Pricing
Asian options
in
the CIR model

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OUTLINE

I. Introduction

II. Distributional results

III. Option pricing

IV. Numerical examples

V. Conclusion

Introduction: Different kinds of Asian options

- Type of average: $T_0 < T_1 < \ldots < T_n = T$
  - ➔ Discrete arithmetic: $Y_T = \frac{1}{N+1} \sum_{n=0}^{N} X_{T_n}$
  - ➔ Discrete geometric: $Y_T = \left(\prod_{n=0}^{N} X_{T_n} \right)^{\frac{1}{N+1}}$
  - ➔ Continuous arithmetic: $Y_T = \frac{1}{T-T_0} \int_{T_0}^{T} X_u du$
  - ➔ Continuous geometric: $Y_T = e^{\frac{1}{T-T_0} \int_{T_0}^{T} \ln(X_u) du}$

- Type of strike:
  - ➔ Fixed: $\text{payoff} = (Y_T - K)^+$
  - ➔ Floating: $\text{payoff} = (Y_T - S_T)^+$

- Average starting time:
  - ➔ Forward-starting: $T_0 > t$
  - ➔ Starting: $T_0 = t$
  - ➔ Backward-started: $T_0 < t$

INTRODUCTION:
LITERATURE REVIEW FOR ASIAN OPTIONS

- Different approaches:

  - ➔ (Semi-)Analytical approaches and Laplace transform analysis
  - ➔ Pseudo-analytical approximations
  - ➔ Numerical PDE solutions
  - ➔ Monte Carlo simulation
  - ➔ Miscellaneous numerical methods

INTRODUCTION: NUMERICAL\ANALYTICAL LAPLACE TRANSFORM INVERSION

- "(Numerical) Laplace transform inversion is still more an art than a science" Davies and Martin (79).

- Numerical inversion methods:
  - ➜ Plethora of algorithms
  - ➜ Most of those have a number of free-parameters.

- Analytical inversion:
  - ➜ Generic technique: the Bromwich integral

$$F(x) = \mathcal{L}^{-1}(L(\mu)) = \lim_{R\to\infty} \frac{1}{2i\pi} \int_{c-iR}^{c+iR} L(\mu) e^{x\mu} d\mu \quad (1)$$

    c at the right of all singularities.

  - ➜ Specific relation with known inverses.

INTRODUCTION:
THE SQUARE-ROOT PROCESS

- Unique solution to the SDE:

$$dX_t = (a - bX_t)dt + \sigma\sqrt{X_t}dW_t \tag{2}$$

with the constraints: $a \geq 0$, $b \in \mathbb{R}$ and $\sigma > 0$.

- The two most common forms in Finance:

➔ Mean-reverting form with $a > 0$ and $b > 0$:

→ Mean-Reversion level: $\frac{a}{b}$

→ Mean: $E(X_t) = \frac{a}{b} + (X_0 - \frac{a}{b})e^{-bt}$

➔ Exploding form with $a = 0$ and $b \leq 0$:

$$dS_t = rS_t dt + \sigma\sqrt{S_t}dW_t \tag{3}$$

→ $r = -b$

→ Mean: $E(S_t) = S_0 e^{rt}$

→ Probability of reaching the origin in finite time: $P(\tau_0 < \infty) \in [0, 1]$

DISTRIBUTIONAL RESULTS: IMPORTANCE OF THE ADDITIVITY PROPERTY

- The additivity property

$$\begin{aligned} &\mathsf{SR}(a^1, b, \sigma, x_0^1) + \mathsf{SR}(a^2, b, \sigma, x_0^2) \\ &\sim \mathsf{SR}(a^1 + a^2, b, \sigma, x_0^1 + x_0^2) \end{aligned} \tag{4}$$

- Application: functional form of the MGF

➔ Definition

$$\mathcal{L}^{a,x_0,b,\sigma}(\lambda, \mu) = E(e^{-\lambda X_t - \mu Y_t}),\ \lambda \geq 0,\ \mu \geq 0 \tag{5}$$

$$Y_t = \int_0^t X_s ds \tag{6}$$

➔ Additivity implies

$$\mathcal{L}^{a,x_0}(\lambda, \mu) = \mathcal{L}^{0,x_0}(\lambda, \mu)\mathcal{L}^{a,0}(\lambda, \mu)$$

➔ Scaling implies

$$\mathcal{L}^{a,x_0}(\lambda, \mu) = e^{-x_0\psi}e^{-a\phi}$$

DISTRIBUTIONAL RESULTS: JOINT MOMENT GENERATING FUNCTION

- Notation

$$\gamma = \sqrt{b^2 + 2\sigma^2\mu} \tag{7}$$

- The joint MGF of (X_t, Y_t) is given by (See Lamberton-Lapeyre):

$$\mathcal{L}(\lambda, \mu) = E(e^{-\lambda X_t - \mu Y_t}) = e^{-X_0\psi(t) - a\phi(t)} \tag{8}$$

$$\begin{cases} \psi(t) \;=\; \frac{\lambda((\gamma-b)+e^{-\gamma t}(\gamma+b))+2\mu(1-e^{-\gamma t})}{\sigma^2\lambda(1-e^{-\gamma t})+(\gamma+b)+e^{-\gamma t}(\gamma-b)} \\ \phi(t) \;=\; \frac{-2}{\sigma^2}\ln\Big(\frac{2\gamma e^{\frac{(b-\gamma)t}{2}}}{\sigma^2\lambda(1-e^{-\gamma t})+(\gamma+b)+e^{-\gamma t}(\gamma-b)}\Big) \end{cases} \tag{9}$$

DISTRIBUTIONAL RESULTS: JOINT MOMENTS

- System of ODE

$$\frac{dE(Y_t^m X_t^k)}{dt} = mE(Y_t^{m-1} X_t^{k+1}) + akE(Y_t^m X_t^{k-1}) \\ -bkE(Y_t^m X_t^k) + k(k-1)\frac{\sigma^2}{2} E(Y_t^m X_t^{k-1}) \qquad (10)$$

- Laplace transform

$$\widehat{M}_{m,n} = \sum_{j=0}^{n} \sum_{i=1}^{I_j^{m,n}} \frac{\alpha_{j,i}^{m,n}}{(\zeta + jb)^i} \qquad (11)$$

- Explicit simple form

$$M_{m,n}(t) = E(Y_t^m X_t^{n-m}) = \sum_{j=0}^{n} e^{-jbt} \Big(\sum_{i=1}^{I_j^{m,n}} \alpha_{j,i}^{m,n} \frac{t^{i-1}}{(i-1)!} \Big) \qquad (12)$$

$$I_j^{m,n} = \min(n+1-j, m+1) \qquad (13)$$

DISTRIBUTIONAL RESULTS: MOMENTS-BASED EXPANSIONS FOR THE INTEGRATED PROCESS

- Joint distribution determined by its moments
- Edgeworth-type expansions
- Laguerre expansions: Dufresne (00)

$$\frac{1}{2}[f(y^+) + f(y^-)] = y^b e^{-dx} \sum_{n=0}^{\infty} c_n L_n^a(x) \tag{14}$$

under some regularity and integrability conditions,

where

$$L_n^a(x) = \sum_{m=0}^{n} (-1)^m \binom{n+a}{n-m} \frac{x^m}{m!} \tag{15}$$

DISTRIBUTIONAL RESULTS: A SIMPLIFYING CHANGE OF MEASURE

• The following process

$$L(t) = e^{\frac{b^2 Y_t}{2\sigma^2} + \frac{b(X_t - X_0)}{\sigma^2} - \frac{abt}{\sigma^2}} \tag{16}$$

is an exponential martingale and $E(L(T)) = 1$.

• Defining T the time horizon and Q^* the measure constructed with the Radon-Nykodim derivative

$$\frac{dQ^*}{dQ} = L(T) \tag{17}$$

the process X_t is a **time-changed squared Bessel process** under Q^*

$$dX_t = adt + \sigma\sqrt{X_t} dW_t^* \tag{18}$$

and has the simplified MGF

$$\mathcal{L}^*(\lambda, \mu) = e^{-x_0 \frac{\lambda\gamma(1+e^{-\gamma t})+2\mu(1-e^{-\gamma t})}{\sigma^2\lambda(1-e^{-\gamma t})+\gamma(1+e^{-\gamma t})}} \tag{19}$$

DISTRIBUTIONAL RESULTS: JOINT DENSITY FOR THE EQUITY CASE

- The non-absorbed part

➔ Noting:

→ $\alpha_n = \frac{s+s_0+(n+1)\sigma^2 t}{2}$

→ $B_n(y)$: a sequence of functions defined by

$$\sum_{p=0}^{n} \frac{\binom{n+1}{n-p}}{p!}\Big(\frac{-s}{\sqrt{2y\alpha}}\Big)^p \sum_{q=0}^{n} \frac{\binom{n+1}{n-q}}{q!}\Big(\frac{-s_0}{\sqrt{2y\alpha}}\Big)^q He_{p+q+3}\Big(\frac{\alpha_n}{\sqrt{2y\alpha}}\Big) e^{-\frac{\alpha_n^2}{4y\alpha}} \quad (20)$$

where

$$He_k(x) = \sum_{s=0}^{\left[\frac{k}{2}\right]} (-1)^s \frac{x^{k-2s}}{2^s} \frac{k!}{(k-2s)!s!} \quad (21)$$

➔ Joint density of (S_t, Y_t) under Q for $X_t > 0$

$$f^{S,Y}(s,y) = \frac{s_0 \alpha}{2\sqrt{2\pi}} \frac{1}{(y\alpha)^2} e^{-\frac{r^2 y}{2\sigma^2} + \frac{r(s-s_0)}{\sigma^2}} \sum_{n=0}^{\infty} \frac{B_n(y)}{n+1} \quad (22)$$

- Leading term

$$e^{-\frac{\sigma^2 t^2}{2y} n^2} \quad (23)$$

DISTRIBUTIONAL RESULTS: INTEREST-RATE CASE

- The marginal density of the integral Y_t

$$f^Y(y) = \sum_{k=0}^{\infty} f_k^Y(y)$$

- Involves

➔ Hermite polynomial $He_k(x)$

➔ Complementary error function $erfc(x)$

- Useful quantities

➔ $\mathcal{L}^{-1}(\frac{E(e^{-(\lambda+\mu)Y_t})}{\mu}) = \mathcal{G}_{a,b,\sigma}(y, \lambda)$

➔ $\mathcal{L}^{-1}(\frac{E(Y_t e^{-(\lambda+\mu)Y_t})}{\mu}) = \hat{\mathcal{G}}_{a,b,\sigma}(y, \lambda)$

OPTION PRICES:

- Guaranteed endowment call options of payoff $(1 - Ke^{Y_T})^+$

$$E(e^{-Y_T} - K)^+ = \mathcal{G}_{a,b,\sigma}(-\ln K, 1) - K\mathcal{G}_{a,b,\sigma}(-\ln K, 0)$$

- Cash binary Asian floor of payoff $1_{\{Y_T \leq K\}}$

$$\text{CBA}^f(K, T) = \mathcal{G}_{a,b,\sigma}(K, 1)$$

- Rate binary Asian floor of payoff $Y_T 1_{\{Y_T \leq K\}}$

$$\text{RBA}^f(K, T) = \hat{\mathcal{G}}_{a,b,\sigma}(K, 1)$$

- Regular Asian Asian floor of payoff $(Y_T - K)1_{\{Y_T \leq K\}}$

$$\text{AO}^f(K, T) = K.\text{CBA}(K, T) - \text{RBA}(K, T)$$

- Call options from Call-Put parity

NUMERICAL EXAMPLES

Maturity	Type	Strike 0.08	0.09	0.10	0.11	0.12
0.1	P	0.1061	0.2936	0.5337	0.7427	0.8790
	CBA^c	0.9631	0.8104	0.4812	0.1763	0.0387
	TM^c	0.0098	0.0085	0.0053	0.0021	0.0006
	RBA^c	0.0097	0.0084	0.0052	0.0021	0.0005
	AO^c	0.0199	0.0109	0.0043	0.0011	0.0002
0.5	P	0.1549	0.3261	0.5280	0.7107	0.8441
	CBA^c	0.8018	0.6377	0.4452	0.2718	0.1459
	TM^c	0.0444	0.0285	0.0275	0.0180	0.0103
	RBA^c	0.0421	0.0351	0.0260	0.0169	0.0097
	AO^c	0.0201	0.0128	0.0074	0.0039	0.0018
1	P	0.1878	0.3535	0.5354	0.6979	0.8209
	CBA^c	0.7301	0.5779	0.4125	0.2662	0.1565
	TM^c	0.0867	0.0726	0.0553	0.0383	0.0242
	RBA^c	0.0777	0.0647	0.0490	0.0337	0.0211
	AO^c	0.0193	0.0127	0.0078	0.0044	0.0023
2	P	0.1789	0.3509	0.5395	0.7050	0.8276
	CBA^c	0.6644	0.5193	0.3633	0.2291	0.1317
	TM^c	0.1744	0.1451	0.1093	0.0746	0.0465
	RBA^c	0.1402	0.1155	0.0859	0.0578	0.0354
	AO^c	0.0170	0.0110	0.0066	0.0037	0.0019

Table 1: Evolution with T, Chacko and Das parameters.

NUMERICAL EXAMPLES

Type	T	.08	.09	.10	.11	.12	T	.08	.09	.10	.11	.12
$\mathcal{G}_0$		50	51	54	57	59		16	19	19	21	21
$\mathcal{G}_1$		49	51	53	56	53		16	19	19	21	21
$\hat{\mathcal{G}}_0$	$.1$	49	51	53	57	53	$.5$	16	19	19	21	21
$\hat{\mathcal{G}}_1$		50	51	54	57	59		16	20	19	21	21
$\mathcal{G}_0$		15	14	12	14	14		11	10	11	10	9
$\mathcal{G}_1$		15	14	12	14	14		11	10	11	10	9
$\hat{\mathcal{G}}_0$	1	15	14	12	15	14	2	11	10	11	10	9
$\hat{\mathcal{G}}_1$		15	14	12	15	14		11	10	11	10	9
$\mathcal{G}_0$		9	7	8	7	8		8	6	6	6	6
$\mathcal{G}_1$		9	3	8	7	6		8	6	6	6	5
$\hat{\mathcal{G}}_0$	5	9	7	8	7	8	10	8	6	6	6	6
$\hat{\mathcal{G}}_1$		9	7	8	7	6		8	6	6	6	6

Table 2: Evolution of the speed of convergence with T.

NUMERICAL EXAMPLES

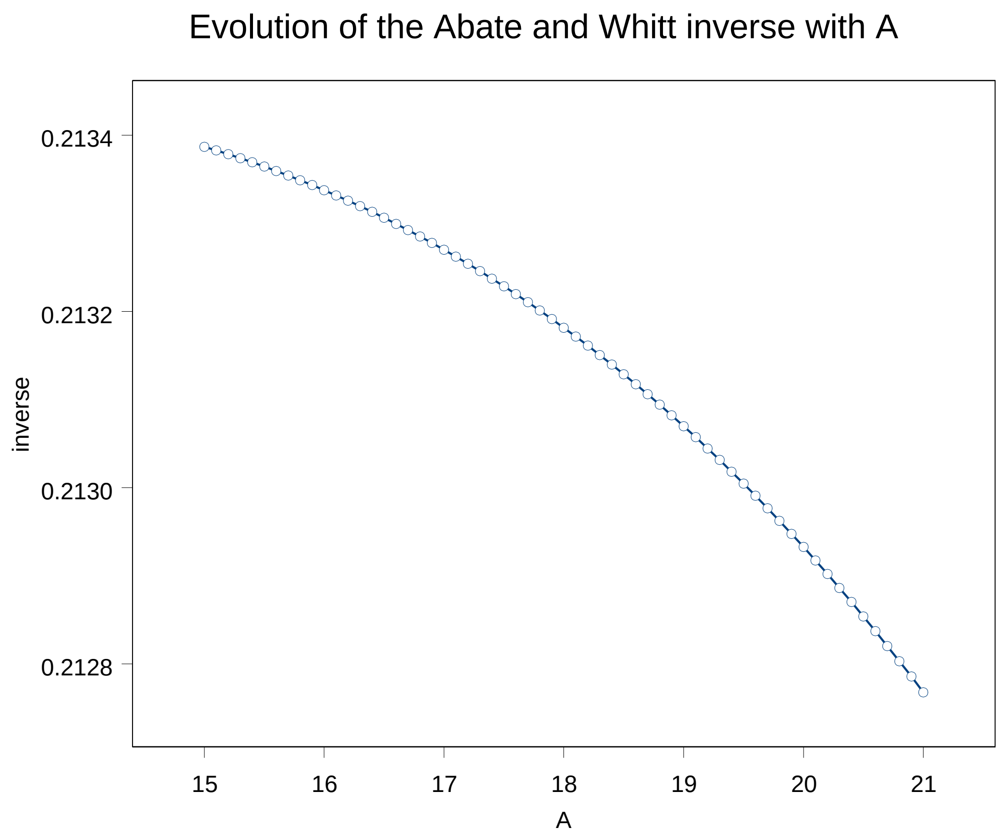

Figure 1: *Parameters:* $r_0 = 0.1$, $a = 0.15$, $b = 1.5$, $\sigma = 0.2$ *and* $T = 10$

- Abate-Whitt

$$F^{\text{AB}}(t) = \frac{e^{\frac{A}{2}}}{2t}\text{Re}(f(\frac{A}{2t})) + \frac{e^{\frac{A}{2}}}{t}\sum_{k=1}^{\infty}\text{Re}(f(\frac{A+2ki\pi}{2t}))$$

CONCLUSION

- Explicit analytical formulae
  - ➔ Systematic implementation
  - ➔ Regions of high maturity and volatility

- Square-root equity Asian options
  - ➔ As alternative equity model
  - ➔ For benchmarking, cross-testing

- Importance of square-root process in Financial modelling $\Longrightarrow$ various applications