$(bdt + \sigma dW)^2 = \sigma^2 dt + 2b\sigma dt dW + b^2(dt)^2 = \sigma^2 dt + \text{higher-order terms}$$

to get finally

$$df = \left(f_t + b f_x + \frac{1}{2}\sigma^2 f_{xx} \right) dt + \sigma f_x dW + \text{higher-order terms.}$$

As above the higher-order terms are irrelevant, and summarising, we obtain *Itô's lemma*, the analogue for the Itô or stochastic calculus of the chain rule for ordinary (Newton-Leibniz) calculus:

Theorem A.1.3 (Itô's Lemma)**.** *If $X(t)$ has stochastic differential given by A.2, then $f = f(t, X(t))$ has stochastic differential*

$$df = \left(f_t + b f_x + \frac{1}{2}\sigma^2 f_{xx} \right) dt + \sigma f_x dW.$$

That is, writing f_0 for $f(0, x_0)$, the initial value of f,

$$f = f_0 + \int_0^t (f_t + b f_x + \frac{1}{2}\sigma^2 f_{xx}) dt + \int_0^t \sigma f_x dW.$$

We will make good use of:

Corollary A.1.1. $I\!E\left(f(t, X(t))\right) = f_0 + \int_0^t I\!E\left(f_t + b f_x + \frac{1}{2}\sigma^2 f_{xx}\right) dt.$

Proof. $\int_0^t \sigma f_2 dW$ is a stochastic integral, so a martingale, so its expectation is constant ($= 0$, as it starts at 0). ■

The differential equation (A.3) above has the unique solution

$$S(t) = S(0) \exp\left\{ \left(\mu - \frac{1}{2}\sigma^2 \right) t + \sigma dW(t) \right\}.$$

For, writing

$$f(t,x) := \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma x\right\},$$

we have

$$f_t = \left(\mu - \frac{1}{2}\sigma^2\right)f, \quad f_x = \sigma f, \quad f_{xx} = \sigma^2 f,$$

and with $x = W(t)$, one has

$$dx = dW(t), \quad (dx)^2 = dt.$$

Thus Itô's lemma gives

$$\begin{aligned} df(t,W(t)) &= f_t dt + f_x dW(t) + \frac{1}{2} f_{xx}(dW(t))^2 \\ &= f\left(\left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW(t) + \frac{1}{2}\sigma^2 dt\right) \\ &= f(\mu dt + \sigma dW(t)), \end{aligned}$$

so $f(t,W(t))$ is a solution of the stochastic differential equation, and the initial condition $f(0,W(0)) = S(0)$ as $W(0) = 0$, giving existence.

Geometric Brownian Motion

Now that we have both Brownian motion W and Itô's Lemma to hand, we can introduce the most important stochastic process for us, a relative of Brownian motion - *geometric* (or *exponential*, or *economic*) Brownian motion.

Suppose we wish to model the time evolution of a stock price $S(t)$ (as we will, in the Black-Scholes theory). Consider how S will change in some small time-interval from the present time t to a time $t + dt$ in the near future. Writing $dS(t)$ for the change $S(t + dt) - S(t)$ in S, the *return* on S in this interval is $dS(t)/S(t)$. It is economically reasonable to expect this return to decompose into two components, a *systematic* part and a *random* part. The systematic part could plausibly be modelled by μdt, where μ is some parameter representing the mean rate of return of the stock. The random part could plausibly be modelled by $\sigma dW(t)$, where $dW(t)$ represents the noise term driving the stock price dynamics, and σ is a second parameter describing how much effect this noise has - how much the stock price fluctuates. Thus σ governs how volatile the price is, and is called the *volatility* of the stock. The role of the driving noise term is to represent the random buffeting effect of the multiplicity of factors at work in the economic environment in which the stock price is determined by supply and demand.

Putting this together, we have the stochastic differential equation

$$dS(t) = S(t)(\mu dt + \sigma dW(t)), \quad S(0) > 0, \tag{A.3}$$

due to Itô in 1944. This corrects Bachelier's earlier attempt of 1900 (he did not have the factor $S(t)$ on the right - missing the interpretation in terms of returns, and leading to negative stock prices!) Incidentally, Bachelier's work served as Itô's motivation in introducing Itô calculus. The mathematical importance of Itô's work was recognised early, and led on to the work of Doob (1953), Meyer (1976) and many others (see the memorial volume Ikeda, Watanabe, M., and Kunita (1996) in honour of Itô's eightieth birthday in 1995). The economic importance of geometric Brownian motion was recognised by Paul A. Samuelson in his work from 1965 on (Samuelson (1965)), for which Samuelson received the Nobel Prize in Economics in 1970, and by Robert Merton (see Merton (1990) for a full bibliography), in work for which he was similarly honoured in 1997.

A.1.4 Girsanov's Theorem

Consider first independent $N(0,1)$ random variables $Z_1,\dots,Z_n$ on a probability space $(\Omega,\mathcal{F},I\!P)$. Given a vector $\gamma=(\gamma_1,\dots,\gamma_n)$, consider a new probability measure $\tilde{I\!P}$ on $(\Omega,\mathcal{F})$ defined by

$$\tilde{I\!P}(d\omega)=\exp\left\{\sum_{i=1}^n \gamma_i Z_i(\omega)-\frac{1}{2}\sum_{i=1}^n\gamma_i^2\right\}I\!P(d\omega).$$

As $\exp\{.\}>0$ and integrates to 1, as $\int\exp\{\gamma_i Z_i\}dI\!P=\exp\{\frac{1}{2}\gamma_i^2\}$, this is a probability measure. It is also equivalent to $I\!P$ (has the same null sets), again as the exponential term is positive. Also

$$\begin{aligned}
&\tilde{I\!P}(Z_i\in dz_i,\quad i=1,\dots,n)\\
=\;&\exp\left\{\sum_{i=1}^n\gamma_i Z_i-\frac{1}{2}\sum_{i=1}^n\gamma_i^2\right\}I\!P(Z_i\in dz_i,\quad i=1,\dots,n)\\
=\;&(2\pi)^{-\frac{n}{2}}\exp\left\{\sum_{i=1}^n\gamma_i z_i-\frac{1}{2}\sum_{i=1}^n\gamma_i^2-\frac{1}{2}\sum_{i=1}^n z_i^2\right\}\prod_{i=1}^n dz_i\\
=\;&(2\pi)^{-\frac{n}{2}}\exp\left\{-\frac{1}{2}\sum_{i=1}^n(z_i-\gamma_i)^2\right\}dz_1\dots dz_n.
\end{aligned}$$

This says that if the Z_i are independent $N(0,1)$ under $I\!P$, they are independent $N(\gamma_i,1)$ under $\tilde{I\!P}$. Thus the effect of the *change of measure* $I\!P\to\tilde{I\!P}$, from the original measure $I\!P$ to the *equivalent* measure $\tilde{I\!P}$, is to *change the mean*, from $0=(0,\dots,0)$ to $\gamma=(\gamma_1,\dots,\gamma_n)$.

This result extends to infinitely many dimensions - i.e., from random vectors to stochastic processes, indeed with random rather than deterministic means. Let $W=(W_1,\dots W_d)$ be a d-dimensional Brownian motion defined on a filtered probability space $(\Omega,\mathcal{F},I\!P,I\!F)$ with the filtration $I\!F$ satisfying the usual conditions. Let $(\gamma(t):0\le t\le T)$ be a measurable, adapted d-dimensional process with $\int_0^T\gamma_i(t)^2dt<\infty\quad a.s.,\ i=1,\dots,d$, and define the process $(L(t):0\le t\le T)$ by

$$L(t)=\exp\left\{-\int_0^t\gamma(s)'dW(s)-\frac{1}{2}\int_0^t\|\gamma(s)\|^2\,ds\right\}. \tag{A.4}$$

Then L is continuous, and, being the stochastic exponential of $-\int_0^t\gamma(s)'dW(s)$, is a local martingale. Given sufficient integrability on the process γ, L will in fact be a (continuous) martingale. For this, *Novikov's condition* suffices:

$$I\!E\left(\exp\left\{\frac{1}{2}\int_0^T\|\gamma(s)\|^2\,ds\right\}\right)<\infty. \tag{A.5}$$

We are now in the position to state a version of Girsanov's theorem, which will be one of our main tools in studying continuous-time financial market models.

Theorem A.1.4 (Girsanov)**.** *Let γ be as above and satisfy Novikov's condition; let L be the corresponding continuous martingale. Define the processes $\tilde{W}_i,\ i=1,\dots,d$ by*

$$\tilde{W}_i(t):=W_i(t)+\int_0^t\gamma_i(s)ds,\quad(0\le t\le T),\quad i=1,\dots,d.$$

Then under the equivalent probability measure $\tilde{I\!P}$ (defined on $(\Omega,\mathcal{F}_T)$) with Radon-Nikodým derivative

$$\frac{d\tilde{I\!P}}{dI\!P}=L(T),$$

the process $\tilde{W}=(\tilde{W}_1,\dots,\tilde{W}_d)$ is d-dimensional Brownian motion.

In particular, for $\gamma(t)$ constant ($= \gamma$), change of measure by introducing the Radon-Nikodým derivative $\exp\left\{-\gamma W(t) - \frac{1}{2}\gamma^2 t\right\}$ corresponds to a change of drift from c to $c - \gamma$. If $\mathbb{F} = (\mathcal{F}_t)$ is the Brownian filtration (basically $\mathcal{F}_t = \sigma(W(s), 0 \leq s \leq t)$ slightly enlarged to satisfy the usual conditions) any pair of equivalent probability measures $\mathbb{Q} \sim \mathbb{P}$ on $\mathcal{F} = \mathcal{F}_T$ is a Girsanov pair, i.e.

$$\left.\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = L(t)$$

with L defined as above. Girsanov's theorem (or the Cameron-Martin-Girsanov theorem) is formulated in varying degrees of generality, discussed and proved, e.g. in Karatzas and Shreve (1991), §3.5, Protter (2004), III.6, Revuz and Yor (1991), VIII, Dothan (1990), §5.4 (discrete time), §11.6 (continuous time).

A.2 Financial Market Models

A.2.1 The Financial Market Model

We start with a general model of a frictionless (compare Chapter 1) security market where investors are allowed to trade continuously up to some fixed finite planning horizon T. Uncertainty in the financial market is modelled by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ satisfying the usual conditions of right-continuity and completeness. We assume that the σ-field $\mathcal{F}_0$ is trivial, i.e. for every $A \in \mathcal{F}_0$ either $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$, and that $\mathcal{F}_T = \mathcal{F}$.
There are $d + 1$ primary traded assets, whose price processes are given by stochastic processes $S_0, \ldots, S_d$. We assume that the processes $S_0, \ldots, S_d$ represent the prices of some traded assets (stocks, bonds, or options).
We have not emphasised so far that there was an implicit numéraire behind the prices $S_0, \ldots, S_d$; it is the numéraire relevant for domestic transactions at time t. The formal definition of a numéraire is very much as in the discrete setting.

Definition A.2.1. *A numéraire is a price process $X(t)$ almost surely strictly positive for each $t \in [0, T]$.*

We assume now that $S_0(t)$ is a non-dividend paying asset, which is (almost surely) strictly positive and use S_0 as numéraire. 'Historically' (see Harrison and Pliska (1981)) the money market account $B(t)$, given by $B(t) = e^{r(t)}$ with a positive deterministic process $r(t)$ and $r(0) = 0$, was used as a numéraire, and the reader may think of $S_0(t)$ as being $B(t)$.
Our principal task will be the pricing and hedging of contingent claims, which we model as $\mathcal{F}_T$-measurable random variables. This implies that the contingent claims specify a stochastic cash-flow at time T and that they may depend on the whole path of the underlying in $[0, T]$ - because $\mathcal{F}_T$ contains all that information. We will often have to impose further integrability conditions on the contingent claims under consideration. The fundamental concept in (arbitrage) pricing and hedging contingent claims is the interplay of self-financing replicating portfolios and risk-neutral probabilities. Although the current setting is on a much higher level of sophistication, the key ideas remain the same.
We call an $\mathbb{R}^{d+1}$-valued predictable process

$$\varphi(t) = (\varphi_0(t), \ldots, \varphi_d(t)), \quad t \in [0, T]$$

with $\int_0^T \mathbb{E}(\varphi_0(t))dt < \infty, \sum_{i=0}^d \int_0^T \mathbb{E}(\varphi_i^2(t))dt < \infty$ a trading strategy (or dynamic portfolio process). Here $\varphi_i(t)$ denotes the number of shares of asset i held in the portfolio at time t - to be determined on the basis of information available *before* time t; i.e. the investor selects his time t portfolio after observing the prices $S(t-)$. The components $\varphi_i(t)$ may assume negative as well as positive values, reflecting the fact that we allow short sales and assume that the assets are perfectly divisible.

Definition A.2.2. *(i) The value of the portfolio φ at time t is given by the scalar product*

$$V_\varphi(t) := \varphi(t) \cdot S(t) = \sum_{i=0}^{d} \varphi_i(t) S_i(t), \quad t \in [0,T].$$

The process $V_\varphi(t)$ is called the value process, or wealth process, of the trading strategy φ.
(ii) The gains process $G_\varphi(t)$ is defined by

$$G_\varphi(t) := \int_0^t \varphi(u) dS(u) = \sum_{i=0}^{d} \int_0^t \varphi_i(u) dS_i(u).$$

(iii) A trading strategy φ is called self-financing if the wealth process $V_\varphi(t)$ satisfies

$$V_\varphi(t) = V_\varphi(0) + G_\varphi(t) \quad \textit{for all}\, t \in [0,T].$$

Remark A.2.1. *(i) The financial implications of the above equations are that all changes in the wealth of the portfolio are due to capital gains, as opposed to withdrawals of cash or injections of new funds.*
(ii) The definition of a trading strategy includes regularity assumptions in order to ensure the existence of stochastic integrals.

Using the special numéraire $S_0(t)$ we consider the discounted price process

$$\tilde{S}(t) := \frac{S(t)}{S_0(t)} = (1, \tilde{S}_1(t), \dots \tilde{S}_d(t))$$

with $\tilde{S}_i(t) = S_i(t)/S_0(t)$, $i = 1, 2, \dots, d$. Furthermore, the discounted wealth process $\tilde{V}_\varphi(t)$ is given by

$$\tilde{V}_\varphi(t) := \frac{V_\varphi(t)}{S_0(t)} = \varphi_0(t) + \sum_{i=1}^{d} \varphi_i(t) \tilde{S}_i(t)$$

and the discounted gains process $\tilde{G}_\varphi(t)$ is

$$\tilde{G}_\varphi(t) := \sum_{i=1}^{d} \int_0^t \varphi_i(t) d\tilde{S}_i(t).$$

Observe that $\tilde{G}_\varphi(t)$ does not depend on the numéraire component φ_0.
It is convenient to reformulate the self-financing condition in terms of the discounted processes:

Proposition A.2.1. *Let φ be a trading strategy. Then φ if self-financing if and only if*

$$\tilde{V}_\varphi(t) = \tilde{V}_\varphi(0) + \tilde{G}_\varphi(t).$$

Of course, $V_\varphi(t) \geq 0$ if and only if $\tilde{V}_\varphi(t) \geq 0$.

The *proof* follows by the numéraire invariance theorem using S_0 as numéraire. ■

Remark A.2.2. *The above result shows that a self-financing strategy is completely determined by its initial value and the components $\varphi_1, \dots, \varphi_d$. In other words, any set of predictable processes $\varphi_1, \dots, \varphi_d$ such that the stochastic integrals $\int \varphi_i d\tilde{S}_i$, $i = 1, \dots, d$ exist can be uniquely extended to a self-financing strategy φ with specified initial value $\tilde{V}_\varphi(0) = v$ by setting the cash holding as*

$$\varphi_0(t) = v + \sum_{i=1}^{d} \int_0^t \varphi_i(u) d\tilde{S}_i(u) - \sum_{i=1}^{d} \varphi_i(t) \tilde{S}_i(t), \; t \in [0,T].$$

A.2.2 Equivalent Martingale Measures

We develop a relative pricing theory for contingent claims. Again the underlying concept is the link between the no-arbitrage condition and certain probability measures. We begin with:

Definition A.2.3. *A self-financing trading strategy φ is called an arbitrage opportunity if the wealth process V_φ satisfies the following set of conditions:*

$$V_\varphi(0) = 0, \quad \mathbb{P}(V_\varphi(T) \geq 0) = 1, \quad \textit{and} \quad \mathbb{P}(V_\varphi(T) > 0) > 0.$$

Arbitrage opportunities represent the limitless creation of wealth through risk-free profit and thus should not be present in a well-functioning market.
The main tool in investigating arbitrage opportunities is the concept of equivalent martingale measures:

Definition A.2.4. *We say that a probability measure $\mathbb{Q}$ defined on $(\Omega, \mathcal{F})$ is an equivalent martingale measure if:*

(i) $\mathbb{Q}$ is equivalent to $\mathbb{P}$,
(ii) the discounted price process $\tilde{S}$ is a $\mathbb{Q}$ martingale.

We denote the set of martingale measures by $\mathcal{P}$.

A useful criterion in determining whether a given equivalent measure is indeed a martingale measure is the observation that the growth rates relative to the numéraire of all given primary assets under the measure in question must coincide. For example, in the case $S_0(t) = B(t)$ we have:

Lemma A.2.1. *Assume $S_0(t) = B(t) = e^{r(t)}$, then $\mathbb{Q} \sim \mathbb{P}$ is a martingale measure if and only if every asset price process S_i has price dynamics under $\mathbb{Q}$ of the form*

$$dS_i(t) = r(t)S_i(t)dt + dM_i(t),$$

where M_i is a $\mathbb{Q}$-martingale.

The *proof* is an application of Itô's formula.
In order to proceed we have to impose further restrictions on the set of trading strategies.

Definition A.2.5. *A self-financing trading strategy φ is called tame (relative to the numéraire S_0) if*

$$\tilde{V}_\varphi(t) \geq 0 \quad \textit{for all}\, t \in [0, T].$$

We use the notation Φ for the set of tame trading strategies.

We next analyse the value process under equivalent martingale measures for such strategies.

Proposition A.2.2. *For $\varphi \in \Phi$ $\tilde{V}_\varphi(t)$ is a martingale under each $\mathbb{Q} \in \mathcal{P}$.*

This observation is the key to our first central result:

Theorem A.2.1. *Assume $\mathcal{P} \neq \emptyset$. Then the market model contains no arbitrage opportunities in Φ.*

Proof. For any $\varphi \in \Phi$ and under any $\mathbb{Q} \in \mathcal{P}$ $\tilde{V}_\varphi(t)$ is a martingale. That is,

$$\mathbb{E}_{\mathbb{Q}}\left(\tilde{V}_\varphi(t)|\mathcal{F}_u\right) = \tilde{V}_\varphi(u), \;\text{ for all }\quad u \leq t \leq T.$$

For $\varphi \in \Phi$ to be an arbitrage opportunity we must have $\tilde{V}_\varphi(0) = V_\varphi(0) = 0$. Now

$$\mathbb{E}_{\mathbb{Q}}\left(\tilde{V}_\varphi(t)\right) = 0, \;\text{ for all }\quad 0 \leq t \leq T.$$

Now φ is tame, so $\tilde{V}_\varphi(t) \geq 0,\ 0 \leq t \leq T$, implying $\mathbb{E}_{\mathbb{Q}}\left(\tilde{V}_\varphi(t)\right) = 0,\ 0 \leq t \leq T$, and in particular $\mathbb{E}_{\mathbb{Q}}\left(\tilde{V}_\varphi(T)\right) = 0$. But an arbitrage opportunity φ also has to satisfy $\mathbb{P}\left(V_\varphi(T) \geq 0\right) = 1$, and since $\mathbb{Q} \sim \mathbb{P}$, this means $\mathbb{Q}\left(V_\varphi(T) \geq 0\right) = 1$. Both together yield

$$\mathbb{Q}\left(V_\varphi(T) > 0\right) = \mathbb{P}\left(V_\varphi(T) > 0\right) = 0,$$

and hence the result follows. ■

A.2.3 Risk-neutral Pricing

We now assume that there exists an equivalent martingale measure $I\!P^*$ which implies that there are no arbitrage opportunities with respect to Φ in the financial market model. Until further notice we use $I\!P^*$ as our reference measure, and when using the term martingale we always assume that the underlying probability measure is $I\!P^*$. In particular, we restrict our attention to contingent claims X such that $X/S_0(T) \in L^1(\mathcal{F}, I\!P^*)$.
We now define a further subclass of trading strategies:

Definition A.2.6. *A self-financing trading strategy φ is called ($I\!P^*$-) admissible if the relative gains process*

$$\tilde{G}_\varphi(t) = \int_0^t \varphi(u) d\tilde{S}(u)$$

is a ($I\!P^*$-) martingale. The class of all ($I\!P^*$-) admissible trading strategies is denoted $\Phi(I\!P^*)$.

By definition $\tilde{S}$ is a martingale, and $\tilde{G}$ is the stochastic integral with respect to $\tilde{S}$. We see that any sufficiently integrable processes $\varphi_1, \ldots, \varphi_d$ give rise to $I\!P^*$-admissible trading strategies.
We can repeat the above argument to obtain

Theorem A.2.2. *The financial market model $\mathcal{M}$ contains no arbitrage opportunities in $\Phi(I\!P^*)$.*

Under the assumption that no arbitrage opportunities exist, the question of pricing and hedging a contingent claim reduces to the existence of replicating self-financing trading strategies. Formally:

Definition A.2.7. *(i) A contingent claim X is called attainable if there exists at least one admissible trading strategy such that*

$$V_\varphi(T) = X.$$

We call such a trading strategy φ a replicating strategy for X.
(ii) The financial market model $\mathcal{M}$ is said to be complete if any contingent claim is attainable.

Again we emphasise that this depends on the class of trading strategies. On the other hand, it does not depend on the numéraire: it is an easy exercise in the continuous asset-price process case to show that if a contingent claim is attainable in a given numéraire it is also attainable in any other numéraire and the replicating strategies are the same.
If a contingent claim X is attainable, X can be replicated by a portfolio $\varphi \in \Phi(I\!P^*)$. This means that holding the portfolio and holding the contingent claim are equivalent from a financial point of view. In the absence of arbitrage the (arbitrage) price process $\Pi_X(t)$ of the contingent claim must therefore satisfy

$$\Pi_X(t) = V_\varphi(t).$$

Of course the questions arise of what will happen if X can be replicated by more than one portfolio, and what the relation of the price process to the equivalent martingale measure(s) is. The following central theorem is the key to answering these questions:

Theorem A.2.3 (Risk-Neutral Valuation Formula)**.** *The arbitrage price process of any attainable claim is given by the risk-neutral valuation formula*

$$\Pi_X(t) = S_0(t) I\!E_{I\!P^*}\left[\left. \frac{X}{S_0(T)} \right| \mathcal{F}_t \right]. \tag{A.6}$$

The uniqueness question is immediate from the above theorem:

Corollary A.2.1. *For any two replicating portfolios $\varphi, \psi \in \Phi(I\!P^*)$ we have*

$$V_\varphi(t) = V_\psi(t).$$

Proof of Theorem A.2.3 Since X is attainable, there exists a replicating strategy $\varphi \in \Phi(\mathbb{P}^*)$ such that $V_\varphi(T) = X$ and $\Pi_X(t) = V_\varphi(t)$. Since $\varphi \in \Phi(\mathbb{P}^*)$ the discounted value process $\tilde{V}_\varphi(t)$ is a martingale, and hence

$$\begin{aligned}
\Pi_X(t) &= V_\varphi(t) = S_0(t)\tilde{V}_\varphi(t) \\
&= S_0(t)\mathbb{E}_{\mathbb{P}^*}\left[\tilde{V}_\varphi(T)\middle|\mathcal{F}_t\right] = S_0(t)\mathbb{E}_{\mathbb{P}^*}\left[\frac{V_\varphi(T)}{S_0(T)}\middle|\mathcal{F}_t\right] \\
&= S_0(t)\mathbb{E}_{\mathbb{P}^*}\left[\frac{X}{S_0(T)}\middle|\mathcal{F}_t\right].
\end{aligned}$$

■

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