BESSEL PROCESSES AND A FUNCTIONAL OF BROWNIAN MOTION[*]

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Abstract

The goal of this paper is to give a concise account of the connection between Bessel processes and the integral of geometric Brownian motion. The latter appears in the pricing of Asian options. Bessel processes are defined and some of their properties are given. The known expressions for the probability density function of the integral of geometric Brownian motion are stated, and other related results are given, in particular the Geman & Yor (1993) Laplace transform for Asian option prices.

1. Introduction

In several papers, Marc Yor has described and applied the properties of Bessel processes that relate to the integral of geometric Brownian motion (called "IGBM" in the sequel). The goal of this paper is to give a concise account of the connection between Bessel processes and IGBM. From the point of view of financial mathematics, the main motivation for the study of IGBM is the pricing of Asian options. Some of the most important results about pricing of Asian options will be given, in particular the Geman-Yor formula for the Laplace transform of Asian option prices and the four known expressions for the probability density function ("PDF" in the sequel) of IGBM. Specialists will not find much that is new, with the possible exception of a somewhat different derivation of the law of IGBM sampled at an exponential time (Section 3). Also, a known result is stated and proved again, namely that the Geman-Yor transform for Asian options is valid for all drifts, not only non-negative ones; this is in response to Carr & Schröder (2003), who noticed that the proof in Geman & Yor (1993) covered only non-negative drifts, and extended the original proof to all drifts. Section 3 shows that there is a much simpler approach, which had already been pointed out (perhaps too briefly) in Dufresne (2000, Section 1), and also in Donati-Martin *et al.* (2001).

Section 2 defines Bessel processes and gives some of their properties. Section 3 describes the relation between IGBM and Asian options, and then shows how time-reversal can be applied to IGBM. A first expression for the PDF of IGBM is given. This expression goes back to Wong (1964), but was apparently unknown in financial mathematics until quite recently. Section 4 uses the link between geometric Brownian motion and Bessel processes (Lamperti's relation) to derive the law of IGBM at an independent exponential time. Some applications of this result are mentioned: (i) a second expression for the PDF of IGBM; (ii) a relationship between IGBM with opposite drifts; (iii) the Geman-Yor formula for the Laplace transform of Asian option prices; (iv) an extension of Pitman's $2M - X$ theorem. In Section 5 formulas for the reciprocal moments of IGBM and two other expressions for the PDF of IGBM are given.

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Vocabulary and notation

"Brownian motion" means "standard Brownian motion" (*i.e.* drift 0, variance t in one dimension; vector of independent one-dimensional standard Brownian motions in higher dimensions). Equality of probability distributions is denoted "$\overset{\mathrm{d}}{=}$".

a.s.	almost surely (same as "with probability one")
A_t	$= A_t^{(0)} = \int_0^t e^{2B_s}\, ds$
$A_t^{(\nu)}$	$= \int_0^t e^{2(\nu t + B_s)}\, ds$
B_t	Brownian motion
B_t^i	i-th component of vector of Brownian motions
$B_t^{(\nu)}$	$= \nu t + B_t$: Brownian motion with drift ν
$BES^\delta(x)$	δ-dimensional Bessel process starting from x
$BES^{(\nu)}(x)$	Bessel process of index ν starting from x (same as $BES^\delta(x)$ if $\nu = \frac{\delta}{2} - 1$)
$BESQ^\delta(x)$	δ-dimensional squared Bessel process starting from x
$BESQ^{(\nu)}(x)$	squared Bessel process of index ν starting from x (same as $BESQ^\delta(x)$ if $\nu = \frac{\delta}{2} - 1$)
BM	1-dimensional Brownian motion starting from 0
BM^δ	δ-dimensional Brownian motion starting from 0
$BM^\delta(x)$	δ-dimensional Brownian motion starting from x
$f_t^\delta(x,y)$	transition density function of $BESQ^\delta(x)$
${}_1F_1(a,b;z)$	$= \sum_{n=0}^\infty \dfrac{(a)_n}{(b)_n} \dfrac{z^n}{n!}$: confluent hypergeometric function
${}_2F_1(a,b,c;z)$	$= \sum_{n=0}^\infty \dfrac{(a)_n (b)_n}{(c)_n} \dfrac{z^n}{n!}$: (Gauss) hypergeometric function
$g_\nu(t,x)$	PDF of $A_t^{(\nu)}$
$H_\nu(z)$	$= \dfrac{2^\nu \Gamma(\frac{1}{2})}{\Gamma\left(\frac{1-\nu}{2}\right)} {}_1F_1\left(-\frac{\nu}{2}, \frac{1}{2}; z^2\right) + \dfrac{2^\nu \Gamma(-\frac{1}{2})}{\Gamma\left(-\frac{\nu}{2}\right)} z\, {}_1F_1\left(\frac{1-\nu}{2}, \frac{3}{2}; z^2\right)$: Hermite function
$I_\nu(z)$	$= \sum_{n=0}^\infty \dfrac{(z/2)^{\nu+2n}}{n!\,\Gamma(n+\nu+1)}$: modified Bessel function of the first kind of order ν
$K_\nu(z)$	$= \dfrac{\pi}{2} \dfrac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu z}$: Macdonald's function of order ν
$L_n^a(z)$	Laguerre polynomial
$p_t^{(\nu)}(x,y)$	transition density function of Bessel process of index ν starting from x
S_λ	exponential random variable, with PDF $\lambda e^{-\lambda t} \mathbf{1}_{\{t>0\}}$
T_u	t such that $A_t^{(\nu)} = u$
$W_{a,b}(z)$	Whittaker function
ρ_t	a Bessel process
ρ_t^2	squared Bessel process

2. Bessel processes: definition, some fundamental results

The quadratic variation of a stochastic process is used below. For a continuous process X which can be expressed as $C + M$, where C has bounded variation and M is a local martingale (X is then a continuous semimartingale), the quadratic variation of X, also called the "bracket of X", is

$$\langle X, X \rangle_t \;=\; \lim_{\max_i |t_{i+1}-t_i| \to 0} \sum_{i=0}^{n-1} (M_{t_{i+1}} - M_{t_i})^2,$$

where $0 = t_0 < t_1 < \cdots < t_n = t$. The quadratic covariation (or "bracket") of two processes X and Y is defined as

$$\langle X, Y \rangle_t \;=\; \frac{1}{4}(\langle X + Y, X + Y \rangle_t - \langle X - Y, X - Y \rangle_t).$$

This definition implies that $\langle \cdot, \cdot \rangle_t$ is symmetric and linear in each argument:

$$\langle X, Y \rangle \;=\; \langle Y, X \rangle, \quad \langle \alpha_1 X^{(1)} + \alpha_2 X^{(2)}, Y \rangle_t \;=\; \alpha_1 \langle X^{(1)}, Y \rangle_t + \alpha_2 \langle X^{(2)}, Y \rangle_t$$

for all $\alpha_1, \alpha_2 \in \mathbb{R}$. If X and Y have Itô differentials

$$dX \;=\; a_X \, dt + \sigma_X \, dB, \quad dY \;=\; a_Y \, dt + \sigma_Y \, dB,$$

which is the same as

$$X_t \;=\; X_0 + \int_0^t a_X(s) \, ds + \int_0^t \sigma_X(s) \, dB_s, \quad Y_t \;=\; Y_0 + \int_0^t a_Y(s) \, ds + \int_0^t \sigma_Y(s) \, dB_s,$$

then

$$\langle X, Y \rangle_t \;=\; \int_0^t \sigma_X(s)\sigma_Y(s) \, ds.$$

The definition of quadratic covariation given above implies that the bracket of two independent Brownian motions is 0. (For more details on quadratic variation, see Revuz & Yor (1999, pp.120-128) and Protter (1990, pp.58, 98).)

We will also use Itô's formula: if $X^{(1)}, \ldots, X^{(n)}$ have Itô differentials, and ifthe function $F(X^{(1)}, \ldots, X^{(n)})$ is a function with continuous second-order partial derivatives in an open set which comprises all the possible values of $X^{(1)}, \ldots, X^{(n)}$, then

$$dF(X^{(1)}, \ldots, X^{(n)}) \;=\; \sum_{i=1}^{n} F_{x_i} \, dX^{(i)} + \frac{1}{2} \sum_{i,j=1}^{n} F_{x_i x_j} d\langle X^{(i)}, X^{(j)} \rangle_t.$$

The following facts regarding Bessel processes are taken from Revuz & Yor (1999, Chapter XI). We first define the squared Bessel processes for integer dimensions. We write $X = \{X_t; t \geq 0\} \sim BM^\delta(x)$ if X is δ-dimensional Brownian motion starting at $x \in \mathbb{R}^\delta$. Let $B = (B^1, \ldots, B^\delta) \sim BM^\delta(x)$, and define $\rho = |B|$. Then Itô's formula implies

$$\rho_t^2 \;=\; \rho_0^2 + 2 \sum_{i=1}^{\delta} \int_0^t B_s^i \, dB_s^i + \delta t.$$

(Here $\rho_0^2 = |x|^2$.) Next, define a new one-dimensional process β as follows:

$$\beta_t = \sum_{i=1}^{\delta} \int_0^t \left(\frac{B_s^i}{\rho_s} \right) dB_s^i$$

For $\delta \geq 1$, the set $\{s \,:\, \rho_s = 0\}$ has Lebesgue measure 0, and $|B_s^i/\rho_s| \leq 1$, so the division by ρ_s above causes no problem. We now appeal to a classical result due to Lévy:

If X is a continuous local martingale with respect to a filtration $\{\mathcal{F}_t\}$ with $X_0 = 0$ and $\langle X, X \rangle_t = t$ for all $t \geq 0$, then it is a $\{\mathcal{F}_t\}$-Brownian motion.

A simple calculation shows that $\langle \beta, \beta \rangle_t = t$, and thus β is Brownian motion:

$$\langle \beta, \beta \rangle_t = \sum_{i=1}^{t} \int_0^t \frac{(B_s^i)^2}{\rho_s^2} \, ds = \int_0^t \frac{\rho_s^2}{\rho_s^2} \, ds = t.$$

We may thus rewrite the stochastic differential of ρ^2 as

$$\rho_t^2 = \rho_0^2 + 2 \int_0^t \rho_s \, d\beta_s + \delta t, \quad \delta = 1, 2, \ldots$$

The Bessel processes are extended to other $\delta \geq 0$ as follows. For $x \in \mathbb{R}_+$, consider the stochastic differential equation ("SDE" in the sequel)

$$Z_t = x + 2 \int_0^t \sqrt{|Z_s|} \, d\beta_s + \delta t.$$

General theorems ensure that this SDE has a unique strong solution for any $x \geq 0, \delta \geq 0$. There are also comparison theorems which say that, since the solution of the equation when $\delta = x = 0$ is $Z_t = 0$ for all $t \geq 0$, then $Z_t \geq 0$ in all cases where $\delta, x \geq 0$. The solution of this SDE will be said to have "dimension" parameter δ, even when δ is not an integer.

Definition. *Let $x, \delta \geq 0$. The unique strong solution of*

$$Z_t = x + 2 \int_0^t \sqrt{Z_s} \, d\beta_s + \delta t$$

*is called the **squared Bessel process of dimension** δ started at x. The **index** of the process is $\nu = \frac{\delta}{2} - 1$. The notation is either $Z \sim BESQ^\delta(x)$ or $Z \sim BESQ^{(\nu)}(x)$. The probability law of this process on the set of continuous functions will be denoted Q_x^δ or $\mathsf{Q}_x^{(\nu)}$.*

(*N.B.* The meaning of the index of a Bessel process will soon become clear.) It is convenient to write $\mathsf{Q}_x^\delta(F(X))$ for $\mathsf{E}F(\rho^2)$, when $\rho^2 \sim BESQ^\delta(x)$. (Here X_t represents the t-coordinate of a sample path of ρ^2, and F is a functional of the process, for example

$$F(X) = X_{t_1} + X_{t_2}^2 \qquad \text{or} \qquad F(X) = \int_0^1 g(X_t) \, dt.)$$

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Additivity property of the squared Bessel process laws. When δ_1, δ_2 are integers, it is easy to see that the sum of two independent $BESQ^{\delta_i}(x_i), i = 1, 2$ must be $BESQ^{\delta_1+\delta_2}(x_1+x_2)$: simply represent the processes as

$$B_1^2 + \cdots + B_{\delta_1}^2 \quad \text{and} \quad B_{\delta_{i+1}}^2 + \cdots + B_{\delta_1+\delta_2}^2,$$

respectively, where the $\{B_1, \ldots, B_{\delta_1+\delta_2}\}$ are independent one-dimensional Brownian motions. Many proofs of properties of Bessel processes become much simpler by using the fact that this additivity property remains valid when the dimensions of the Bessel processes are not both integers. The proof is as follows. Suppose $\beta^{(1)}, \beta^{(2)}$ are two independent Brownian motions, and that

$$dX_t^{(i)} = 2\sqrt{X_t^{(i)}} \, d\beta_t^{(i)} + \delta_i \, dt, \qquad \delta_i, X_0^{(i)} \geq 0, \qquad i = 1, 2.$$

The processes $X^{(1)}, X^{(2)}$ are then independent. Define $X = X^{(1)} + X^{(2)}$. Let $\beta^{(3)}$ be a third independent Brownian motion and define a new process

$$\beta_t = \int_0^t \mathbf{1}_{\{X_s>0\}} \sqrt{\frac{X_s^{(1)}}{X_s}} \, d\beta_s^{(1)} + \int_0^t \mathbf{1}_{\{X_s>0\}} \sqrt{\frac{X_s^{(2)}}{X_s}} \, d\beta_s^{(2)} + \int_0^t \mathbf{1}_{\{X_s=0\}} \, d\beta_s^{(3)}.$$

Then $\langle \beta, \beta \rangle_t = t$, β is Brownian motion and

$$dX_t = 2\sqrt{X_t} \, d\beta_t + (\delta_1 + \delta_2) \, dt,$$

which mean that $X \sim BESQ^{\delta_1+\delta_2}(x_1 + x_2)$. This may be written more succinctly as

$$\mathsf{Q}_{x_1}^{\delta_1} * \mathsf{Q}_{x_2}^{\delta_1} = \mathsf{Q}_{x_1+x_2}^{\delta_1+\delta_2}, \qquad \delta_i, x_i \geq 0, \qquad i = 1, 2.$$

Scaling property of **BESQ**. One-dimensional Brownian motion has the familiar scaling property, which says that if $\{B_t; t \geq 0\}$ is Brownian motion, then so is $\{cB_{t/c^2}; t \geq 0\}$. This carries over immediately to the square of Brownian motion, and then to sums of independent squared Brownian motions. The same scaling property also holds for arbitrary $BESQ$ processes: for any $c > 0$,

$$\{X_t; t \geq 0\} \sim BESQ^\delta(x) \quad \text{implies} \quad \{cX_{t/c}; t \geq 0\} \sim BESQ^\delta(cx).$$

The distribution of ρ_t^2. If $\rho^2 \sim BESQ^\delta(x)$, then

$$\mathsf{E}\, e^{-\lambda\rho_t^2} = \mathsf{Q}_x^\delta(e^{-\lambda X_t}) = \phi(x, \delta).$$

The additivity property of squared Bessel processes implies that

$$\phi(x_1 + x_2, \delta_1 + \delta_2) = \phi(x_1, \delta_1)\phi(x_2, \delta_2) \quad \text{for all} \quad x_1, x_2, \delta_1, \delta_2 \geq 0.$$

In particular, this implies $\phi(x_1 + x_2, 0) = \phi(x_1, 0)\phi(x_2, 0)$; given that $\phi(0, 0) = 1$, we find that $\phi(x) = A^x$ for some $A > 0$. The same reasoning leads to $\phi(0, \delta) = C^\delta$ for some $C > 0$. Since $\phi(x, \delta) = \phi(x, 0)\phi(0, \delta)$, we get

$$\phi(x, \delta) = A^x C^\delta.$$

In order to find A and C, suppose $\{B_t; t \geq 0\}$ is $BM^1(\sqrt{x})$ and calculate

$$\phi(x, 1) = \mathsf{E}\, e^{-\lambda B_t^2} = \frac{1}{(1 + 2\lambda t)^{\frac{1}{2}}} e^{-\lambda x/(1+2\lambda t)}.$$

This implies that, in general,

$$\phi(x, \delta) = \frac{1}{(1 + 2\lambda t)^{\frac{\delta}{2}}} e^{-\lambda x/(1+2\lambda t)}.$$

When $x = 0$ and $\delta > 0$, the exponential disappears from this expression, and ρ_t^2 has a gamma distribution with PDF

$$f_t^\delta(0, y) = \frac{y^{\frac{\delta}{2}-1}}{(2t)^{\frac{\delta}{2}} \Gamma(\frac{\delta}{2})} e^{-\frac{y}{2t}} \mathbf{1}_{\{y>0\}}.$$

When $x, \delta > 0$, the exponential factor in the Laplace transform above corresponds to a compound Poisson/Exponential distribution:

$$e^{-\lambda x/(1+2\lambda t)} = \exp\left[\frac{x}{2t}(M(\lambda) - 1)\right], \quad M(\lambda) = \frac{1}{(1 + 2\lambda t)},$$

and thus

$$\mathsf{E}\, e^{-\lambda \rho_t^2} = e^{-\frac{x}{2t}} \sum_{n=0}^{\infty} \frac{x^n}{(2t)^n n!} M(\lambda)^{n+\frac{\delta}{2}}.$$

The PDF of ρ_t^2 can thus be found by summing the densities of the **Gamma**$(n + \frac{\delta}{2}, \frac{1}{2t})$ which appear when inverting this expression term by term:

$$f_t^\delta(x, y) = \sum_{n=0}^{\infty} \frac{x^n y^{\frac{\delta}{2}+n-1}}{n! \Gamma(\frac{\delta}{2} + n)(2t)^{\frac{\delta}{2}+2n}} e^{-(x+y)/2t}.$$

Next, the reason for the name "Bessel" process, as well as for the definition of the "index" of the process are finally given: recall the "modified Bessel function of the first kind of order ν"

$$I_\nu(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{\nu+2n}}{n! \Gamma(n + \nu + 1)}, \qquad \nu, z \in \mathbb{C}.$$

Then

$$f_t^\delta(x, y) = \frac{1}{2t}\left(\frac{y}{x}\right)^{\frac{\nu}{2}} e^{-(x+y)/2t} I_\nu\left(\frac{\sqrt{xy}}{t}\right) \mathbf{1}_{\{y>0\}}, \qquad \nu = \frac{\delta}{2} - 1.$$

The case $x > 0, \delta = 0$ is special, because the distribution of ρ_t^2 is then compound Poisson/Exponential with no independent gamma component added, and so there is a positive probability that $\rho_t^2 = 0$. A slight modification of the above calculation yields the probability mass at the origin and the density of the continuous part of the distribution:

$$\mathsf{P}(\rho_t^2 = 0) = e^{-x/2t}$$

$$\mathsf{P}(\rho_t^2 \in dy) = \frac{1}{2t}\left(\frac{y}{x}\right)^{-\frac{1}{2}} e^{-(x+y)/2t} I_1\left(\frac{\sqrt{xy}}{t}\right) \mathbf{1}_{\{y>0\}}\, dy.$$

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Behaviour of trajectories of squared Bessel processes. We have just seen that there is a positive probability that $\rho_t^2 = 0$ if, and only if, $\delta = 0$. This is related to the following facts about the trajectories of $BESQ^\delta$ (see Revuz & Yor, 1999, Chapter XI, for more details):

(i) for $\delta = 0$, the point $x = 0$ is absorbing (after it reaches 0, the process will stay there forever);

(ii) for $0 < \delta < 2$, the point $x = 0$ is reflecting (the process inmmediately moves away from 0);

(iii) for $\delta \leq 1$, the point $x = 0$ is reached a.s. ;

(iv) for $\delta \geq 2$, the point $x = 0$ is unattainable.

Definition. *For $x \geq 0$, the square root of $BESQ^\delta(x^2)$ is called the Bessel process of dimension δ started at x, denoted $BES^\delta(x)$ or $BES^{(\nu)}(x)$.*

When $x > 0$ and $\delta \geq 2$ the point 0 is unattainable, and we may apply Itô's formula, with $\rho = \sqrt{\rho^2}$, to find that BES^δ satisfies the SDE

$$\rho_t \;=\; x + \beta_t + \frac{\delta - 1}{2} \int_0^t \frac{ds}{\rho_s} \;=\; x + \beta_t + \left(\nu + \frac{1}{2}\right) \int_0^t \frac{ds}{\rho_s}.$$

For $\delta < 2$, the squared Bessel process reaches 0, and thus the conditions needed to apply Itô's formula are not satisfied; see Revuz & Yor (1999, pp.446, 451). In the sequel we will restrict our attention to Bessel processes of order greater than or equal to 2.

3. A functional of Brownian motion: the integral of geometric Brownian motion

Motivation: Asian options. In the Black-Scholes framework, let r be the risk-free rate of interest, $t \in [0, T]$, $B \sim BM$, and

$$dS_t \;=\; \mu S_t\, dt + \sigma S_t\, dB_t.$$

Asian (or average) options have payoffs which involve some average of the prices of of the risky asset S. For instance, an Asian call option would have a payoff such as

$$\left(\frac{1}{N} \sum_{i=1}^N S_{t_i} - K\right)_+ \qquad \text{where} \quad 0 \leq t_1 < t_2 < \cdots < t_N \leq T.$$

The no-arbitrage theory of derivative pricing says that, in this model, the price of a European option is the expected discounted value of the payoff under the risk-free measure. A problem arises with Asian options because the distribution of the sum in this expression does not have a simple form, which implies numerical difficulties in computing the N-fold integral which gives the price of the option. Various approximations exist, but few error bounds are available; even simulation is not very efficient (Vázquez-Abad & Dufresne, 1998; Su & Fu, 2000).

If the average contains enough prices, and if they are computed at evenly spread time points, then the discrete average above is very close to the continuous average

$$\frac{1}{T} \int_0^T S_u\, du \;=\; \frac{S_0}{T} \int_0^T e^{(r - \frac{\sigma^2}{2})u + \sigma \widetilde{B}_u}\, du.$$

(Here $\widetilde{B}$ is BM under the equivalent martingale measure.) There are three parameters in the above integral, r, σ and T. By the scaling property of Brownian motion, we can fix one of the parameters. Marc Yor chose to set $\sigma = 2$ (for reasons which will become apparent later), and thus define

$$A_t^{(\nu)} = \int_0^t e^{2(\nu s + B_s)} \, ds, \qquad B \sim BM.$$

(We will also write $A_t = A_t^{(0)}$.) The conversion rule is

$$\int_0^T e^{\mu s + \sigma B_s} \overset{\mathrm{d}}{=} \frac{4}{\sigma^2} A_t^{(\nu)}, \qquad t = \frac{\sigma^2 T}{4}, \qquad \nu = \frac{2\mu}{\sigma^2}.$$

($N.B.$ In the case where $\mu = r - \sigma^2/2$, it can be seen that the normalized drift ν can be positive or negative, large or small.)

Geman & Yor (1993) appear to have been the first to study the pricing of continuous-average Asian options, though the process $A_t^{(\nu)}$ had been studied before, as the integral of geometric Brownian motion occurs in various contexts.

Moments. The law of $A_t^{(\nu)}$ has all moments finite; this is because

$$t e^{-2t\nu_- + 2\underline{B}_t} \leq A_t^{(\nu)} \leq t e^{2t\nu_+ + 2\overline{B}_t},$$

where $\underline{B}$ and $\overline{B}$ are respectively the running minimum and maximum of B. All moments (positive or negative) of the variables on either sides of the inequality are finite, and so

$$\mathsf{E}(A_t^{(\nu)})^r < \infty \qquad \forall \, r \in \mathbb{R}.$$

Dufresne (1989) and Yor (1992a) independently derived the general expression for the integral moments of $A_t^{(\nu)}$, but it has since been found (Yor, 2001, p.54) that the following formula had appeared in a paper about an astronomical model, see Ramakrishnan (1954): for $n = 1, 2, \ldots,$

$$\mathsf{E}(A_t^{(\nu)})^n = \sum_{k=0}^n b_{n,k} e^{a_k t}, \quad a_k = 2k\nu + 2k^2, \quad b_{n,k} = n! \Big/ \prod_{\substack{j=0 \\ j \neq k}}^n (a_j - a_k).$$

These moments can be found directly or by using time reversal (see below)

Remark. By the scaling property of Brownian motion, the above formula also gives

$$\mathsf{E}\Big(\int_0^T e^{\mu s + \sigma B_s} \, ds \Big)^n, \qquad n = 1, 2 \ldots$$

Time reversal. The following result (Dufresne, 1989) has been extended to Lévy processes by Carmona *et al.* (1997):

For each fixed $t > 0$,

$$Y_t^{(\nu)} \overset{\mathrm{def}}{=} \int_0^t e^{2[(t-s) + B_t - B_s]} \, ds \overset{\mathrm{d}}{=} A_t^{(\nu)}.$$

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To prove this, note that the two continuous processes

$$U \; = \; \{B_s; 0 \le s \le t\}, \qquad V \; = \; \{B_t - B_{t-s}; 0 \le s \le t\}$$

have the same finite-dimensional distributions, and thus have the same law (as random elements of $C[0,t]$). Define a functional $I : C[0,t] \to \mathbb{R}$ by

$$I(x) \; = \; \int_0^t e^{2(\nu s + x_s)} \, ds.$$

Then $I(U)$ and $I(V)$ have the same distribution, which means that

$$\int_0^t e^{2(\nu s + B_s)} \, ds \; \overset{\mathrm{d}}{=} \; \int_0^t e^{2(\nu s + B_t - B_{t-s})} \, ds \; = \; \int_0^t e^{2[(t-u) + B_t - B_u]} \, du.$$

By Itô's formula,
$$dY_t^{(\nu)} \; = \; [2(\nu + 1)Y_t^{(\nu)} + 1]dt + 2Y_t^{(\nu)} \, dB_t.$$

This SDE yields ordinary differential equations for the moments of $Y_t^{(\nu)}$. Moreover, the PDE associated with this diffusion may be solved, to give the eigenfunction expansion for the PDF of $Y_t^{(\nu)}$. This was done by Wong (1964). Monthus and Comtet (1994) write the eigenfunction expansion as

$$g_\nu(t,x) \; = \; e^{-\frac{1}{2x}} \Bigg[\sum_{0 \le n < -\nu/2} e^{2tn(\nu+n)} \frac{(-1)^{n+1}(\nu+2n)}{\Gamma(1-\nu-n)} \left(\frac{1}{2x}\right)^{1-\nu-n} L_n^{-\nu-2n}\left(\frac{1}{2x}\right)$$
$$+ \frac{1}{2\pi^2} \int_0^\infty ds \, e^{-\frac{t}{2}(\nu^2+s^2)} s \sinh(\pi s) \left| \Gamma\left(\frac{\nu+is}{2}\right) \right|^2 \left(\frac{1}{2x}\right)^{\frac{1-\nu}{2}} W_{(1-\nu)/2, is/2}\left(\frac{1}{2x}\right) \Bigg],$$

where $W_{a,b}$ is Whittaker's function. Details about eigenfunction expansions for Asian options may be found in Linetsky (2001).

4. The law of $A_t^{(\nu)}$ at an independent exponential time

One of the classical tools of probability theory is the **resolvent**: given a process $\{X_t; t \ge 0\}$, this is the application which associates to a function f the integral

$$\mathsf{E} \int_0^\infty e^{-\lambda t} f(X_t) \, dt.$$

This is of course a Laplace transform in the time parameter t. A very well known example is Brownian motion starting at x, for which

$$\int_0^\infty e^{-\lambda t} f(B_t) \, dt \; = \; \int_{-\infty}^\infty \frac{1}{\sqrt{2\lambda}} e^{-\sqrt{2\lambda}|y-x|} f(y) \, dy.$$

Here the function $\frac{1}{\sqrt{2\lambda}} e^{-\sqrt{2\lambda}|y-x|}$ is the density of the resolvent. An equivalent way of viewing the resolvent is to think of it as

$$\frac{1}{\lambda} \mathsf{E} \, f(X_{S_\lambda}),$$

where $S_\lambda \sim \textbf{Exponential}(\lambda)$ is independent of $\{X_t; t \geq 0\}$. Finding the density of the resolvent is then equivalent to finding the distribution of X_{S_λ}. From what was said above, the PDF of B_{S_λ} is $(\sqrt{2\lambda}/2)e^{-\sqrt{2\lambda}|y-x|}$, or, said another way, $B_{S_\lambda} - x \sim \textbf{DoubleExponential}(\sqrt{2\lambda})$.

The distribution of $A_{S_\lambda}^{(\nu)}$ was found by Marc Yor. The derivation below is a little different from the original papers (Yor, 1992a, 1992b; Yor, 2001, Chapter 6); other derivations may be found in Donati-Martin $al.$ (2001) and Dufresne (2001b). Here is the main result of this section (Yor, 1992a):

The distribution of $A_{S_\lambda}^{(\nu)}$ is the same as that of $B/2G$, where B and G are independent,
$B \sim \textbf{Beta}(1,\alpha)$, $G \sim \textbf{Gamma}(\beta,1)$, *with*

$$\alpha = \frac{\gamma + \nu}{2}, \qquad \beta = \frac{\gamma - \nu}{2}, \qquad \gamma = \sqrt{2\lambda + \nu^2}.$$

The proof presented below rests on a relationship between geometric Brownian motion and Bessel processes, due to Lamperti (1972) (see Revuz & Yor, 1999, p.452). Two other required results are, first, that the transition density of the $BES^{(\nu)}(x)$ is

$$p_t^{(\nu)}(x,y) = \frac{y}{t}\left(\frac{y}{x}\right)^\nu e^{-(x^2+y^2)/2t} I_\nu\left(\frac{xy}{t}\right) \mathbf{1}_{\{y>0\}}, \qquad x,t > 0, \; \nu \geq 0.$$

and, second, that for $\rho \sim BES^{(0)}(x)$,

$$\mathsf{E}\left(e^{-v^2 \int_0^t \frac{ds}{\rho_s^2}} \Big| \rho_t = r\right) = \frac{I_{|v|}(rx/t)}{I_0(rx/t)} \qquad \forall \, v \in \mathbb{R}.$$

The first result is a direct consequence of the expression previously given for $f_t^\delta(x,y)$, while the second one follows from Girsanov's Theorem (Revuz & Yor, 1999, p.450; Yor, 1992a). Only the case $\nu = 0$ is needed in the sequel.

First, let us state and prove the result due to Lamperti:

Let $B \sim BM$, $\delta \geq 2$ and $\nu = \frac{\delta}{2} - 1$. There exists $\rho \sim BES^\delta(1)$ such that

$$e^{\nu t + B_t} = \rho_{A_t^{(\nu)}}, \qquad A_t^{(\nu)} = \int_0^t e^{2(\nu s + B_s)} \, ds.$$

To prove the existence of such a ρ, observe that the process $A_t^{(\nu)}$ is continuous and strictly increasing in $t \geq 0$, $A_0^{(\nu)} = 0$, and, since $\nu \geq 0$, $\lim_{t \to \infty} A_t^{(\nu)} = \infty$. Hence, for any $u \geq 0$, one may define variable T_u by

$$A_{T_u}^{(\nu)} = u.$$

Differentiating each side with respect to u yields

$$\frac{dT_u}{du} = e^{-2(\nu s + B_s)}\Big|_{s=T_u}.$$

By Itô's formula,

$$e^{\nu t + B_t} = 1 + (\nu + \tfrac{1}{2}) \int_0^t e^{\nu s + B_s} \, ds + M_t, \qquad M_t = \int_0^t e^{\nu s + B_s} \, dB_s$$

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which implies

$$e^{\nu T_u + B_{T_u}} = 1 + (\nu + \tfrac{1}{2}) \int_0^{T_u} e^{\nu s + B_s}\, ds + \beta_u, \qquad \beta_u = M_{T_u}.$$

The process M is a continuous martingale, $\langle M, M \rangle_t = A_t^{(\nu)}$, and $\langle M, M \rangle_\infty = \infty$. We may then use the well-known result, due to Dambis, Dubins and Schwarz (see Revuz & Yor, 1999, p.181), which says that if these conditions hold then M is a time-transformed Brownian motion, as the process $\{\beta_u; u \geq 0\}$ is Brownian motion and $M_t = \beta_{\langle M, M \rangle_t}$ for $t \geq 0$. (The filtrations of the Brownian motions B and β are however different.)

Moreover, from the formula for dT_u / du, we also find

$$\int_0^{T_u} e^{\nu s + B_s}\, ds = \int_0^u e^{\nu T_y + B_{T_y}}\, dT_y = \int_0^u \frac{dy}{e^{\nu T_y + B_{T_y}}}.$$

Hence, if we define $\rho_u = e^{\nu T_u + B_{T_u}}$, then

$$\rho_u = 1 + (\nu + \tfrac{1}{2}) \int_0^u \frac{dy}{\rho_y} + \beta_u,$$

which means that $\rho \sim BES^{(\nu)}(1)$. Lamperti's relation is proved.

Next, let us derive the law of $A_{S_\lambda}^{(\nu)}$.

Step 1. First, find the joint law of $(e^{B_{S_\lambda}^{(\nu)}}, A_{S_\lambda}^{(\nu)})$. The idea is to look for a function $h(\cdot, \cdot)$ such that for any non-negative functions f, g,

$$\mathsf{E}[f(e^{B_{S_\lambda}^{(\nu)}}) g(A_{S_\lambda}^{(\nu)})] = \iint f(x) g(y) h(x, y)\, dx\, dy.$$

The function $h(\cdot, \cdot)$ is then the PDF of $(e^{B_{S_\lambda}^{(\nu)}}, A_{S_\lambda}^{(\nu)})$.

By the Cameron-Martin theorem,

$$\mathsf{E}[f(e^{B_t^{(\nu)}}) g(A_t^{(\nu)})] = e^{-\nu^2 t / 2} \mathsf{E}[e^{\nu B_t} f(e^{B_t}) g(A_t)],$$

and so

$$\mathsf{E}[f(e^{B_{S_\lambda}^{(\nu)}}) g(A_{S_\lambda}^{(\nu)})] = \lambda \mathsf{E} \int_0^\infty e^{-\gamma^2 t / 2} e^{\nu B_t} f(e^{B_t}) g(A_t)\, dt$$

$$= \lambda \mathsf{E} \int_0^\infty e^{-\gamma^2 T_u / 2} e^{\nu B_{T_u}} f(e^{B_{T_u}}) g(A_{T_u})\, dT_u.$$

As we have seen before, $A_{T_u}^{(\nu)} = u$, and $\{\rho_u = e^{B_{T_u}}; u \geq 0\}$ is $BES^{(0)}(1)$, so the last expression is equal to

$$\lambda \mathsf{E} \int_0^\infty e^{-\gamma^2 T_u / 2} \rho_u^\nu f(\rho_u) g(u)\, dT_u = \lambda \mathsf{E} \int_0^\infty e^{-\frac{\gamma^2}{2} \int_0^u \frac{ds}{\rho_s^2}} \rho_u^{\nu - 2} f(\rho_u) g(u)\, du,$$

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where we have used $dT_u/du = e^{-2B_{T_u}} = \rho_u^{-2}$.

Next, by conditioning on ρ_u,

$$
\begin{aligned}
\mathsf{E}\big[e^{-\frac{\gamma^2}{2}\int_0^u \frac{ds}{\rho_s^2}}\rho_u^{\nu-2}f(\rho_u)\big] &= \mathsf{E}\big\{f(\rho_u)\rho_u^{\nu-2}\mathsf{E}\big[e^{-\frac{\gamma^2}{2}\int_0^u \frac{ds}{\rho_s^2}}\big|\rho_u\big]\big\} \\
&= \mathsf{E}\Big[f(\rho_u)\rho^{\nu-2}\frac{I_\gamma(\rho_u/u)}{I_0(\rho_u/u)}\Big] \\
&= \int_0^\infty f(r)r^{\nu-2}\frac{I_\gamma(r/u)}{I_0(r/u)}p_u^{(0)}(1,r)\,dr \\
&= \frac{1}{u}\int_0^\infty f(r)r^{\nu-1}e^{-(1+r^2)/2u}I_\gamma\big(\tfrac{r}{u}\big)\,dr.
\end{aligned}
$$

Hence,

$$
\mathsf{E}[f(e^{B_{S_\lambda}^{(\nu)}})g(A_{S_\lambda}^{(\nu)})] = \int_0^\infty\int_0^\infty f(r)g(u)\frac{\lambda}{u}r^{\nu-1}e^{-(1+r^2)/2u}I_\gamma\big(\tfrac{r}{u}\big)\,dr\,du
$$

and the joint distribution we are seeking is

$$
\begin{aligned}
\mathsf{P}\big(e^{B_{S_\lambda}^{(\nu)}}\in dr, A_{S_\lambda}^{(\nu)}\in du\big) &= \frac{\lambda}{u}r^{\nu-1}e^{-(1+r^2)/2u}I_\gamma\big(\tfrac{r}{u}\big)\,dr\,du\,\mathbf{1}_{\{r,u>0\}} \\
&= \lambda r^{\nu-2-\gamma}p_u^{(\gamma)}(1,r)\,dr\,du\,\mathbf{1}_{\{r>0\}}.
\end{aligned}
$$

Step 2. From the preceding formula, we can find the PDF of $A_{S_\lambda}^{(\nu)}$ by integrating out r. This may be achieved by expressing the Bessel function as a series and then integrating term by term. Thus

$$
\mathsf{P}\big(A_{S_\lambda}^{(\nu)}\in du\big) = \lambda(2u)^{(\nu-\gamma)/2-1}\frac{\Gamma\big(\frac{\nu+\gamma}{2}\big)}{\Gamma(\gamma+1)}e^{-1/2u}\,{}_1F_1\big(\tfrac{\nu+\gamma}{2},\gamma+1,\tfrac{1}{2u}\big).
$$

From the formula $e^{-z}{}_1F_1(a,b;z) = {}_1F_1(b-a,b;-z)$, this can be rewritten as

$$
\mathsf{P}\big(A_{S_\lambda}^{(\nu)}\in du\big) = \lambda(2u)^{(\nu-\gamma)/2-1}\frac{\Gamma\big(\frac{\nu+\gamma}{2}\big)}{\Gamma(\gamma+1)}\,{}_1F_1\big(\tfrac{\gamma-\nu}{2}+1,\gamma+1,-\tfrac{1}{2u}\big).
$$

Now, if $B\sim\mathbf{Beta}(c,d)$ and $G\sim\mathbf{Gamma}(f,1)$ are independent, then, by conditioning on B, the PDF of B/G is seen to be

$$
g(x) = \frac{x^{-f-1}\Gamma(c+d)}{\Gamma(c)\Gamma(d)\Gamma(f)}\int_0^1 e^{-y/x}y^{c+f-1}(1-y)^{d-1}\,dy\,\mathbf{1}_{\{x>0\}}.
$$

From the well-known integral formula for confluent hypergeometric functions (Lebedev, 1972, p.266),

$$
{}_1F_1(a,b;z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 e^{zt}t^{a-1}(1-t)^{b-a-1}\,dt, \quad \mathrm{Re}\,b>\mathrm{Re}\,a>0,
$$

the PDF of B/G can also be expressed as

$$
g(x) = \frac{x^{-f-1}\Gamma(c+d)\Gamma(c+f)}{\Gamma(c)\Gamma(f)\Gamma(c+d+f)}\,{}_1F_1(c+f,c+d+f;-\tfrac{1}{x})\,\mathbf{1}_{\{x>0\}}.
$$

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From this formula, it can be seen that the law of $A_{S_\lambda}^{(\nu)}$ is that of the ratio of a beta variable to an independent gamma variable, with the parameters given in the statement of the theorem.

We now turn to some applications of Yor's theorem.

Application 1: A second expression for the PDF of $A_t^{(\nu)}$. Based on the law of $A_{S_\lambda}^{(\nu)}$, Yor (1992a) found that

$$\mathsf{P}(A_t^{(\nu)} \in du \,|\, B_t + \nu t = x) = a_t(x, u)\, du$$
$$= \frac{\sqrt{2\pi t}}{u}\, e^{\frac{x^2}{2t} - \frac{1}{2u}(1 + e^{2x})} \theta_{e^x/u}(t)\, du,$$

where

$$\theta_r(t) = \frac{r e^{\frac{\pi^2}{2t}}}{\sqrt{2\pi^3 t}} \int_0^\infty \exp(-y^2/2t) \exp(-r \cosh y)(\sinh y) \sin\left(\frac{\pi y}{t}\right) dy.$$

This means that the PDF of $A_t^{(\nu)}$ is the integral of $a_t(\cdot, u)$ times the normal density function with mean νt and variance t. The PDF is thus a double integral, with the function $\sin\left(\frac{\pi y}{t}\right)$ making the computation possibly more difficult when t is small.

Application 2: A relationship between $A_t^{(\nu)}$ and $A_t^{(-\nu)}$. The appearance of beta and gamma distributions in the law of $A_{S_\lambda}^{(\nu)}$ makes it possible to apply the following result (Dufresne, 1998):

If all the variables below are independent, with $B_{r,s} \sim \mathbf{Beta}(r, s)$, $G_r \sim \mathbf{Gamma}(r, 1)$, $a, b, c > 0$, then

$$\frac{G_a}{B_{b,a+c}} + G_c' \stackrel{\mathrm{d}}{=} \frac{G_{a+c}}{B_{b,a}}.$$

This identity implies the following result, but only for fixed t (Dufresne, 2001a); the more general result for processses was proved by Matsumoto & Yor (2001a).

If $G_\nu \sim \mathbf{Gamma}(\nu, 1)$ is independent of $\{A_t^{(\nu)}; t \geq 0\}$, then the proceses below have the same law:

$$\left\{ \frac{1}{2A_t^{(\nu)}} + G_\nu; t \geq 0 \right\} \stackrel{\mathrm{d}}{=} \left\{ \frac{1}{2A_t^{(-\nu)}}; t \geq 0 \right\}.$$

The next result (Dufresne, 1990) was originally obtained by very different means, and has other proofs as well, but it is perhaps easiest to view it as a corollary of the relationship between IGBMs of different drifts: since $A_\infty^{(\nu)} = \infty$ when $\nu \geq 0$, we find

For any $\nu > 0$, $\dfrac{1}{2A_\infty^{(-\nu)}} \sim \mathbf{Gamma}(\nu, 1)$.

N.B. From the scaling property of Brownian motion, this is the same as

$$\frac{2}{\sigma^2} \left(\int_0^\infty e^{-\mu t + \sigma B_t}\, dt \right)^{-1} \sim \mathbf{Gamma}\left(\tfrac{2\mu}{\sigma^2}, 1\right) \qquad \forall\, \mu, \sigma > 0.$$

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Application 3: A formula for Asian options. Geman & Yor (1993) derived the Laplace transform (in the time parameter) for continuous-averaging Asian option prices. The result was first proved directly from the connection between Bessel processes and IGBM and apparently only allowed $\nu \geq 0$, though both sides of the formula could easily be seen to be analytic in ν, which implies that the result holds for all ν. However, a much simpler proof is obtained (Dufresne, 2000, p.409) if we take the law of $A_{S_\lambda}^{(\nu)}$ as given, and this shorter proof works for all ν. (Carr & Schröder (2003) extend the original proof in Geman & Yor (1993) to $\nu \in \mathbb{R}$.) Finally, observe that the Laplace transform below is in fact equivalent to Yor's theorem about the law of $A_{S_\lambda}^{(\nu)}$, because the function

$$k \;\to\; \mathsf{E}(X - k)_+ \;=\; \int_k^\infty \mathsf{P}(X > x)\, dx, \qquad k \geq 0,$$

uniquely determines the distribution of a non-negative variable X. Here is the result by Geman & Yor (1993):

For all $\nu \in \mathbb{R}$, $q > 0$, $\lambda > 2(\nu + 1)$,

$$\int_0^\infty e^{-\lambda t} \mathsf{E}[(A_t^{(\nu)} - q)_+]\, dt \;=\; \frac{(2q)^{1-\beta}}{2\lambda(\alpha + 1)\Gamma(\beta)} \int_0^1 u^{\beta-2}(1 - u)^{\alpha+1} e^{-u/2q}\, du.$$

This formula is obtained by noting that its left hand side equals

$$\frac{1}{2} \int_0^\infty e^{-\lambda t} \mathsf{E}(2A_t^{(\nu)} - 2q)_+\, dt$$

$$= \;\frac{1}{2\lambda} \mathsf{E}(2A_{T_\lambda}^{(\nu)} - 2q)_+ \;=\; \frac{1}{2\lambda} \mathsf{E}\left(\frac{B_{1,\alpha}}{G_\beta} - 2q \right)_+$$

$$= \;-\frac{1}{2\lambda} \int_0^1 \int_0^{u/2q} \left(\frac{u}{x} - 2q \right) \frac{x^{\beta-1}}{\Gamma(\beta)} e^{-x}\, dx\, d_u(1 - u)^\alpha$$

$$= \;-\frac{1}{2\lambda(\alpha + 1)\Gamma(\beta)} \int_0^1 \int_0^{u/2q} x^{\beta-2} e^{-x}\, dx\, d_u(1 - u)^{\alpha+1}.$$

A further integration by parts yields the result.

From the usual integral expression for the confluent hypergeometric function (noted previously), the formula in Theorem 2 is seen to be the same as

$$\int_0^\infty e^{-\lambda t} \mathsf{E}(A_t^\nu - q)_+\, dt$$

$$= \;\frac{(2q)^{1-\beta}\Gamma(\alpha + 1)}{2\lambda(\beta - 1)\Gamma(\alpha + \beta + 1)}\, {}_1F_1\left(\beta - 1, \alpha + \beta + 1, -\frac{1}{2q} \right).$$

The latter formula was pointed out by Donati-Martin *et al.* (2001).

An alternative expression for the Laplace transform of $\mathsf{E}(A_t^\nu - q)_+$ can be found in Schröder (1999); an explicit integral expression for $\mathsf{E}(A_t^\nu - q)_+$ is given in Schröder (2002).

Application 4: An extension of Pitman's $2M - X$ Theorem. The relationship between $A_t^{(\nu)}$ and $A_t^{(-\nu)}$ is one of the tools used to derive a generalization of Pitman's $2M - X$ Theorem (Revuz & Yor, 1999, p.253; Matsumoto & Yor, 1999, 2000, 2001b):

For $\nu \in \mathbb{R}$ and $c > 0$, the process

$$\{c \log \big(\int_0^t e^{\frac{2}{c}B_s^{(\nu)}} \, ds \big) - B_t^{(\nu)} - c \log(c^2); t \geq 0\}$$

is a diffusion process with infinitesimal generator

$$\frac{1}{2}\frac{d^2}{dx^2} + b^{\nu,c}(x)\frac{d}{dx}, \qquad b^{\nu,c}(x) = \frac{d}{dx} \log K_{\nu c}(e^{-x/c}).$$

As $c \to 0+$, the process above tends to $\{2M_t^{(\nu)} - B_t^{(\nu)}; t \geq 0\}$ and $b^{\nu,c}(x)$ tends to $\nu \coth(\nu x)$, where $M^{(\nu)}$ is the runing maximum of $B^{(\nu)}$, which relate to the original results by Pitman and by Rogers and Pitman.

5. Another expression for the PDF of $A_t^{(\nu)}$

The moments $\mathsf{E}(A_t^{(\nu)})^r$ can be found by inverting

$$\mathsf{E}(A_{S_\lambda}^{(\nu)})^r = \mathsf{E}\left(\frac{B_{1,\alpha}}{2G_\beta}\right) = \frac{\Gamma(t+1)\Gamma(\alpha+1)\Gamma(\beta-t)}{2^t\Gamma(\alpha+t+1)\Gamma(\beta)},$$

where $\alpha = \frac{1}{2}(\nu + \sqrt{2\lambda + \nu^2})$ and $\beta = \alpha - \nu$ are as before. This yields (Dufresne, 2000):

For all $\nu \in \mathbb{R}$ and $r > -3/2$,

$$\mathsf{E}(A_t^{(\nu)})^r = \frac{e^{-\nu^2 t/2}}{\sqrt{2\pi t^3}} \int_0^\infty y e^{-y^2/2t} \psi_\nu(r,y) \, dy$$

$$\psi_\nu(r,y) = \frac{\Gamma(1+r)}{\Gamma(2+2r)} e^{-\nu y}(1 - e^{-2y})^{1+2r} {}_2F_1(\nu+1+2r, 1+r, 2+2r; 1-e^{-2y})$$

$$\psi_\nu(-1,y) = \frac{\cosh[(\nu-1)y]}{\sinh(y)}.$$

Schröder (2000) has shown that the integral moments of $1/A_t^{(\nu)}$ may alternatively be expressed as theta integrals: for $n = 1, 2, \ldots$,

$$\mathsf{E}(A_t^{(\nu)})^{-n} = e^{-\nu^2/2} \sum_{k=1}^n a_{n,k}\, T_k(t)$$

with

$$T_k(t) = 2^{-\frac{3}{2}} \int_0^\infty \vartheta(\tfrac{\nu}{2} \,|\, y)\frac{y^{k-1}}{(yt+\frac{1}{2})^{k+\frac{1}{2}}} \, dy,$$

where ϑ is Riemann's theta function.

It is easy to see that $\mathsf{E}\exp(sA_t^{(\nu)}) = \infty$ for all $s > 0$, and it has been proved that the distribution of $A_t^{(\nu)}$ is not determined by its moments (Hörfelt, 2004); this makes sense intuitively,

as the same situation prevails for the lognormal distribution. By contrast, and a little surprisingly, $\mathsf{E}\exp(s/2A_t^{(\nu)}) < \infty$ if $s < 1$ (Dufresne, 2001b). This implies that the distribution of $1/A_t^{(\nu)}$ is determined by its moments. These facts suggest there might be some advantage in concentrating on $1/A_t^{(\nu)}$, rather than on $A_t^{(\nu)}$. It can be shown (Dufresne, 2000) that the PDF of $A_t^{(\nu)}$ can be expressed as a Laguerre series in which the weights are combinations of the moments of $1/A_t^{(\nu)}$: a third expression for the PDF of $A_t^{(\nu)}$ is

$$g_\nu(t,x) \;=\; 2^{-b-1}c^{a+1}x^{-b-2}e^{-c/2x}\sum_{n=0}^{\infty}a_n(t)L_n^a(c/2x),$$

where $a > -1, b \in \mathbb{R}, 0 < c < e^{-\nu_- t}$, and

$$
\begin{aligned}
a_n(t) \;&=\; \frac{n!}{\Gamma(n+a+1)}\mathsf{E}\,L_n^a\left(\frac{c}{2A_t^{(\nu)}}\right)\\
&=\; \sum_{k=0}^{n}\frac{n!(-c)^k}{\Gamma(k+a+1)k!(n-k)!}\mathsf{E}(2A_t^{(\nu)})^{-(a-b+k)}.
\end{aligned}
$$

Similar series are given in Dufresne (2000) for Asian option prices. The Laguerre series are further improved by Schröder (2000).

The hypergeometric PDE. The appearance of hypergeometric functions is not fortuitous, as the next result shows (Dufresne, 2001b):

Let $h^{\nu,r}(s,t) := e^{\nu^2 t/2}\mathsf{E}(A_t^{(\nu)})^{-r}e^{s/2A_t^{(\nu)}}$. Then $h = h^{\nu,r}$ satisfies the PDE

$$-\tfrac{1}{2}h_t \;=\; -\tfrac{1}{4}(\nu-2r)^2 h + [r+(\nu-2r-1)s]h_s + s(1-s)h_{ss},$$

(subscripts indicate partial derivatives).

The right hand side is the Gauss hypergeometric operator. Properties of the law of $A_t^{(\nu)}$ may be obtained from this PDE; we cite two: a relation between $A_t^{(\nu)}$ for different drifts ν, and a fourth expression for the density of $1/A_t^{(\nu)}$.

For all $\mu, r \in \mathbb{R}$, $\mathrm{Re}(s) < 1$, $t > 0$,

$$h^{\mu,r}(s,t) \;=\; (1-s)^{\mu-r}h^{2r-\mu,r}(s,t).$$

(This property includes $1/2A_t^{(\mu)} + G_\mu =^{\mathrm{d}} 1/2A_t^{(-\mu)}$ as a particular case.)

Let $\nu \in \mathbb{R}$, $t, x > 0$ and

$$q(y,t) \;=\; \frac{e^{\frac{\pi^2}{8t}-\frac{y^2}{2t}}}{\pi\sqrt{2t}}\,\cosh y\,.$$

The PDF of $1/(2A_t^{(\nu)})$ is $g_\nu(t,x) \;=\; e^{-\nu^2 t/2}\tilde{g}_\nu(t,x)$ with

$$\tilde{g}_\nu(t,x) = 2^{-\nu}x^{-\frac{\nu+1}{2}}\int_{-\infty}^{\infty}e^{-x\cosh^2 y}q(y,t)\cos\left(\tfrac{\pi}{2}(\tfrac{y}{t}-\nu)\right)H_\nu(\sqrt{x}\sinh y)\,dy.$$

Here H_ν is the Hermite function (a generalization of the Hermite polynomials, see Lebedev (1972), Chapter 10). An interesting aspect of this expression is that it boils down to a single integral when $\nu = 0, 1, 2, \ldots$. Other equivalent expressions are:

(a) If $\nu \neq -1, -3, \ldots$:

$$
\begin{aligned}
\tilde{g}_\nu(t,x) &= 2x^{-\frac{\nu+1}{2}} e^{-x} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \int_0^\infty q(y,t) \cos\left(\frac{\pi y}{2t}\right) \\
&\quad \times \int_0^\infty q(y,t) \cos\left(\frac{\pi y}{2t}\right) {}_1F_1\left(\frac{\nu+1}{2}, \frac{1}{2}; -x \sinh^2 y\right) dy \\
&= 2x^{-\frac{\nu+1}{2}} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \\
&\quad \times \int_0^\infty e^{-x \cosh^2 y} q(y,t) \cos\left(\frac{\pi y}{2t}\right) {}_1F_1\left(-\frac{\nu}{2}, \frac{1}{2}; x \sinh^2 y\right) dy.
\end{aligned}
$$

(b) If $\nu \neq -2, -4, \ldots$:

$$
\begin{aligned}
\tilde{g}_\nu(t,x) &= 2x^{-\frac{\nu}{2}} e^{-x} \frac{\Gamma\left(\frac{\nu}{2} + 1\right)}{\Gamma\left(\frac{3}{2}\right)} \\
&\quad \times \int_0^\infty q(y,t) \sinh y \, \sin\left(\frac{\pi y}{2t}\right) {}_1F_1\left(\frac{\nu}{2} + 1, \frac{3}{2}; -x \sinh^2 y\right) dy \\
&= 2x^{-\frac{\nu}{2}} \frac{\Gamma\left(\frac{\nu}{2} + 1\right)}{\Gamma\left(\frac{3}{2}\right)} \\
&\quad \times \int_0^\infty e^{-x \cosh^2 y} q(y,t) \sinh y \, \sin\left(\frac{\pi y}{2t}\right) {}_1F_1\left(\frac{1-\nu}{2}, \frac{3}{2}; x \sinh^2 y\right) dy.
\end{aligned}
$$

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