Exercise 7

1. Suppose $X_1, \cdots, X_n$ are a random sample from Cauchy distribution with the density function
$$f(x, \theta) = \frac{1}{\pi}\{1 + (x - \theta)^2\}.$$
Find the equation of which the MLE of θ is the solution. Write down the Newton-Raphson algorithm for calculating the MLE numerically. How would you choose an initial value for the algorithm?

2. Let $X_1, \cdots, X_n$ be a random sample from an exponential distribution with the density function $f(x, \lambda) = \lambda e^{-\lambda x} I(x \geq 0)$, where $\lambda > 0$ is an unknown parameter. Suppose the last $n - m$ $(m < n)$ observations are *censored* in the sense that we only known $X_j \in [u_j, v_j]$ for some given $v_j > u_j > 0$ while X_j itself is unknown, $j = m+1, \cdots, n$. Outline an EM algorithm for estimating the parameter $\lambda > 0$.

3. Let $X_1, \cdots, X_n$ be a random sample from Bernoulli distribution, i.e. $P(X_1 = 1) = p = 1 - P(X_1 = 0)$, where $p \in (0, 1)$ is unknown. Let $\theta = p^2$.
   (a) Find the Cramér-Rao lower bound for the variance of unbiased estimators for θ.
   (b) Find the MLE $\widehat{\theta}$ for the parameter θ.
   (c) Show that $E(\widehat{\theta}) \neq \theta$.
   (d) Outline a bootstrap procedure for estimating the bias of $\widehat{\theta}$.

4. Given a random sample of n observations from the N(0,θ) distribution, construct a most powerful test of the hypothesis $H_0 : \theta = 1$ against $H_1 : \theta = 2$. Find the power of the test of the size 0.05 when $n = 10$.

5. Let $(X_1, \cdots, X_n)$ be a random sample from $N(0, \sigma^2)$. Find the UMPT for testing hypotheses $H_0 : \sigma^2 \leq \sigma_0^2$ against $H_1 : \sigma^2 > \sigma_0^2$, where $\sigma_0^2 > 0$ is a given constant.

