THE VALUE OF AN ASIAN OPTION

by

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Abstract: This paper approaches the problem of computing the price of an Asian option in two different ways. Firstly, exploiting a scaling property, we reduce the problem to the problem of solving a parabolic PDE in two variables. Secondly, we provide a lower bound which is so accurate that it is essentially the true price. Both methods can be implemented in real time.

Keywords: Asian option, Brownian motion, fixed strike, floating strike.

1. Introduction

Let us suppose that the price at time t, S_t, of some risky asset is given by

$$S_t = S_0 \exp(\sigma B_t - \frac{1}{2}\sigma^2 t + ct), \tag{1.1}$$

where B is a standard one-dimensional Brownian motion, and c is some constant whose value matters little for now. The problem of computing the value of an Asian (call) option with maturity T and strike price K, written on this risky asset, is mathematically equivalent to calculating

$$\mathbb{E}\Big(Y - K\Big)^+, \tag{1.2}$$

where we define Y by

$$Y \equiv \int_0^T S_u \mu(du) \tag{1.3}$$

and assume that $c = r$, the riskless interest rate. In the case of the 'fixed strike' Asian option, the measure μ is given by $\mu(du) = T^{-1} I_{[0,T]}(u)du$, although other candidates for μ are of interest and can be handled without difficulty. For example, if we take $\mu(u) = \delta_T(du)$, then we have the classical European call option, and if we take $\mu(du) = T^{-1} I_{[0,T]}(u)du - \delta_T(du)$ together with $K = 0$, then we have the 'floating strike' Asian option, whose price at time zero is therefore

$$\mathbb{E}\Big(T^{-1}\int_0^T S_u\, du - S_T\Big)^+. \tag{1.4}$$

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Asian options are commonly traded; they were introduced in part to avoid a problem common to European options, that by manipulating the price of an asset near to the maturity date, speculators could drive up the gains from the option. Despite this, there is not yet (and probably never will be) a simple analytical expression for the value, in contrast to the situation for a European call, where the famous Black-Scholes formula is available.

Previous work on this problem has been of three broad types. Firstly, there are numerical studies, such as the work of Kemna & Vorst [6] who use Monte Carlo techniques, and Carverhill & Clewlow [1] who use a Fourier transform method to compute the law of the average. Secondly, there are methods which replace the law of the average (which is hard to specify) by something more tractable; the work of Ruttiens [10], Vorst [12], Levy [7], [8], Levy & Turnbull [9], Turnbull & Wakeman [10] is of this nature, though it has to be admitted that these approaches offer little control on the error produced by the ansatz. Thirdly, there is the determined analysis of Yor [13], and Geman & Yor [3], [4]. This has produced notable expressions for the price as a triple integral, and for the price of an Asian option with 'independent exponential maturity'. Numerical inversion of this Laplace transform seems likely to be slow, and no simple analytic inversion has been found to date.

The present paper is a combination of analysis and numerics. In Section 2, we exploit a scaling property to reduce the calculation of the price of an Asian option to the solution of a parabolic PDE in two variables, rather that the three which at first sight appear necessary. We have learned that a similar scaling property for the floating strike Asian option was observed already by Ingersoll [5], p377; one can use this scaling behaviour equally well for fixed strike Asian options, and indeed the formalism we use covers also options whose averaging period starts at a time other than 0, or whose averaging is with respect to a quite general weight function. Numerical solution of this PDE in real time is a practical possibility, and we discuss in Section 4 how this is done. We find that (even without great effort to lubricate the programs) it is possible to compute a value accurate to about 5% in 3 seconds on a SUN SPARC 2 station, provided σ is not too small.

The approach of Section 3 is to try to obtain bounds on the price. We propose only trivial methods for this, based on conditioning firstly on some variable Z. Thus to obtain a lower bound, we have

$$\mathbb{E}(Y^+) = \mathbb{E}(\mathbb{E}(Y^+ \mid Z)) \geq \mathbb{E}\left(\mathbb{E}(Y \mid Z)^+\right). \tag{1.5}$$

We investigated numerically several possible choices for Z, some of them bivariate. However, for the fixed strike Asian option, by far the best choice turned out to be

$$Z = \int_0^T X_u du.$$

We obtain a two-dimensional integral as a lower bound, which (at least for a wide range of values of S_0, K, σ, r) is *staggeringly accurate*! (The yardstick here is the set of results of the PDE method of Section 2, which agree with the Monte Carlo results of Kemna & Vorst [7]). The reason why this bound is so good is quite general, and is applicable also to the floating strike Asian option, or an Asian option where the payoff Y is computed as a discrete average of the prices throughout the period. It rests on an analysis of the error committed in making the estimate (1.5), which simplifies very pleasingly.

As we discuss in the final section, Section 4, this bound also computes very quickly, taking less than one second, on a SUN SPARC 2 station, to get a value accurate to about 1%. This estimate for the price has also been used by Curran [2] in the case of fixed strike Asian options.

2. A PDE for the price of an Asian option

We shall assume until further notice that the maturity of the option is T, fixed, and that the probability measure μ has a density ρ_t in $(0,T)$. There is no essential loss of generality in this. If we define

$$\phi(t,x) \equiv \mathbb{E}\Big[\Big(\int_t^T S_u \mu(du) - x\Big)^+ \mid S_t = 1\Big], \tag{2.1}$$

where S is given as at (1.1), then we develop the martingale

$$\begin{aligned} M_t &\equiv \mathbb{E}\Big[\Big(\int_0^T S_u \mu(du) - K\Big)^+ \mid \mathcal{F}_t\Big] \\ &= \mathbb{E}\Big[\Big(\int_t^T S_u \mu(du) - \Big(K - \int_0^t S_u \mu(du)\Big)\Big)^+ \mid \mathcal{F}_t\Big] \\ &= S_t \mathbb{E}\Big[\Big(\int_t^T \frac{S_u}{S_t} \mu(du) - \frac{K - \int_0^t S_u \mu(du)}{S_t}\Big)^+ \mid \mathcal{F}_t\Big] \\ &= S_t \phi(t, \xi_t), \end{aligned} \tag{2.2}$$

where

$$\xi_t \equiv \frac{K - \int_0^t S_u \mu(du)}{S_t}. \tag{2.3}$$

It is immediate from the definition of ϕ that ϕ is jointly continuous, decreasing in t, and decreasing convex in x. Now by Itô's formula,

$$d\xi_t = -\rho_t dt + \xi_t(-\sigma dB_t - r dt + \sigma^2 dt),$$

so assuming that ϕ has enough smoothness to apply Itô's formula to (2.2), we have (with "$\doteq$" signifying that the two sides differ by a local martingale)

$$\begin{aligned} dM &= \phi dS + S\Big(\dot\phi dt + \phi' d\xi + \frac{1}{2}\phi'' d[\xi]\Big) + dS d\phi \\ &\doteq r\phi S dt + S\Big(\dot\phi + \phi'(-\rho_t - r\xi + \sigma^2\xi) + \frac{1}{2}\sigma^2\xi^2\phi''\Big)dt - \sigma S \cdot \phi' \sigma \xi dt \\ &= S\Big[r\phi + \dot\phi - (\rho_t + r\xi)\phi' + \frac{1}{2}\sigma^2\xi^2\phi''\Big]dt, \end{aligned}$$

which implies that

$$0 = \dot\phi + r\phi + \frac{1}{2}\sigma^2\xi^2\phi'' - (\rho_t + r\xi)\phi', \tag{2.4}$$

If we now write $f(t,x) \equiv e^{-r(T-t)}\phi(t,x)$, we find that f solves

$$\dot f + \mathcal{G}f = 0, \tag{2.5}$$

where $\mathcal{G}$ is the operator

$$\mathcal{G} \equiv \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} - (\rho_t + rx)\frac{\partial}{\partial x}.$$

The boundary conditions depend on the problem; in the case of the fixed strike Asian option,

$$f(T,x) = x^-, \tag{2.6}$$

whereas in the case of the floating strike Asian option, we shall have

$$f(T,x) = (1+x)^-. \tag{2.7.}$$

Now the PDE (2.5) is quite simple and can be solved numerically, as we shall discuss in Section 4. Let us denote the solution to the PDE (2.4) with the (fixed strike) boundary condition (2.6) by ϕ, and let the solution to (2.4) with the (floating strike) boundary condition (2.7) be denoted by ψ. Thus in the case where μ is uniform on $[0,T]$, the price of the Asian option with maturity T, fixed strike price K, and initial price S_0 is

$$\begin{aligned} e^{-rT}\mathbb{E}\Big(\int_0^T (S_u - K)\frac{du}{T}\Big)^+ &= S_0 f(0, KS_0^{-1}) \\ &\equiv e^{-rT}S_0\phi(0, KS_0^{-1}), \end{aligned}$$

and the price of the Asian option with maturity T and floating strike is simply

$$e^{-rT}\mathbb{E}\Big(\int_0^T S_u\frac{du}{T} - S_T\Big)^+ = e^{-rT}S_0\psi(0,0).$$

Notice that in this case, for $x \le 0$,

$$\phi(t,x) = r^{-1}(e^{r(T-t)} - 1) - x,$$

which makes the solution of the PDE easier in this case. Also, for large negative x, $\psi(t,x)$ is very close to

$$\mathbb{E}\Big(\int_t^T S_u\frac{du}{T} - S_t - x\Big) = \frac{e^{r(T-t)} - 1}{r} - e^{r(T-t)} - x,$$

which helps to set boundary values for numerical methods. Other formulae can be derived simply from these; for example, for an Asian option with strike K and maturity T, but whose average is computed over the interval $[T-t,T]$, $0 < t < T$, the price is

$$e^{-rT}\int_0^\infty \mathbb{P}(S_{T-t}\in dx)x\phi(T-t,K/x),$$

where ϕ is computed using the measure μ which is uniform on $[T-t,T]$, and this is easily computed once the function $\phi(T-t,\cdot)$ is known. The case where μ puts equal weight on a finite sequence of equally-spaced time-points is just as easy.

3. Lower bounds.

If we condition the process X on some zero-mean Gaussian variable Z, it remains a Gaussian process, and this is the heart of the lower-bound method used here. To set up some notation, let us write

$$\mathbb{E}(B_t|Z)=m_tZ,\quad \mathrm{cov}(B_s,B_t|Z)=v_{st}. \tag{3.1}$$

It is well known that

$$m_t=\mathbb{E}(B_tZ)/\mathbb{E}(Z^2),\;\; v_{st}=s\wedge t-\mathbb{E}(B_sZ)\mathbb{E}(B_tZ)/\mathbb{E}(Z^2). \tag{3.2}$$

In most cases of practical interest, it will prove to be quite easy to compute explicitly what these functions are. For example, if we fix $T=1$ and take $Z=\int_0^1 B_tdt$ then

$$m_t=3t(2-t)/2,\quad v_{st}=s\wedge t-3st(2-s)(2-t)/4, \tag{3.3}$$

and when $Z=\int_0^1 B_tdt-B_1=\int_0^1 tdB_t$, we obtain likewise

$$m_t=-3t^2/2,\quad v_{st}=s\wedge t-3s^2t^2/4. \tag{3.4}$$

The lower bound of (1.5) is not guaranteed to be good, but we can estimate the error made as follows. For any random variable U, we have

$$\begin{aligned}0&\le \mathbb{E}(U^+)-\mathbb{E}(U)^+\\&=\tfrac12(\mathbb{E}(|U|)-|\mathbb{E}(U)|)\\&\le\tfrac12\mathbb{E}(|U-\mathbb{E}(U)|)\\&\le\tfrac12\mathrm{var}(U)^{1/2}.\end{aligned}$$

Accordingly,

$$\begin{aligned}0&\le\mathbb{E}[\mathbb{E}(Y^+|Z)-\mathbb{E}(Y|Z)^+]\\&\le\tfrac12\mathbb{E}[\mathrm{var}(Y|Z)^{1/2}],\end{aligned} \tag{3.5}$$

and it is the variance of Y given Z which we propose to estimate. Firstly, the mean of Y given Z is

$$\mathbb{E}[\int_0^1 \exp(\sigma B_t-\tfrac12\sigma^2t+rt)\,\mu(dt)|Z]=\int_0^1\exp(\sigma m_tZ-\tfrac12\sigma^2vm_t^2+rt)\mu(dt), \tag{3.6}$$

where we have made the abbreviation $v \equiv \mathrm{var}(Z)$, and then similarly we have that

$$\begin{aligned}&\mathrm{var}(Y|Z)\\&\quad= \int_0^1 \mu(ds)\int_0^1 \mu(dt)\exp(\sigma Z(m_s+m_t) - \tfrac{1}{2}\sigma^2 v(m_s^2+m_t^2) + r(s+t))(e^{\sigma^2 v_{st}} - 1),\end{aligned}$$

after a few rearrangements. Let us now consider what we would have if we used the approximation $e^x \approx 1+x$ in this integral; we would obtain

$$V \equiv \int_0^1 \mu(ds)\int_0^1 \mu(dt)(1+\sigma Z(m_s+m_t) - \tfrac{1}{2}\sigma^2 v(m_s^2+m_t^2) + r(s+t))\sigma^2 v_{st},$$

which *vanishes if* $Z = \int_0^1 B_t\mu(dt)$. To see this, note that the first factor in the integrand may be written in the form

$$1 + f(s,Z) + f(t,Z) \equiv 1 + \{rs + \sigma Z m_s - \tfrac{1}{2}\sigma^2 v m_s^2\} + \{rt + \sigma Z m_t - \tfrac{1}{2}\sigma^2 v m_t^2\},$$

and that

$$\int_0^1 v_{st}\mu(ds) = \mathrm{cov}(\int_0^1 B_s\mu(ds), B_t|Z) = \mathrm{cov}(Z, B_t|Z) = 0.$$

Accordingly, we may estimate

$$\mathrm{var}(Y|Z) = \mathrm{var}(Y|Z) - V \le I_1 + I_2, \tag{3.7}$$

where

$$\begin{aligned}I_1 &= \int_0^1 \mu(ds)\int_0^1 \mu(dt)\ (e^{f(s,Z)+f(t,Z)} - 1 - f(s,Z) - f(t,Z)).|e^{\sigma^2 v_{st}} - 1|\\I_2 &= \int_0^1 \mu(ds)\int_0^1 \mu(dt)\ (e^{\sigma^2 v_{st}} - 1 - \sigma^2 v_{st}).|1 + f(s,Z) + f(t,Z)|.\end{aligned}$$

Now the estimate (3.7) depends on Z, and will not be small for all values of Z. However, we have (using (3.5) and (3.7)) the estimates

$$0 \le \mathbb{E}[\mathbb{E}(Y^+|Z) - \mathbb{E}(Y|Z)^+] \le \tfrac{1}{2}(\mathbb{E}(I_1+I_2))^{1/2}, \tag{3.8}$$

so our task is to estimate $\mathbb{E}I_1$ and $\mathbb{E}I_2$. We shall use the inequalities

$$|e^x - 1| \le |x|e^{|x|}, \quad |e^x - 1 - x| \le \tfrac{1}{2}x^2 e^{|x|},$$

valid for all real x. We shall also write $g(s) \equiv rs - \frac{1}{2}\sigma^2 v m_s$ in what follows, and let c, γ_1, γ_2 be constants such that for all $0 \le s,\ t \le 1$,

$$|v_{st}| \le c, \quad (m_s+m_t)^2 \le \gamma_1, \quad |g_s + g_t| \le \gamma_2.$$

Now we estimate

$$\begin{aligned}
\mathbb{E}I_1 &\leq c\sigma^2 e^{c\sigma^2} \int_0^1 \int_0^1 |\mu|(ds)|\mu|(dt)\mathbb{E}[\exp(g_s + g_t + Z\sigma(m_s + m_t)) \\
&\qquad\qquad\qquad\qquad\qquad - 1 - g_s - g_t - Z\sigma^2(m_s + m_t)] \\
&= c\sigma^2 e^{c\sigma^2} \int_0^1 \int_0^1 |\mu|(ds)|\mu|(dt)[\exp(g_s + g_t + \tfrac{1}{2}\sigma^2 v(m_s + m_t)^2) - 1 - g_s - g_t] \\
&= c\sigma^2 e^{c\sigma^2} \int_0^1 \int_0^1 |\mu|(ds)|\mu|(dt)[\{e^{\frac{1}{2}\sigma^2 v(m_s+m_t)^2} - 1\}e^{g_s+g_t} + e^{g_s+g_t} - 1 - g_s - g_t] \\
&\leq c\sigma^2 e^{c\sigma^2+\gamma_2}\left[\tfrac{1}{2}\sigma^2\gamma_1 v e^{\frac{1}{2}\sigma^2\gamma_1 v} + \tfrac{1}{2}\gamma_2^2\right] \int_0^1 \int_0^1 |\mu|(ds)|\mu|(dt). \\
&= c\sigma^2 e^{c\sigma^2+\gamma_2}\left[\tfrac{1}{2}\sigma^2\gamma_1 v e^{\frac{1}{2}\sigma^2\gamma_1 v} + \tfrac{1}{2}\gamma_2^2\right] M,
\end{aligned}$$

where we have abbreviated

$$M \equiv \int_0^1 \int_0^1 |\mu|(ds)|\mu|(dt).$$

As for the other term, we have

$$\begin{aligned}
\mathbb{E}I_2 &= \int_0^1 \int_0^1 |\mu|(ds)|\mu|(dt)(e^{\sigma^2 v_{st}} - 1 - v_{st}).|1 + g_s + g_t| \\
&\leq \tfrac{1}{2}\sigma^4 c^2 e^{\sigma^4 c^2}(1 + \gamma_2)M.
\end{aligned}$$

Assembling this, the estimate on the right of (3.8) becomes

$$\tfrac{1}{2}\left[c\sigma^2 e^{c\sigma^2+\gamma_2}\left[\tfrac{1}{2}\sigma^2\gamma_1 v e^{\frac{1}{2}\sigma^2\gamma_1 v} + \tfrac{1}{2}\gamma_2^2\right] + \tfrac{1}{2}\sigma^4 c^2 e^{\sigma^4 c^2}(1 + \gamma_2)\right]^{1/2} M^{1/2}.$$

Let us now see how these estimates shape up in the two examples of main interest to us, the fixed and floating strike Asian options.

Firstly for the fixed strike, we see that $v \equiv \mathrm{var}(Z) = \frac{1}{3}$ and easily that we may take $\gamma_1 = 9$, $\gamma_2 = r + \sigma^2/4$. It is also not hard to establish that we may take $c = \frac{1}{3}$. The integral of $|\mu(ds)|\ |\mu(dt)|$ over the square is 1.

For the floating strike Asian option, the constants are the same except that now we take $c = 3^{2/3}/4$ and the integral of the measure over the square comes to 4.

It is now clear that for typical values, say r and σ of the order of 10^{-1}, we shall have that $\mathbb{E}I_1$ is bounded by something that is of the order of 10^{-4} and $\mathbb{E}I_2$ is bounded by something which is of the order of 10^{-4} also, so we can expect that the bound on the error will be something of the order of 10^{-2} at worst; in the next section we discuss the outcome of numerics on the bounds.

4. Computational aspects

Throughout this section, we assume that $T = 1$ and $S_0 = 100$. We tried a variety of methods for solving the PDE (2.5), and report here on what worked well, and on what worked less well. Neither of the authors is an expert in numerical methods, and it was not the goal to obtain the most rapid possible program. However, even the clumsy computing which we carried out showed that it is possible to obtain accuracies of a few percent in times of the order of a few seconds.

It turns out that by treating (2.5) simply as a parabolic PDE and solving it with the NAG routine D03PAF (as the coefficient $\sigma^2 x^2/2$ vanishes at $x = 0$, we replace it by $\sigma^2 x^2/2 + 10^{-60}$ to justify the use of D03PAF), quite acceptable combinations of accuracy and speed were obtained. Tables 1.1, 1.2 and 1.3 give some specimen results for typical values of the parameters for the fixed strike problem, with maturity time $T = 1$ (computations carried out on a SUN SPARC 2 station). We also took the step size $h = 1/1000$; the values were very close to $h = 1/300$, but the times were longer. The effect of σ on the speed and accuracy is very noticable. For $\sigma = 0.05$, it requires 11 seconds to get the first two significant figures correctly, for small values of r, whereas for $\sigma = 0.3$, it takes less than one second. We carried out the calculations for volatilities σ bigger than 0.3, and the speed and accuracy were as good as for $\sigma = 0.3$.

Table 2 gives the corresponding results for the floating-strike problem, and once again it takes only a couple of seconds to get a rather precise answer when either the volatility σ or the interest rate r is not too small, and the calculation becomes longer when both σ and r are very close to zero.

Table 3 gives some specimen upper and lower bounds for typical values of the parameters with maturity time $T = 1$, initial price $S_0 = 100$ and interest rate $r = 0.09$. Computations have been carried out on a SUN SPARC 2 station, using estimations described respectively in Sections 2 and 3(ii) and it took about ten seconds to get an upper bound, and less than one second a lower one. In practical computations, it is more than sufficient to condition on $X_T = x$ (or on $\int_0^T X_u du = x$ for the lower bound) for $x \in [-6, 6]$. In both situations, the NAG routine D01BAF(D01BAZ) was used for the first integral, and then we summerized over $x \in [-6, 6]$ with step size 10^{-2}, which turned out to be a satisfactory at both speed and precision level (one can manage so that the larger the step size is, the less accurate the estimations become). Comparisons are given with respect to Monte Carlo results given in Levy & Turnbull [9].

Table 1.1 (σ=0.05)

interest rate r	strike price K	step size	PDE result	time (sec.)
0.02	95	0.020	5.97	1
		0.010	5.47	3
		0.005	5.83	11
		0.003	5.88	25
	100	0.020	2.17	1
		0.010	1.78	3
		0.005	1.68	11
		0.003	1.68	25
	105	0.020	0.96	1
		0.010	0.38	3
		0.005	0.21	11
		0.003	0.17	25
0.09	95	0.020	8.83	1
		0.010	8.67	3
		0.005	8.82	11
		0.003	8.81	25
	100	0.020	3.91	1
		0.010	3.93	3
		0.005	4.18	11
		0.003	4.26	25
	105	0.020	1.97	1
		0.010	1.19	3
		0.005	1.01	11
		0.003	0.98	25
0.28	95	0.020	16.48	1
		0.010	15.35	3
		0.005	15.42	11
		0.003	15.42	25
	100	0.020	11.19	1
		0.010	11.78	3
		0.005	11.64	11
		0.003	11.64	25
	105	0.020	7.74	1
		0.010	7.69	3
		0.005	7.88	11
		0.003	7.86	25

Table 1.2 (σ=0.10)

interest rate r	strike price K	step size	PDE result	time (sec)
0.02	90	0.020	10.34	1
		0.010	10.77	3
		0.005	10.82	11
		0.003	10.83	25
	100	0.020	2.92	1
		0.010	2.79	3
		0.005	2.79	11
		0.003	2.80	25
	110	0.020	0.48	1
		0.010	0.28	3
		0.005	0.22	11
		0.003	0.20	25
0.09	90	0.020	13.12	1
		0.010	13.38	3
		0.005	13.38	11
		0.003	13.38	25
	100	0.020	4.69	1
		0.010	4.80	3
		0.005	4.89	11
		0.003	4.90	25
	110	0.020	1.00	1
		0.010	0.74	3
		0.005	0.66	11
		0.003	0.64	25
0.28	90	0.020	19.34	1
		0.010	19.20	3
		0.005	19.20	11
		0.003	19.20	25
	100	0.020	11.32	1
		0.010	11.63	3
		0.005	11.65	11
		0.003	11.65	25
	110	0.020	4.47	1
		0.010	4.58	3
		0.005	4.66	11
		0.003	4.67	25

Table 1.3 (σ=0.30)

interest rate r	strike price K	step size	PDE result	time (sec.)
0.02	90	0.020	13.13	1
		0.010	13.17	3
		0.005	13.18	11
		0.003	13.18	25
	100	0.020	7.30	1
		0.010	7.31	3
		0.005	7.31	11
		0.003	7.31	25
	110	0.020	3.68	1
		0.010	3.65	3
		0.005	3.64	11
		0.003	3.64	25
0.09	90	0.020	14.93	1
		0.010	14.97	3
		0.005	14.98	11
		0.003	14.98	25
	100	0.020	8.80	1
		0.010	8.82	3
		0.005	8.83	11
		0.003	8.83	25
	110	0.020	4.72	1
		0.010	4.70	3
		0.005	4.70	11
		0.003	4.70	25
0.28	90	0.020	19.62	1
		0.010	19.66	3
		0.005	19.67	11
		0.003	19.67	25
	100	0.020	13.28	1
		0.010	13.32	3
		0.005	13.33	11
		0.003	13.33	25
	110	0.020	8.29	1
		0.010	8.30	3
		0.005	8.31	11
		0.003	8.31	25

Table 2

volatility σ	interest rate r	step size	PDE result	time (sec.)
0.05	0.02	0.0200	1.39	3
		0.0100	0.97	7
		0.0080	0.89	11
		0.0025	0.72	60
	0.08	0.0200	0.67	3
		0.0100	0.33	7
		0.0080	0.26	11
		0.0025	0.13	56
	0.14	0.0200	0.29	3
		0.0100	0.09	7
		0.0080	0.06	11
		0.0025	0.01	48
0.10	0.02	0.0200	2.04	2
		0.0100	1.86	8
		0.0080	1.84	12
		0.0025	1.82	50
	0.08	0.0200	1.13	2
		0.0100	0.90	8
		0.0080	0.86	13
		0.0025	0.80	45
	0.14	0.0200	0.56	3
		0.0100	0.38	8
		0.0080	0.35	12
		0.0025	0.30	45
0.15	0.02	0.0200	2.98	2
		0.0100	2.96	9
		0.0080	2.95	12
		0.0025	2.95	45
	0.08	0.0200	1.88	2
		0.0100	1.80	9
		0.0080	1.79	13
		0.0025	1.77	45
	0.14	0.0200	1.13	2
		0.0100	1.02	9
		0.0080	0.99	11
		0.0025	0.97	46

Table 3

volatility σ	strike price K	upper bound	M-C result	lower bound
0.10	95	8.95	8.91	8.91
	100	5.10	4.91	4.92
	105	2.34	2.06	2.07
0.30	90	15.23	14.96	14.98
	100	9.39	8.81	8.83
	110	5.37	4.68	4.70
0.50	90	18.52	18.14	18.18
	100	13.69	12.98	13.02
	110	9.97	9.10	9.18

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