The square-root process and Asian options

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Abstract. Although the square-root process has long been used as an alternative to the Black-Scholes geometric Brownian motion model for option valuation, the pricing of Asian options on this diffusion model has never been studied analytically. However, the additivity property of the square-root process makes it a very suitable model for the analysis of Asian options. In this paper, we develop explicit prices for digital and regular Asian options. We also obtain distributional results concerning the square-root process and its average over time, including analytic formulae for their joint density and moments. We also show that the distribution is actually determined by those moments.

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1. Introduction

Asian options, whose payoff is based on the arithmetic average $\frac{\int_0^T S_u du}{T}$ of the underlying equity price S_t over a given period of time $(0, T)$, have known a very large success over the past years. Although the valuation of these options in the Black and Scholes framework has triggered the interest of financial mathematicians for over a decade in an intricate interplay between theoretical (Yor [40], Geman and Yor [41], Schröder [36], Milevsky and Posner [28], [32], Dufresne [16], etc.) and computational (Caverhill and Clewlow [6], Fu *et al* [18], Rogers and Shi [34], Zvan [42], Vecer [38], etc.) approaches, the pricing of such options under alternative models has not been much explored so far, with the noticeable exception of Andreasen [1] for jump-diffusions and Vecer and Xu [39], very recently, for a general treatment of semi-martingale models in terms of integro-differential equations. The square-root process offers a very tractable mathematical structure, which enables simple pricing of important exotic options in explicit form. This has, for example, been highlighted by Lo *et al* [26] in the case of barrier options (analysis extended by Davydov and Linetsky [13] and [14] for constant elasticity of variance models). In the case of Asian options, although little attention has been paid to this in the literature, the additivity property of the square-root process makes it a specially well-suited model for pricing options on arithmetic averages. We show in this paper that the square-root process leads to simple explicit expressions for Asian options prices, unlike in the Black and Scholes framework.

The popularity of the square-root process in all main branches of financial modeling can be explained by its desirable property of positivity and its richness of behaviour. As a result, it has been used to model equities (Cox-Ross [10] alternative process), interest rates (CIR [9] interest rate model and its time-inhomogeneous [27], multivariate [7] and other derivatives) , stochastic volatility (Heston [22] model and its various extensions [2], [15], [30]) and other financial quantities. In this paper, we will focus on the Cox-Ross form of the process used for modelling equities

$$dS_t = rS_t dt + \sigma\sqrt{S_t} dW_t \tag{1}$$

and on its temporal integral $Y_t = \int_0^t S_u du$. The corresponding analysis for the mean-reverting form

$$dX_t = (a - bX_t)dt + \sigma\sqrt{X_t} dW_t \tag{2}$$

and its temporal integral can be found in our companion paper [12] although some of the results presented here are valid[+] for both forms (2) and (1). The Cox-Ross model was first proposed as an alternative to the Black-Scholes model, exhibiting a specific stochastic volatility. The instantaneous variance of the percentage price change $\frac{\sigma^2}{S_t}$ is an inverse function of the stock price. Along with the square-root model, Cox and Ross ([10] and [8]) also introduced a class of processes, the constant elasticity of variance

[+] Results valid for both forms will be quoted in terms of X whereas results specific to the equity form (1) will be quoted in terms of S.

processes, sharing similar properties. Numerous subsequent studies proved the empirical adequacy of these models to stock prices movements as well as to the pricing of European (Beckers [4], Schröder [35]) and exotic (Boyle and Tian [5], Davydov and Linetsky [13]) derivatives. The square-root equity model on its own has raised the interest of researchers (see Lo *et al* [26], Hauser and Lauterbach [21]).

In order to evaluate Asian derivatives, we first study the distributional properties of the process. After recalling some important properties of the square-root process in the second section, we proceed to derive the joint moments of (X_T, Y_T) in the third one. By solving a triangular set of partial differential equations, we prove that these moments take a simple explicit form, which can be practical. Those moments have indeed an important informational content, since they are proven to actually determine the distribution of these processes and are needed for moments-base expansions of the Laguerre type [16]. In the fourth section, we also derive the joint distribution of the process and its integral by analytic Laplace transform inversion, using a simplifying measure change relating the square-root process to a time-changed square Bessel process. In the fifth section, we apply these results to the pricing of fixed-strike Asian option and test the numerical behaviour of these results in the last and sixth section. The formulae derived for these prices take the form of very fast converging series, which proves to be not only faster but also more robust than numerical Laplace inversion in regions of high maturities and volatilities. They are also more suited to systematic implementation, as will be explained later.

2. The square-root process: some reminders

For any positive initial value S_0, there is a unique strong solution to the stochastic differential equation (1) and as S is positive, Y is positive and increasing. The joint distribution of (S_T, Y_T) can be characterised by its moment generating function $E\left(e^{-\lambda X_t} e^{-\mu \int_0^t X_u du} | X_0 = x_0\right)$ (see Lamberton and Lapeyre [23] for instance) for the general form (2)). As mentioned earlier, it is the additivity of the process which plays a key role in the derivation of this simple MGF. Indeed, the MGF, as a function of the time and the initial value, follows the partial differential equation

$$\frac{\partial \tilde{L}^{a,x_0}}{\partial t} = \frac{\sigma^2}{2} x \frac{\partial^2 \tilde{L}^{a,x_0}}{\partial^2 x^2} + (a - bx) \frac{\partial \tilde{L}^{a,x_0}}{\partial x} - \mu x \tilde{L}^{a,x_0} \tag{3}$$

subject to the initial condition

$$\tilde{L}^{a,x_0}(0, x) = e^{-\lambda x}$$

The additivity property, corollary of the additivity property of the square Bessel processes (see Revuz and Yor [33]), states that the sum of two independent square-root processes with the coefficients a_1 and a_2, the initial values x_0^1 and x_0^2 respectively and the same σ and b is another square-root process with the coefficient $a_1 + a_2$, the initial value $x_0^1 + x_0^2$ and the same σ and b. This implies that $\tilde{L}^{a,x_0} = \tilde{L}^{0,x_0} \tilde{L}^{a,0}$ by convolution. The scaling property of the process, on the other hand, leads to the form $\tilde{L}^{0,x_0} = e^{-a\phi}$

and $\tilde{L}^{0,x_0} = e^{-x_0\psi}$ where ϕ and ψ are some positive functions of the others parameters. The additivity property thus leads to the fundamental decomposition $\tilde{L}^{a,x_0} = e^{-x_0\psi - a\phi}$. Using this functional form in the partial differential equation (3) transforms it into two simple coupled ordinary differential equations, one being a Ricatti equation and the other a direct differentiation equation. Simplifying to the equity case (1) and solving for the Ricattti equation gives the following expressions, where we change the notation $\tilde{L}^{0,s_0}$ to $\mathcal{L}^{S,Y}(\lambda,\mu)$ to emphasise the importance of the MGF arguments μ and λ.

$$\mathcal{L}^{S,Y}(\lambda,\mu) = E\Big(e^{-\lambda S_t} e^{-\mu\int_0^t S_u du} | S_0 = s_0\Big) = e^{-s_0\psi} \tag{4}$$

with

$$\psi = \frac{\lambda((\gamma - b) + e^{-\gamma t}(\gamma + b)) + 2\mu(1 - e^{-\gamma t})}{\sigma^2\lambda(1 - e^{-\gamma t}) + (\gamma + b) + e^{-\gamma t}(\gamma - b)} \tag{5}$$

and

$$\gamma = \sqrt{b^2 + 2\mu\sigma^2} \tag{6}$$

This result stemming from the additivity of the square-root process is the core from which we will derive our new results concerning the distribution of (S, Y). The transform $\mathcal{L}^{S,Y}(\lambda,\mu)$ perfectly characterises the distribution. However, it is desirable to obtain densities and moments in as simple a form as possible for practical purposes such as parameters estimation or derivative pricing. This is the object of the following section.

3. Joint moments

The first quantities of interest when studying a random variable are its moments. In the literature concerning stochastic volatility, the moments of the average $\frac{Y_T}{T}$ have been used either to compute approximations for option prices (see Ball and Roma [3]) or to gain insight in the distribution of the stock (see Das [11]). However, only the four first moments have been given in these texts since they were computed through successive differentiation of the moment-generating function. Although it is theoretically possible to obtain all these moments through repeated differentiation, this method remains tedious and even with formal calculus packages like Mathematica or Maple, only the first ones can be handled in this quite time-consuming way. We show here that it is actually possible to obtain all of them analytically, since it turns out that they have a relatively simple form.

In the case of the square-root process and its integral, moments convey total information since they determine the joint distribution. Indeed, the function equal to the joint MGF for positive λ and μ, i.e. the rightmost part of equation (4), is analytic for a range of negative values. From Beppo Levi's theorem, the MGF, written as the expectation of an entire series, converges monotonically to and coincides with the analytic function in (4) inside this range of negative values. Therefore, knowing the joint moments of (X_t, Y_t) is equivalent to knowing their joint distribution.

The following results concerning the joint moments of (X_t, Y_t) are general and also valid for the mean-reverting form. For this section, we hence use the parametrisation of (2).

Theorem 3.1. *The joint moments of X_t and Y_t are given by*

$$M_{m,n}(t) = E(Y_t^m X_t^{n-m}) = \sum_{j=0}^{n} e^{-jbt}\Big(\sum_{i=1}^{I_j^{m,n}} \alpha_{j,i}^{m,n} \frac{t^{i-1}}{(i-1)!}\Big) \tag{7}$$

where

$$I_j^{m,n} = \min(n+1-j, m+1) \tag{8}$$

The coefficients $\alpha_{j,i}^{m,n}$ can be obtained by recursion through the relations

- *For $j \neq n-m$*

$$\begin{aligned}\alpha_{j,i}^{m,n} &= m \sum_{i'=i}^{I_j^{m-1,n}} \frac{(-1)^{i'-i}\alpha_{j,i'}^{m-1,n}}{((n-m-j)b)^{i'-i+1}} \\ &+(n-m)\Big(a+(n-m-1)\frac{\sigma^2}{2}\Big)\sum_{i'=i}^{I_j^{m,n-1}} \frac{(-1)^{i'-i}\alpha_{j,i'}^{m,n-1}}{((n-m-j)b)^{i'-i+1}}\end{aligned} \tag{9}$$

- *For $j = n-m$*
  - *For $i = 1$*

$$\begin{aligned}\alpha_{n-m,1}^{m,n} &= c_{m,n} + m\sum_{j=0}^{n}\sum_{i=1}^{I_{n-m}^{m-1,n}} \frac{\alpha_{j,i}^{m-1,n}}{((j-n+m)b)^i} \\ &+(n-m)\Big(a+(n-m-1)\frac{\sigma^2}{2}\Big)\sum_{j=0}^{n-1}\sum_{i=1}^{I_{n-m}^{m,n-1}} \frac{\alpha_{j,i}^{m,n-1}}{((j-n+m)b)^i}\end{aligned} \tag{10}$$

  - *For $i > 1$*

$$\alpha_{n-m,i}^{m,n} = m\alpha_{n-m,i-1}^{m-1,n} + (n-m)\Big(a+(n-m-1)\frac{\sigma^2}{2}\Big)\alpha_{n-m,i-1}^{m,n-1} \tag{11}$$

where

$$c_{m,n} = r_0^n 1_{\{m=0\}} \tag{12}$$

- *Initial condition*

$$\alpha_{0,1}^{0,0} = 1 \tag{13}$$

Proof. Assessing first the issue of existence, the joint MGF (see Lamberton and Lapeyre [23] for the general case) is infinitely differentiable in a neighbourhood of $(0,0)$, implying the joint moments of X_t and Y_t exist for any positive order for any finite t.

Therefore,

$$\frac{dE(Y_t^m X_t^k)}{dt} = mE(Y_t^{m-1}X_t^{k+1}) + akE(Y_t^m X_t^{k-1})$$

$$-bkE(Y_t^m X_t^k) + k(k-1)\frac{\sigma^2}{2}E(Y_t^m X_t^{k-1}) \tag{14}$$

It should be noticed that the computation of positive order moments does not actually involve the moments of the reciprocals of either X_t or Y_t.

Denoting $\widehat{M}_{m,n}(\zeta)$ the Laplace transform of $M_{m,n}(t) = E(Y_t^m X_t^{n-m})$ with respect to time for $\zeta \in \mathbb{R}^+$, $\widehat{M}_{m,n}(\zeta) = \int_0^\infty e^{-\zeta t} M_{m,n}(t)dt$, the ordinary differential equation (14) becomes

$$\widehat{M}_{m,n}(\zeta)[\zeta + b(n-m)] - M_{m,n}(0) = m\widehat{M}_{m-1,n}(\zeta) + d(n,m)\widehat{M}_{m,n-1}(\zeta) \tag{15}$$

with

$$d(n,m) = (n-m)\Big(a + (n-m-1)\frac{\sigma^2}{2}\Big) \tag{16}$$

(7) can then be shown through induction, assuming that for a given $n > 0$ and for all integers $m < n$, the joint moments have the form

$$\widehat{M}_{m,n} = \sum_{j=0}^{n} \sum_{i=1}^{I_j^{m,n}} \frac{\alpha_{j,i}^{m,n}}{(\zeta + jb)^i} \tag{17}$$

The inversion then comes from the classical result

$$\int_0^\infty e^{-\zeta t - jbt} \frac{t^{i-1}}{(i-1)!} dt = \frac{1}{(\zeta + jb)^i}$$

$\square$

The mathematical importance of those moments along with their appealing form make them a useful tool as they allow us to state explicitly the density and other functionals and expectations in an analytical form with methods like Laguerre series (see Dufresne [16]).

4. Laplace transform inversion and density

The moments-based expansion methods that can be derived from the results of the previous section remain quite general and their actual efficiency depends on the specific distribution. In the following, we will derive a series representation for this density by exploiting a change of measure relating the square-root process to a square-Bessel process. Although moments-based expansions remain useful for complex payoff, this series representation is valuable and simpler for pricing options with simple payoff like call or put options. Just like the square-root process density is an infinite weighted average of gamma densities (see Feller [17]), this density will be represented in a series form.

Theorem 4.1. *Under the measure Q^* given by the Radon-Nykodim derivative $\frac{dQ^*}{dQ} = L(T)$, with $L(t) = e^{\frac{b^2 Y_t}{2\sigma^2} + \frac{b(X_t - x_0)}{\sigma^2} - \frac{abt}{\sigma^2}}$, the process $X(t)$ is a.s. proportional to a square Bessel process*

$$dX_t = adt + \sigma\sqrt{X_t}dW_t^* \tag{18}$$

W_t^* being a Brownian motion under the Q^*-measure. For the equity form (1), this translates to $L(t) = e^{\frac{r^2 Y_t}{2\sigma^2} - \frac{r(S_t - s_0)}{\sigma^2}}$ and

$$dS_t = \sigma\sqrt{S_t}dW_t^* \tag{19}$$

Proof. From the SDE defining X,

$$L_t = e^{\int_0^t \frac{b}{\sigma}\sqrt{X_u}dW_u - \int_0^t \frac{b^2}{2\sigma^2} X_u du}$$

Since the Novikov condition $E(e^{\frac{b^2}{2\sigma^2} Y_t}) < \infty$ is verified (the rightmost part of (4) is analytic for $\lambda = 0$ and $\mu \geq -\frac{b^2}{2\sigma^2}$), L is an exponential martingale with mean 1. Girsanov's theorem then implies that the process W^* defined by $W_t^* = W_t - \int_0^t \frac{b}{\sigma}\sqrt{X_u}dW_u$ is a Brownian motion under Q^*. □

If X might also be connected with Bessel processes through other transformations, the change of measure proposed here is a simple result, easy to manipulate and suited to the analysis of the path-dependent integral Y_t, which requires path properties to be exploitable. For a detailed analysis of the properties of Bessel processes, please refer to Pitman and Yor [31], Shiga and Watanabe [37] and Revuz and Yor [33] and the references within.

Theorem 4.2. *Denoting $\alpha = \frac{\sigma^2}{8}$, the joint density of S_t and Y_t, for $S_t > 0$ under Q, for the equity process (1), is given by*

$$f^{S,Y}(s,y) = \frac{s_0\alpha}{2\sqrt{2\pi}} \frac{1}{(y\alpha)^2} e^{-\frac{r^2 y}{2\sigma^2} + \frac{R(s-s_0)}{\sigma^2}} \sum_{n=0}^{\infty} \frac{B_n^{(X,Y)}(y)}{n+1} \tag{20}$$

with the terms $B_n^{(X,Y)}(y)$ defined as

$$\sum_{p=0}^{n} \frac{\binom{n+1}{n-p}}{p!} \Big(\frac{-s}{\sqrt{2y\alpha}}\Big)^p \sum_{q=0}^{n} \frac{\binom{n+1}{n-q}}{q!} \Big(\frac{-s_0}{\sqrt{2y\alpha}}\Big)^q He_{p+q+3}\Big(\frac{\alpha_n}{\sqrt{2y\alpha}}\Big) e^{-\frac{\alpha_n^2}{4y\alpha}} \tag{21}$$

He_k being the k^{th} Hermite polynomial $\frac{\partial^k e^{-\frac{x^2}{2}}}{\partial x^k} = (-1)^k He_k(x) e^{-\frac{x^2}{2}}$ and

$$\alpha_n = \frac{x + x_0 + ((n+1)\sigma^2 t)}{2} \tag{22}$$

The density of Y_t conditional on $S_t = 0$ is $\frac{f_0^Y(y)}{P_Q(S_t=0)}$ where

$$f_0^Y(y) = \frac{e^{-\frac{r^2 y}{2\sigma^2} - \frac{rs_0}{\sigma^2}}}{y\sqrt{2\pi}} \sum_{n=0}^{\infty} \sum_{p=0}^{n} \frac{\binom{n}{n-p}}{p!} \Big(\frac{-s_0}{\sqrt{2y\alpha}}\Big)^p \Big(He_{p+1}\Big(\frac{\beta_n}{\sqrt{2y\alpha}}\Big) e^{-\frac{\beta_n^2}{4y\alpha}}$$

$$-He_{p+1}\Big(\frac{\beta_{n+1}}{\sqrt{2y\alpha}}\Big) e^{-\frac{\beta_{n+1}^2}{4y\alpha}}\Big) \tag{23}$$

with

$$\beta_n = \frac{s_0 + nt\sigma^2}{2} \tag{24}$$

and

$$P_Q(S_t = 0) = e^{-\frac{s_0 r(1+e^{-rT})}{\sigma^2(1-e^{-rT})}} \tag{25}$$

Proof. Under the Q^*-measure, from the previous theorem

$$\mathcal{L}^{*S,Y}(\lambda,\mu) = e^{-s_0\frac{\gamma(1+e^{-\gamma t})}{\sigma^2(1-e^{-\gamma t})}} \sum_{n=0}^{\infty} \frac{s_0^n}{n!} \frac{\left(\frac{4e^{\gamma t}\gamma^2}{\sigma^4(1-e^{-\gamma t})^2}\right)^n}{\left(\lambda + \frac{\gamma(1+e^{-\gamma t})}{\sigma^2(1-e^{-\gamma t})}\right)^n} \tag{26}$$

Inverting this MGF with respect to λ (convergence ensured by Beppo Levi theorem) leads to

$$\delta_x(0) + I_1\Big(\frac{2\sqrt{4ss_0e^{\gamma t}\gamma^2}}{\sigma^2(1-e^{-\gamma t})}\Big)\sqrt{\frac{s_0}{s}}\frac{2\gamma e^{-s_0\frac{\gamma t}{2}}}{\sigma^2(1-e^{-\gamma t})}e^{-(s+s_0)\frac{\gamma(1+e^{-\gamma t})}{\sigma^2(1-e^{-\gamma t})}}$$

since (26) is the sum of a constant and weighted gamma MGFs. $\delta_s(0)$ stands here for the Dirac delta function which is null everywhere except at 0 where it is infinite.

For non-null s, this can be rewritten (see Gradshteyn and Ryzhik [20]) as

$$s_0\frac{4\gamma^2e^{-\gamma t}}{\sigma^4}e^{-(s+s_0)\frac{\gamma}{\sigma^2}}\sum_{n=0}^{\infty}\frac{n!L_n^1(\frac{2s\gamma}{\sigma^2})L_n^1(\frac{2s_0\gamma}{\sigma^2})e^{-\gamma nt}}{(n+1)!} \tag{27}$$

This expression is actually an eigenfunction expansion (see Davydov and Linetsky [14] and [25] for applications of eigenfunction expansions in Finance and similar expansions in Laguerre polynomials for the CIR model).

It is known that the inverse of $(\frac{2\gamma}{\sigma^2})^{2\kappa}e^{-\frac{2q\gamma}{\sigma^2}}$ for q and κ positive is (see Gradshteyn and Ryzhik [20])

$$\alpha\sqrt{\frac{2}{\pi}}(2y\alpha)^{-\kappa-1}e^{-\frac{q^2}{8y\alpha}}D_{2\kappa+1}\Big(\frac{q}{\sqrt{2y\alpha}}\Big) \tag{28}$$

and D_ν, the parabolic cylinder functions of order ν, simplify to Hermite polynomials for integer indices. Convergence of the series of inverses to the general inverse is ensured by the presence of the factor $e^{-(s+s_0)\frac{\gamma}{\sigma^2}}$ and the uniform convergence of (27).

In the case absorption occurs, treating

$$\lim_{\lambda\to\infty}\mathcal{L}^{*S,Y}(\lambda,\mu) = E^{Q^*}(e^{-\mu Y_t}I_{\{S_t=0\}}) = e^{-s_0\frac{\gamma(1+e^{-\gamma t})}{\sigma^2(1-e^{-\gamma t})}}$$

in the same way leads to the result. □

Remarks.

1. This series is fast-converging, as the leading term is roughly of order $e^{-\frac{\sigma^2t^2}{2y}n^2}$.

2. The Hermite polynomial can be easily computed with $He_0 = 1$ and the recursions

$$He_{k+1}(x) = xHe_k(x) - kHe_{k-1}(x) \tag{29}$$

3. In the mean-reverting case (2), the joint density can be derived in the same way in terms of D_ν the parabolic cylinder function of order ν,

$$f^{X,Y}(x,y) = \frac{\left(\frac{x}{\sqrt{2}}\right)^{\frac{2a}{\sigma^2}-1}}{2\sqrt{\pi}(\sqrt{y\alpha})^{\frac{2a}{\sigma^2}+2}} e^{-\frac{b^2 y}{2\sigma^2}-\frac{b(x-x_0)}{\sigma^2}+\frac{abt}{\sigma^2}} \sum_{n=0}^{\infty} \frac{n!\alpha}{\Gamma(n+\frac{2a}{\sigma^2})} N_n(y) \tag{30}$$

with the term $N_n(y)$ defined as

$$\sum_{p=0}^{n} \frac{\binom{n+\frac{2a}{\sigma^2}-1}{n-p}}{p!} \left(\frac{-x}{\sqrt{2y\alpha}}\right)^p \sum_{q=0}^{n} \frac{\binom{n+\frac{2a}{\sigma^2}-1}{n-q}}{q!} \left(\frac{-x_0}{\sqrt{2y\alpha}}\right)^q D_\nu\left(\frac{\alpha_n}{\sqrt{2y\alpha}}\right) e^{-\frac{\alpha_n^2}{8y\alpha}} \tag{31}$$

where

$$\alpha_n = \frac{x + x_0 + (a+n\sigma^2)t}{2} \tag{32}$$

$$\nu = p + q + \frac{2a}{\sigma^2} + 1 \tag{33}$$

5. Regular and digital Asian options

The price of digital and regular Asian options as well as other related distributional results can be derived from the following preliminary result.

Theorem 5.1. *For $\lambda \geq \mu$ and $\mu > 0$, the inverse Laplace transform of the modified MGF* $\frac{\partial^j}{\partial \lambda^j} \frac{E(e^{-(\lambda+\mu)Y_T})}{\mu^i}$ with respect to μ, $\mathrm{MMI}^p(K,\lambda,i,j) + \mathrm{MMI}^a(K,\lambda,i,j)$ - where $\mathrm{MMI}^a(K,\lambda,i,j)$ is the component related to absorption- can be written as a sum of elements of leading order $e^{-\frac{\sigma^2 t^2}{2y}n^2}$.*

More precisely, for $i,j = 0,1$, we will denote

$$\mathrm{MMI}^p(K,\lambda,i,j) = S_0 \frac{e^{-\frac{r(2S_0+rK)}{2\sigma^2}}}{2\beta(\lambda)} \sum_{n=0}^{\infty} \frac{m_n^p(K,\lambda,i,j)}{n+1} \tag{34}$$

and

$$\mathrm{MMI}^a(K,\lambda) = \sum_{n=0}^{\infty} \frac{m_n^a(K,\lambda,i,j)}{2\beta(\lambda)}$$

with $\beta(\lambda) = \sqrt{\frac{4r^2}{\sigma^4} + \frac{8\lambda}{\sigma^2}}$.

The expression for $m_n^p(K,\lambda,i,j)$ and $m_n^a(K,\lambda,i,j)$ for the cases needed for main applications will be given in Appendix, the other cases can easily be computed by applying the same methodology.

Proof. See Appendix. □

The previous results permit the calculation of the following different quantities.

* MMI standing for **M**odified **M**GF **I**nverse

Theorem 5.2. *The density of Y_T is given by*

$$f^Y(y) = \mathrm{MMI}(y, 0, 0, 0) \tag{35}$$

The price of a digital put Asian option in cash, of payoff $1_{\{Y_T \le KT\}}$, is worth

$$e^{-rT} P(Y_T \le KT) = e^{-rT} \mathrm{MMI}(KT, 0, 1, 0) \tag{36}$$

The asset-digital put Asian option, defined by the payoff $\frac{Y_T}{T}.1_{\{Y_T \le KT\}}$, is given by

$$\frac{e^{-rT}}{T} P(Y_T \le KT) = -\frac{e^{-rT}}{T} \mathrm{MMI}(KT, 0, 1, 1) \tag{37}$$

The regular fixed strike put Asian option, of payoff $(K - \frac{Y_T}{T}).1_{\{Y_T \le K\}}$ is

$$K e^{-rT} \mathrm{MMI}(KT, 0, 1, 0) + \frac{e^{-rT}}{T} \mathrm{MMI}(KT, 0, 1, 1) \tag{38}$$

Remarks.

1. The corresponding call options can be retrieved from the previous results from the call-put parity. For example, the regular call Asian option is related to the put through

$$\mathrm{AC} = \mathrm{AP} + S_0 \frac{1 - e^{-rT}}{r} - e^{-rT} \tag{39}$$

2. It would be possible to deduce the price of floating-strike options in the same way.

6. Numerical performance

We choose to illustrate this series method for the joint density with an adaptation of the textbook Black-Scholes regular example $S_0^{\mathrm{BS}} = 100$, $r^{\mathrm{BS}} = 0.05$ and $\sigma^{\mathrm{BS}} = 20\%$. A square-root process with comparable parameters would be $S_0 = 100$, $a = 0$, $b = -0.05$ and $\sigma = 2$. With this choice of parameters, Figure 1 draws the joint density surface of (X_1, Y_1) when no absorption occurred.

For the same reference parameters values, we observe how the speed of convergence is affected by the variation of some key-parameters. Basically, this evolution is coherent with our previous remark highlighting that the main leading term is $e^{-\frac{\sigma^2 t^2}{2y} n^2}$. In the following tables, N represents the number of terms needed for the absolute difference♯ between the limit (series truncated at 50 terms) and the series truncated at N terms or more to be less than 10^{-4}.

We thus observe in Table 1 that N increases with y and the evolution is indeed rather quadratic than linear in N.

For increasing volatilities ($y = E(Y_1) \approx 102.54$), the decrease in N is also pronounced in Table 2.

To observe the evolution with t, a better understanding of the series behaviour can be obtained by studying the density at the moving point $y = E(Y_t) = X_0 \frac{e^{rt} - 1}{r}$, which exhibits the expected increase in speed (Table 3).

♯ We prefer holding the absolute difference as the stopping criteria rather than the relative difference, since the density can reach values quite close to 0.

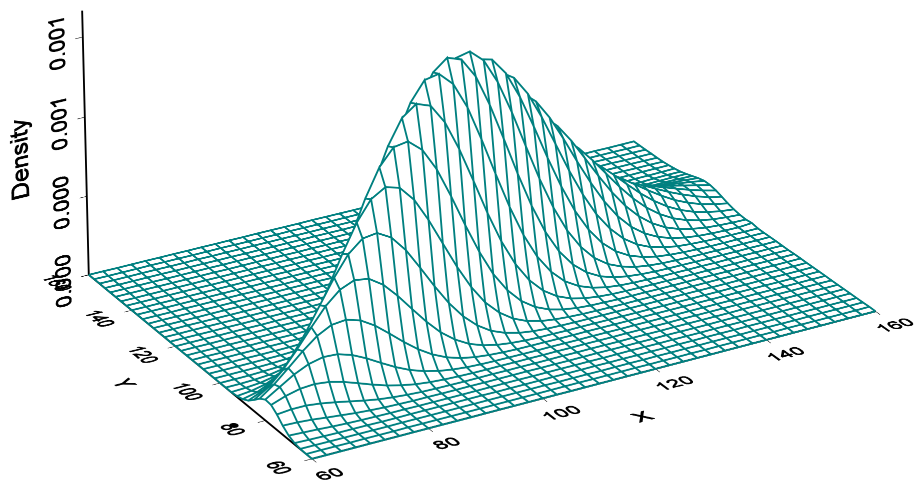

Figure 1. $S_0 = 100,\ a = 0,\ b = -0.05,\ \sigma = 2$ *and* $t = 1$

Y	80	90	100	110	120	130	140	150	160
N	1	20	31	37	42	46	51	55	57

Table 1. Evolution with y at $x = 100$.

σ	2	3	4	5	6
N	31	12	6	4	1

Table 2. Evolution with σ at $x = 100$ and $y = 102.54$.

t	1	2	3	4
$y = X_0 \frac{e^{rt}-1}{r}$	102.54	210.34	323.66	442.80
N	33	15	9	1

Table 3. Evolution with t at $x = 100$ and $y = E(Y_t)$.

To study Asian options, we prefer considering the reference cases widely used for benchmarking in the literature (see Geman and Eydeland [19], Dufresne [16], etc.) for the Black-Scholes model and adapt the diffusion parameters of the geometric Brownian motion to obtain similar levels of stock price and instantaneous local variance for the square-root process.

Case	r	σ	T	K	S_0	Moment	Intrinsic	Option_SR	Option_BS	N
1	0.05	0.69	1	2	1.9	1.8533	0	0.1902	0.1932	4
2	0.05	0.72	1	2	2.1	2.0484	0.1459	0.3098	0.3062	5
3	0.02	0.14	1	2	2	1.9801	0.0197	0.0197	0.05606	0
4	0.18	0.42	1	2	2	1.8303	0.1598	0.2189	0.2184	11
5	0.01	0.35	2	2	2	1.9752	0.0246	0.1725	0.1723	7
6	0.05	0.71	2	2	2	1.9033	0.0936	0.3339	0.3501	3

Table 4. Fixed-strike Asian options.

Table 4 collects the moments of $E(\frac{Y_T}{T})$, the intrinsic values and the prices of the Asian call options both for the square-root process (SR) and the geometric Brownian motion case (BS), the latter having been computing by numerical integration using the explicit formulae derived in Linetsky [24]. N represents the number of terms needed for the error to be inferior to 10^{-4}. Two points spring to the eyes from this table. Firstly, the series converges very rapidly for a broad range of cases. Secondly, the prices obtained for the square-root process are quite close to the Black and Scholes Asian ones. The differences between the models are to be computed in basis points rather than in percents, except in cases 3 and 6. The greater differences appearing in these two cases could be accounted for by the relatively greater importance of the tails there as the synthetic parameter $\sigma\sqrt{T}$ becomes very small for case 3 and quite large for case 6.

The series converges more and more rapidly as T or σ increase. This can be seen from Table 5 and Table 6. The results were cross-tested against two numerical Laplace inversion. We observed that these numerical inversion algorithms get strained and even become unreliable and unstable as the volatility or the maturity increase.

r	σ	T	K	S_0	Moment	Intrinsic	Option	N
0.05	0.71	0.1	2	2	1.9950	0.0050	0.0751	30
0.05	0.71	0.5	2	2	1.9752	0.0246	0.1725	7
0.05	0.71	1	2	2	1.9508	0.0484	0.2468	5
0.05	0.71	2	2	2	1.9033	0.0936	0.3339	3
0.05	0.71	5	2	2	1.7696	0.2120	0.3733	2

Table 5. Evolution with the maturity.

r	σ	T	K	S_0	Moment	Intrinsic	Option	N
0.05	0.1	1	2	2	1.9508	0.0484	0.0484	0
0.05	0.3	1	2	2	1.9508	0.0484	0.1207	18
0.05	0.5	1	2	2	1.9508	0.0484	0.1827	7
0.05	0.7	1	2	2	1.9508	0.0484	0.2446	5

Table 6. Evolution with the volatility.

Table 4 highlighted a strange or rather unexpected behaviour of N with respect to the volatility. The analysis of Table 6 enables us to explain it. The series converges slowly for small σ. But for too tiny σ, as nothing can really happen for such values, the put option is almost worthless and the series converges in one term.

We finally analyse the evolution with respect to the strike K. Table 7 shows results in agreement with our observations concerning the series representing the joint density. In fact, all the results obtained here corroborate the observations and comments related to the effect of the different parameters on the speed of convergence. This is due to the fact that the leading term remains the same.

r	σ	T	K	S_0	Moment	Intrinsic	Option	N
0.05	0.71	1	1	2	1.9508	0.9996	1.0017	2
0.05	0.71	1	1.5	2	1.9508	0.5240	0.5644	4
0.05	0.71	1	2	2	1.9508	0.0484	0.2468	5
0.05	0.71	1	2.5	2	1.9508	-0.4273	0.0822	6
0.05	0.71	1	3	2	1.9508	-0.9029	0.0210	8

Table 7. Evolution with the strike.

7. Conclusion

This paper provides simple analytical pricing formulae for fixed-strike arithmetic Asian options under the square-root process. Before any comparison concerning the actual numerical performance of our series representations, it should be highlighted that these formulae have the first advantage over numerical Laplace inversion that they can be used for systematic implementation. Indeed, the numerical Laplace inversion methods depend sometimes critically on free parameters whose optimal values can vary according to the problem parameters values. Moreover, analysing the numerical evaluation of this series, we found them very rapidly convergent in general but also more specially for large volatilities and maturities. As these formulae turn out to be simpler than in the Black and Scholes model, this approach is not only interesting on its own as a mean to capture the prices under this alternative model constituted by the square-root process but it can also be used as a benchmark to test against the numerics of the Black and Scholes model. As a final remark, it should be noticed that the valuation of floating-strike Asian options can also be derived from our results.

Appendix

To present the formulae for $m_n^a(K,\lambda,i,j)$ and $m_n^p(K,\lambda,i,j)$, we first need to define the following expressions.

▶Definitions and notations.

With the parameters $D_\xi = -\frac{r}{\sigma^2} - \frac{\xi}{2}$ and $\tilde{C} = \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{2K\alpha}}e^{\frac{(S_0+(n+1)\sigma^2 T)\xi}{2}}$, we build two multiply-indexed series G and M, M being built from another series $\tilde{B}$ with the following procedure.

- Induction rules
  - For $M_{p,q,n}$,

$$M_{p,q,n}(\xi,K)=\sqrt{\frac{2}{\pi}}\frac{e^{-\xi^2K\alpha}}{\sqrt{2K\alpha}^{p+q+3}}\tilde{B}_{p,q-1,n}(K)-\xi M_{p,q-1,n}(\xi,K) \tag{A.1}$$

  - For $\tilde{B}_{p,q,n}$,

$$\tilde{B}_{p,q,n}(K)=\Big\{1_{\{p=0\}}\Big[He_{q+2}\Big(\frac{S_0+(n+1)\sigma^2T}{2\sqrt{2K\alpha}}\Big)e^{-\frac{(S_0+(n+1)\sigma^2T)^2}{16K\alpha}}\Big]$$
$$+p\tilde{B}_{p-1,q,n}(K)+\frac{r}{\sigma^2}\tilde{B}_{p,q-1,n}(K)\Big\}2\sqrt{2K\alpha} \tag{A.2}$$

  - For $G_{p,n}(\xi,K)$,

$$G_{p,n}(\xi,K)=\sqrt{\frac{2}{\pi}}\frac{e^{-\xi^2K\alpha}}{\sqrt{2K\alpha}^{p+1}}He_p\Big(\frac{S_0+nT\sigma^2}{2\sqrt{2K\alpha}}\Big)e^{-\frac{(S_0+nT\sigma^2)^2}{16K\alpha}}$$
$$-\xi G_{p-1,n}(\xi,K) \tag{A.3}$$

  - For $DM_{p,q,n}$,

$$DM_{p,q,n}(\xi,K)=-\frac{4K\alpha}{r}M_{p,q+1,n}(\xi,K)-\frac{2(p+q+3)}{r}M_{p,q-1,n}(\xi,K)$$
$$+(\frac{S_0+(n+1)\sigma^2T-4K\alpha\xi}{r})M_{p,q,n}(\xi,K)+\frac{1}{r}M_{p+1,q-1,n}(\xi,K) \tag{A.4}$$

  - For $DG_{p,n}$,

$$DG_{p,n}(\xi,K)=-G_{p-1,n}(\xi,y)-\xi.DG_{p-1,n}(\xi,K)$$
$$-2K\xi e^{-\xi^2K\alpha}\sqrt{\frac{2}{\pi}}\frac{\alpha}{\sqrt{2K\alpha}^{p+1}}He_p(\frac{S_0+nT\sigma^2}{2\sqrt{2K\alpha}})e^{-\frac{(S_0+nT\sigma^2)^2}{16K\alpha}} \tag{A.5}$$

- Initialisations
  - For $M_{p,-p-3,n}$,
    - *If* $\xi\neq-\frac{2r}{\sigma^2}$,

$$\begin{aligned}M_{p,-p-3,n}(\xi,K)&=\tfrac{p}{D_\xi}M_{p-1,-p-2,n}(\xi,K)-\tfrac{\tilde{C}}{2D_\xi}\check{B}_{p,n}(D_\xi,K)\\ M_{0,-3,n}(\xi,K)&=\tfrac{\tilde{C}}{2D_\xi}(\check{B}_{0,n}(0,K)-\check{B}_{0,n}(D_\xi,K))\end{aligned} \tag{A.6}$$

∗ *If* $\xi = -\frac{2r}{\sigma^2}$,

$$M_{p,-p-3,n}\Big(-\frac{2r}{\sigma^2}, K\Big) = \frac{\tilde{C}}{2}\frac{1}{p+1}\check{B}_{p+1,n}(0,K) \tag{A.7}$$

∘ For $\tilde{B}_{p,-p-3,n}$,

$$\tilde{B}_{p,-p-3,n}(K) = -\Big(S_0 + (n+1)\sigma^2 T - 8K\alpha\frac{r}{\sigma^2}\Big)\tilde{B}_{p-1,-p-2,n}(K)$$

$$+8K\alpha\Big\{1_{\{p=1\}}e^{-\frac{(S_0+(n+1)\sigma^2 T)^2}{16K\alpha}} + (p-1)\tilde{B}_{p-2,-p-1,n}(K)\Big\} \tag{A.8}$$

and

$$\tilde{B}_{0,-3,n}(K) = (2\sqrt{\pi K\alpha})\mathrm{erfc}\Big(\frac{S_0 + (n+1)\sigma^2 T - 8K\alpha\frac{r}{\sigma^2}}{4\sqrt{K\alpha}}\Big) \tag{A.9}$$

∘ For the additional initialising terms $\check{B}_{p,n}(D_\xi, K)$,

$$\check{B}_{p,n}(D_\xi, K) = -(S_0 + (n+1)\sigma^2 T + 4K\alpha(\xi + 2D_\xi))\check{B}_{p-1,n}(D_\xi, K)$$

$$+8K\alpha\Big\{1_{\{p=1\}}e^{-\frac{(S_0+(n+1)\sigma^2 T+4K\alpha\xi)^2}{16K\alpha}} + (p-1)\check{B}_{p-2,n}(D_\xi, K)\Big\} \tag{A.10}$$

and

$$\check{B}_{0,n}(D_\xi, K) = (2\sqrt{\pi K\alpha})\mathrm{erfc}\Big(\frac{S_0 + (n+1)\sigma^2 T + 4K\alpha(\xi + 2D_\xi)}{4\sqrt{K\alpha}}\Big) \tag{A.11}$$

∘ For $G_{-1,n}(\xi, K)$,

$$G_{-1,n}(\xi, K) = \Big(e^{\frac{(S_0+nT\sigma^2)\xi}{2}}\Big)\mathrm{erfc}\Big(\frac{S_0 + nT\sigma^2 + 4K\alpha\xi}{4\sqrt{K\alpha}}\Big) \tag{A.12}$$

∘ For $DM_{p,-p-3,n}$,

$$DM_{p,-p-3,n}(\xi, K) = \frac{S_0 + (n+1)\sigma^2 T}{r}M_{p,-p-3,n}(\xi, K)$$

$$+\frac{1}{r}M_{p+1,-p-4,n}(\xi, K) - \frac{4K\alpha}{r}\tilde{C}\check{B}_{p,n}(D_\xi, K) \tag{A.13}$$

∘ For $DG_{-1,n}$,

$$DG_{-1,n}(\xi, K) = \sqrt{\frac{2}{\pi}}e^{\frac{S_0+nT\sigma^2}{2}\xi}e^{-\frac{(S_0+nT\sigma^2+4K\alpha\xi)^2}{16K\alpha}}$$

$$+\frac{S_0 + nT\sigma^2}{2}(e^{\frac{S_0+nT\sigma^2}{2}\xi})\mathrm{erfc}(\frac{S_0 + nT\sigma^2 + 4K\alpha\xi}{4\sqrt{K\alpha}}) \tag{A.14}$$

▶**The actual representation.**

We list below the cases needed for the valuation of the quantities given in Theorem 5.2.

♦ <u>**Case** $i = 0,\ j = 0$</u> This case was actually partly treated in Theorem 4.2 for the absorption case. For $m_n^p(y,0,0,0)$, it can simply be obtained by integration of $f^{S,Y}(s,y)$ against s on $\mathbb{R}^+$. Direct calculation leads to the following relations and to the recursions given above for $\tilde{B}$.

$$m_n^p(y,0,0,0) = \sum_{p=0}^{n} \frac{\binom{n+1}{n-p}}{p!}\Big(\frac{-1}{\sqrt{2y\alpha}}\Big)^p \sum_{q=0}^{n} \frac{\binom{n+1}{n-q}}{q!}\Big(\frac{-S_0}{\sqrt{2y\alpha}}\Big)^q \tilde{B}_{p,q,n}(y) \qquad \text{(A.15)}$$

and

$$m_n^a(y,0,0,0) = f_0^Y(y)$$

♦ <u>**Case** $i = 1,\ j = 0$</u>

The series terms are

$$m_n^p(K,\lambda,1,0) = \sum_{p=0}^{n} \frac{\binom{n+1}{n-p}}{p!} \sum_{q=0}^{n} \frac{\binom{n+1}{n-q}}{q!}(-S_0)^q(M_{p,q,n}(-\beta(\lambda),K) - M_{p,q,n}(\beta(\lambda),K)) \qquad \text{(A.16)}$$

and

$$m_n^a(K,\lambda,1,0) = \sum_{p=0}^{n} \frac{\binom{n}{p}(-S_0)^p}{p!}(G_{p,n}(-\beta(\lambda),K) - G_{p,n}(\beta(\lambda),K)$$
$$-G_{p,n+1}(-\beta,K) + G_{p,n+1}(\beta,K)) \qquad \text{(A.17)}$$

◊ <u>Sketch of the inversion</u>

For the non-absorbed part, we want to invert the Laplace transform

$$e^{-\frac{r^2y}{2\sigma^2} - \frac{r(S_0-s)}{\sigma^2}} \frac{S_0}{\mu} \frac{4\gamma^2 e^{-\gamma T}}{\sigma^4} e^{-(s+S_0)\frac{\gamma}{\sigma^2}} \sum_{n=0}^{\infty} \frac{L_n^1\left(\frac{2s\gamma}{\sigma^2}\right) L_n^1\left(\frac{2S_0\gamma}{\sigma^2}\right)}{n+1} e^{-\gamma n t} \qquad \text{(A.18)}$$

with

$$\gamma = \sqrt{r^2 + 2\sigma^2(\lambda+\mu)}$$

Exploiting the relation

$$\frac{1}{\mu} = \frac{8}{\sigma^2(\frac{2\gamma}{\sigma^2} - \beta(\lambda))(\frac{2\gamma}{\sigma^2} + \beta(\lambda))} = \frac{4}{\beta(\lambda)\sigma^2}\left(\frac{1}{\frac{2\gamma}{\sigma^2} - \beta(\lambda)} - \frac{1}{\frac{2\gamma}{\sigma^2} + \beta(\lambda)}\right)$$

we are mainly concerned about inverses of elements of the type

$$\Big(\frac{2\gamma}{\sigma^2}\Big)^{p+q+2} \frac{e^{-\alpha_n \frac{2\gamma}{\sigma^2}}}{\frac{2\gamma}{\sigma^2} + \xi} \qquad \text{(A.19)}$$

specialising to $\xi^2 = (\pm\beta(\lambda))^2 = \frac{4r^2}{\sigma^4} + \frac{8\lambda}{\sigma^2}$. Given that the inverse of $(\sqrt{\mu})^n e^{-\nu\sqrt{\mu}}$ is $\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{2y}^{n+2}} He_{n+1}(\frac{\nu}{\sqrt{2y}}) e^{-\frac{\nu^2}{4y}}$, then it follows that the inverse of (A.19) is

$$\int_0^{\infty} e^{-\nu\xi} \sqrt{\frac{2}{\pi}} \frac{\alpha}{(\sqrt{2y\alpha})^{p+q+4}} e^{-\xi^2 y\alpha} He_{p+q+3}\Big(\frac{\alpha_n+\nu}{\sqrt{2y\alpha}}\Big) e^{-\frac{(\alpha_n+\nu)^2}{4y\alpha}} d\nu \qquad \text{(A.20)}$$

Calculating this integral leads to the results and the recursions given above for M.

♦ **Case** $i = 1,\ j = 1$

This case is treated by differentiation with respect to λ of the expressions given for the case $i = 1,\ j = 0$. Straightforward calculation gives

$$m_n^p(K,0,1,1) = \sum_{p=0}^{n} \frac{\binom{n+1}{n-p}}{p!} \sum_{q=0}^{n} \frac{\binom{n+1}{n-q}}{q!} (-S_0)^q (DM_{p,q,n}(-\beta(\lambda),K) - DM_{p,q,n}(\beta(\lambda),K)) \quad \text{(A.21)}$$

and

$$m_n^a(K,0,1,1) = \sum_{p=0}^{n} \frac{\binom{n}{p}(-S_0)^p}{p!} (DG_{p,n}(-\beta(\lambda),K) - DG_{p,n}(\beta(\lambda),K)$$
$$-DG_{p,n+1}(-\beta,K) + DG_{p,n+1}(\beta,K)) \quad \text{(A.22)}$$

For the detail of intermediate calculations, please refer to [29].

As a final remark, we notice that the methodology outlined to obtain these three cases can be applied for other values of i and j.

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