Pricing Asian options in the CIR model: Distributional results and option pricing

Pricing

Asian options

in

the CIR model

Angelos DASSIOS
Jayalaxshmi NAGARADJASARMA
Statistics Department
Pricing Asian options in the CIR model: Distributional results and option pricing

OUTLINE

I. Introduction

II. Distributional results

III. Option pricing

IV. Numerical examples

V. Conclusion
INTRODUCTION:
DIFFERENT KINDS OF ASIAN OPTIONS

- Type of average: \( T_0 < T_1 < \ldots < T_n = T \)
  - Discrete arithmetic: \( Y_T = \frac{1}{N+1} \sum_{n=0}^{N} X_{T_n} \)
  - Discrete geometric: \( Y_T = \left( \prod_{n=0}^{N} X_{T_n} \right)^{\frac{1}{N+1}} \)
  - Continuous arithmetic: \( Y_T = \frac{1}{T-T_0} \int_{T_0}^{T} X_u du \)
  - Continuous geometric: \( Y_T = e^{\frac{1}{T-T_0} \int_{T_0}^{T} \ln(X_u) du} \)

- Type of strike:
  - Fixed: payoff = \( (Y_T - K)^+ \)
  - Floating: payoff = \( (Y_T - S_T)^+ \)

- Average starting time:
  - Forward-starting: \( T_0 > t \)
  - Starting: \( T_0 = t \)
  - Backward-started: \( T_0 < t \)
INTRODUCTION:
LITERATURE REVIEW FOR ASIAN OPTIONS

• Different approaches:

  ➣ (Semi-)Analytical approaches and Laplace transform analysis

  ➣ Pseudo-analytical approximations

  ➣ Numerical PDE solutions

  ➣ Monte Carlo simulation

  ➣ Miscellaneous numerical methods
**Introduction:** Numerical Analytical Laplace Transform Inversion

- "(Numerical) Laplace transform inversion is still more an art than a science" Davies and Martin (79).

- Numerical inversion methods:
  - Plethora of algorithms
  - Most of those have a number of free-parameters.

- Analytical inversion:
  - Generic technique: the Bromwich integral
    \[
    F(x) = \mathcal{L}^{-1}(L(\mu)) = \lim_{R \to \infty} \frac{1}{2i\pi} \int_{c-iR}^{c+iR} L(\mu)e^{x\mu}d\mu \tag{1}
    \]
    \(c\) at the right of all singularities.
  - Specific relation with known inverses.
INTRODUCTION:
THE SQUARE-ROOT PROCESS

• Unique solution to the SDE:

\[ dX_t = (a - bX_t)dt + \sigma \sqrt{X_t}dW_t \tag{2} \]

with the constraints: \( a \geq 0, \ b \in \mathbb{R} \) and \( \sigma > 0 \).

• The two most common forms in Finance:

enguish form with \( a > 0 \) and \( b > 0 \):

→ Mean-Reversion level: \( \frac{a}{b} \)
→ Mean: \( E(X_t) = \frac{a}{b} + (X_0 - \frac{a}{b})e^{-bt} \)

• Exploding form with \( a = 0 \) and \( b \leq 0 \):

\[ dS_t = rS_t dt + \sigma \sqrt{S_t}dW_t \tag{3} \]

→ \( r = -b \)
→ Mean: \( E(S_t) = S_0e^{rt} \)
→ Probability of reaching the origin in finite time:

\[ P(\tau_0 < \infty) \in [0, 1] \]
DISTRIBUTIONAL RESULTS:
IMPORTANCE OF THE ADDITIVITY PROPERTY

• The additivity property

\[
\text{SR}(a^1, b, \sigma, x^1_0) + \text{SR}(a^2, b, \sigma, x^2_0) \\
\sim \text{SR}(a^1 + a^2, b, \sigma, x^1_0 + x^2_0)
\] (4)

• Application: functional form of the MGF

porate definition

\[
\mathcal{L}^{a,x_0,b,\sigma}(\lambda, \mu) = E(e^{-\lambda X_t - \mu Y_t}), \quad \lambda \geq 0, \quad \mu \geq 0
\] (5)

\[
Y_t = \int_0^t X_s ds
\] (6)

• Additivity implies

\[
\mathcal{L}^{a,x_0}(\lambda, \mu) = \mathcal{L}^{0,x_0}(\lambda, \mu) \mathcal{L}^{a,0}(\lambda, \mu)
\]

• Scaling implies

\[
\mathcal{L}^{a,x_0}(\lambda, \mu) = e^{-x_0 \psi} e^{-a \phi}
\]
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**DISTRIBUTIONAL RESULTS:**

**JOINT MOMENT GENERATING FUNCTION**

- Notation

\[
\gamma = \sqrt{b^2 + 2\sigma^2 \mu} \quad (7)
\]

- The joint MGF of \((X_t, Y_t)\) is given by (See Lamberton-Lapeyre):

\[
\mathcal{L}(\lambda, \mu) = E(e^{-\lambda X_t - \mu Y_t}) = e^{-X_0 \psi(t) - a \phi(t)} \quad (8)
\]

\[
\begin{align*}
\psi(t) &= \frac{\lambda((\gamma-b)+e^{-\gamma t}(\gamma+b)) + 2\mu(1-e^{-\gamma t})}{\sigma^2 \lambda(1-e^{-\gamma t})(\gamma+b)+e^{-\gamma t}(\gamma-b)} \\
\phi(t) &= \frac{-2}{\sigma^2} \ln \left( \frac{2\gamma e^{\frac{(b-\gamma)t}{2}}}{\sigma^2 \lambda(1-e^{-\gamma t})(\gamma+b)+e^{-\gamma t}(\gamma-b)} \right) \quad (9)
\end{align*}
\]
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**DISTRIBUTIONAL RESULTS: JOINT MOMENTS**

- **System of ODE**

\[
\frac{dE(Y_t^m X_t^k)}{dt} = mE(Y_t^{m-1} X_t^{k+1}) + a k E(Y_t^m X_t^{k-1})
- bk E(Y_t^m X_t^k) + k(k - 1) \frac{\sigma^2}{2} E(Y_t^m X_t^{k-1})
\]  

(10)

- **Laplace transform**

\[
\hat{M}_{m,n} = \sum_{j=0}^{n} \sum_{i=1}^{I_{j}^{m,n}} \frac{\alpha_{j,i}^{m,n}}{(\zeta + jb)^i}
\]

(11)

- **Explicit simple form**

\[
M_{m,n}(t) = E(Y_t^m X_t^{n-m}) = \sum_{j=0}^{n} e^{-jbt} \left( \sum_{i=1}^{I_{j}^{m,n}} \alpha_{j,i}^{m,n} \frac{t^{i-1}}{(i-1)!} \right)
\]

(12)

\[
I_{j}^{m,n} = \min(n + 1 - j, m + 1)
\]

(13)
DISTRIBUTIONAL RESULTS:
MOMENTS-BASED EXPANSIONS FOR THE INTEGRATED PROCESS

• Joint distribution determined by its moments

• Edgeworth-type expansions

• Laguerre expansions: Dufresne (00)

\[
\frac{1}{2} [f(y^+) + f(y^-)] = y^b e^{-dx} \sum_{n=0}^{\infty} c_n L_n^a(x) \tag{14}
\]

under some regularity and integrability conditions, where

\[
L_n^a(x) = \sum_{m=0}^{n} (-1)^m \binom{n + a}{n - m} \frac{x^m}{m!} \tag{15}
\]
### DISTRIBUTIONAL RESULTS: A SIMPLIFYING CHANGE OF MEASURE

- The following process

\[
L(t) = e^{\frac{b^2 Y_t}{2} + \frac{b(X_t - X_0)}{\sigma^2} - \frac{abt}{\sigma^2}}
\]  

(16)

is an exponential martingale and \( E(L(T)) = 1 \).

- Defining \( T \) the time horizon and \( Q^* \) the measure constructed with the Radon-Nykodim derivative

\[
\frac{dQ^*}{dQ} = L(T)
\]

(17)

the process \( X_t \) is a **time-changed squared Bessel process** under \( Q^* \)

\[
dX_t = adt + \sigma \sqrt{X_t} dW^*_t
\]

(18)

and has the simplified MGF

\[
L^*(\lambda, \mu) = e^{-x_0 \frac{\lambda(1 + e^{-\gamma t}) + 2\mu(1 - e^{-\gamma t})}{\sigma^2 \lambda(1 - e^{-\gamma t}) + \gamma(1 + e^{-\gamma t})}}
\]

(19)
DISTRIBUTIONAL RESULTS:

J OINT DENSITY FOR THE EQUITY CASE

- The non-absorbed part

- Noting:

\[ \alpha_n = \frac{s + s_0 + (n+1)\sigma^2 t}{2} \]

- \( B_n(y) \): a sequence of functions defined by

\[
\sum_{p=0}^{n} \frac{(n-p)}{p!} \left( \frac{-s}{\sqrt{2y\alpha}} \right)^p \sum_{q=0}^{n} \frac{(n-q)}{q!} \left( \frac{-s_0}{\sqrt{2y\alpha}} \right)^q H e_{p+q+3} \left( \frac{\alpha_n}{\sqrt{2y\alpha}} \right) e^{-\frac{\alpha_n^2}{4y\alpha}} \] (20)

where

\[
H e_k(x) = \sum_{s=0}^{[\frac{k}{2}]} (-1)^s x^{k-2s} 2^s \frac{k!}{(k-2s)!s!} \] (21)

- Joint density of \((S_t, Y_t)\) under \(Q\) for \(X_t > 0\)

\[
f_{S,Y}(s, y) = \frac{s_0\alpha}{2\sqrt{2\pi}(y\alpha)^2} e^{-\frac{r^2 y^2}{2\sigma^2} + \frac{r(s-s_0)}{\sigma^2}} \sum_{n=0}^{\infty} \frac{B_n(y)}{n+1} \] (22)

- Leading term

\[
e^{-\frac{\sigma^2 t^2}{2y} n^2} \] (23)
DISTRIBUTIONAL RESULTS: INTEREST-RATE CASE

• The marginal density of the integral $Y_t$

$$f^Y(y) = \sum_{k=0}^{\infty} f^Y_k(y)$$

• Involves
  ➤ Hermite polynomial $H e_k(x)$
  ➤ Complementary error function $erfc(x)$

• Useful quantities
  ➤ $\mathcal{L}^{-1}\left(\frac{E(e^{-(\lambda+\mu)Y_t})}{\mu}\right) = \mathcal{G}_{a,b,\sigma}(y, \lambda)$
  ➤ $\mathcal{L}^{-1}\left(\frac{E(Y_t e^{-(\lambda+\mu)Y_t})}{\mu}\right) = \mathcal{\hat{G}}_{a,b,\sigma}(y, \lambda)$
OPTION PRICES:

• Guaranteed endowment call options of payoff $$(1 - Ke^{Y_T})^+$$

$$E(e^{-Y_T} - K)^+ = G_{a,b,\sigma}(-\ln K, 1) - KG_{a,b,\sigma}(-\ln K, 0)$$

• Cash binary Asian floor of payoff $$1_{\{Y_T \leq K\}}$$

$$CBA_f(K, T) = G_{a,b,\sigma}(K, 1)$$

• Rate binary Asian floor of payoff $$Y_T1_{\{Y_T \leq K\}}$$

$$RBA_f(K, T) = \wedge G_{a,b,\sigma}(K, 1)$$

• Regular Asian Asian floor of payoff $$(Y_T - K)1_{\{Y_T \leq K\}}$$

$$AO_f(K, T) = K.CBA(K, T) - RBA(K, T)$$

• Call options from Call-Put parity
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**Numerical Examples**

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Type</th>
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<th>Strike 0.10</th>
<th>Strike 0.11</th>
<th>Strike 0.12</th>
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Table 1: Evolution with $T$, Chacko and Das parameters.
### Numerical Examples

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Table 2: Evolution of the speed of convergence with T.
**NUMERICAL EXAMPLES**

Evolution of the Abate and Whitt inverse with $A$

![Graph showing the evolution of the Abate and Whitt inverse with $A$.]

Figure 1: *Parameters: $r_0 = 0.1$, $a = 0.15$, $b = 1.5$, $\sigma = 0.2$ and $T = 10$*

- **Abate-Whitt**

\[
F_{AB}^{AB}(t) = \frac{e^{\frac{A}{2t}}}{t} \text{Re}(f(\frac{A}{2t})) + \frac{e^{\frac{A}{2t}}}{t} \sum_{k=1}^{\infty} \text{Re}(f(\frac{A + 2k\pi i}{2t}))
\]
CONCLUSION

• Explicit analytical formulae
  ➔ Systematic implementation
  ➔ Regions of high maturity and volatility

• Square-root equity Asian options
  ➔ As alternative equity model
  ➔ For benchmarking, cross-testing

• Importance of square-root process in Financial modelling ➔ various applications