Pricing of catastrophe reinsurance & derivatives using the Cox process with shot noise intensity

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Abstract

We use the Cox process (or a doubly stochastic Poisson process) to model the claim arrival process for catastrophic events. The shot noise process is used for the claim intensity function within the Cox process. The Cox process with shot noise intensity is examined by piecewise deterministic Markov process theory. We apply the model to price stop-loss catastrophe reinsurance contract & catastrophe insurance derivatives. The asymptotic distribution of the claim intensity is used to derive pricing formulae for stop-loss reinsurance contract for catastrophic events & catastrophe insurance derivatives. We assume that there is an absence of arbitrage opportunities in the market to obtain the gross premium for stop-loss reinsurance contract and arbitrage-free prices for insurance derivatives. This can be achieved by using an equivalent martingale probability measure in the pricing models. The Esscher transform is used for this purpose.

Keywords: The Cox process; Shot noise process; Piecewise deterministic Markov process; Stop-loss reinsurance contract; Catastrophe insurance derivatives; Equivalent martingale probability measure; Esscher transform.

JEL Classification: G13, G22

Mathematics Subject Classification (2000): 60G10, 60G44, 60G55 60G57, 60J75

1. Introduction

Insurance companies have traditionally used reinsurance contracts to hedge themselves against losses from catastrophic events. During the last decade, the high level of worldwide catastrophe losses in terms of frequency and severity had a marked effect on the reinsurance market. The catastrophes such as Storm Daria (Europe 1990), Hurricane Andrew (USA 1992) and the Kobe earthquake (Japan 1995) have impacted the profitability and capital bases of reinsurance companies. Some of these companies have withdrawn from the market while others have reduced the level of catastrophe cover they are willing to provide (Booth, 1997).

In the early 1990s, some believed that there was under-capacity provided by the reinsurance market. Some investment banks, particularly in the US, recognised the opportunities that existed in the reinsurance market. Through their large capital bases the investment banks were able to offer alternative reinsurance products. One of the alternative reinsurance products is catastrophe insurance futures and options on catastrophe insurance futures, traded on a quarterly basis (Jan-Mar, Apr-June, July-Sep and Oct-Dec), introduced by The Chicago Board of Trade in December 1992.

The CBOT devised a loss ratio index as the underlying instrument for catastrophe insurance futures and options contracts. The Insurance Service Office calculates the index from loss data reported by at least 25 selected companies (CBOT, 1994, 1995a and 1995b). The loss ratio index is the reported losses incurred in a given quarter and reported by the end of the following quarter, L_t, divided by one fourth of the premiums received in the previous year, Π , i.e. L_t/Π.

The value of an insurance future, F_t, at maturity t is the nominal contract value, US$25,000, times the loss ratio index capped at 2, i.e.

$$F_t = 25,000 \times Min\left(\frac{L_t}{\Pi}, 2\right). \tag{1.1}$$

The CBOT capped the maximum loss ratio at 200% in order to limit the credit risk from unexpected huge losses and to make the contract look like a non-proportional reinsurance policy. However, to date there has not been an incident where the maximum loss ratio has been reached; the highest estimated loss ratio being 179% for Hurricane Andrew. Therefore ignoring the maximum loss ratio, the value of a catastrophe insurance call option on the future of the option, P_t, at maturity t is given by

$$P_t = Max\,(F_t - K, 0) = (F_t - K)^+ = \left(25,000 \times \frac{L_t}{\Pi} - K\right)^+ = \frac{25,000}{\Pi}\,(L_t - B)^+ \tag{1.2}$$

where K is the exercise price and $B = \frac{\Pi K}{25,000}$.

Let Z_i, $i = 1, 2, \cdots$, be the claim amounts, which are assumed to be independent and identically distributed with distribution function $H(z)$ $(z > 0)$. The total loss excess over b, which is a retention limit, up to time t is

$$(C_t - b)^+ \tag{1.3}$$

where $C_t = \sum_{i=1}^{N_t} Z_i$, N_t is the number of claims up to time t and $(C_t - b)^+ = Max\,(C_t - b, 0)$. Therefore the stop-loss reinsurance premium at present time 0 is

$$E\left\{(C_t - b)^+\right\} \tag{1.4}$$

where the expectation is calculated under an appropriate probability measure. Throughout the paper, for simplicity, we assume interest rates to be constant.

If we assume that $L_t = C_t$, the price of the insurance future at time 0 is

$$E\left[25,000 \times Min\left(\frac{C_t}{\Pi}, 2\right)\right] \tag{1.5}$$

and ignoring the maximum loss ratio, the price at time 0 of the call option on the insurance future is

$$\frac{25,000}{\Pi} E\left[(C_t - B)^+\right] \tag{1.6}$$

where the expectations are calculated under an appropriate probability measure. If we substitute 'b' with 'B' in the formula of the stop-loss reinsurance premium at time 0 excluding $\frac{25,000}{\Pi}$, the two formulae (1.4) and (1.6) are equivalent.

There has been discussion and research into the possibility of using catastrophe insurance futures and options contracts rather than conventional reinsurance contracts (Lomax & Lowe, 1994, Smith, 1994, Ryan, 1994, Sutherland, 1995, Kielholz & Durrer, 1997 and Smith, Canelo & Di Dio, 1997). The competitiveness of the reinsurance market emphasises the need for an appropriate pricing model for reinsurance contracts and catastrophe insurance derivatives. This also causes reinsurance companies to assess their strategies for the type of products offered to the market.

2. Doubly stochastic Poisson process and shot noise process

In insurance modelling, the Poisson process has been used as a claim arrival process. Extensive discussion of the Poisson process, from both applied and theoretical viewpoints, can be found in Cramér (1930), Cox & Lewis (1966), Bühlmann (1970), Cinlar (1975), and Medhi (1982). However there has been a significant volume of literature that questions the appropriateness of the Poisson process in insurance modelling (Seal, 1983 and Beard et al., 1984) and more specifically for rainfall modelling (Smith, 1980 and Cox & Isham, 1986).

For catastrophic events, the assumption that resulting claims occur in terms of the Poisson process is inadequate as it has deterministic intensity. Therefore an alternative point process needs to be used to generate the claim arrival process. We will employ a doubly stochastic Poisson process, or the Cox process (Cox, 1955, Bartlett, 1963, Serfozo, 1972, Grandell, 1976, 1991, Bremaud, 1981 and Lando, 1994). Under a doubly stochastic Poisson process, or the Cox process, the claim intensity function is assumed to be stochastic.

The doubly stochastic Poisson process provides flexibility by letting the intensity not only depend on time but also allowing it to be a stochastic process. Therefore the doubly stochastic Poisson process can be viewed as a two step randomisation procedure. A process λ_t is used to generate another process N_t by acting as its intensity. That is, N_t is a Poisson process conditional on λ_t which itself is a stochastic process (if λ_t is deterministic then N_t is a Poisson process).

Many alternative definitions of a doubly stochastic Poisson process can be given. We will offer the one adopted by Bremaud (1981).

Definition 2.1 *Let (Ω, F, P) be a probability space with information structure given by $F = \{\Im_t,\ t \in [0,T]\}$. Let N_t be a point process adapted to F. Let λ_t be a non-negative process adapted to F such that*

$$\int_0^t \lambda_s ds < \infty \quad \textit{almost surely} \quad (\textit{no explosions}).$$

If for all $0 \leq t_1 \leq t_2$ *and* $u \in \Re$

$$E\left\{e^{iu\left(N_{t_2}-N_{t_1}\right)}|\Im_{t_2}^{\lambda}\right\} = \exp\left\{\left(e^{iu}-1\right)\int_{t_1}^{t_2}\lambda_s ds\right\} \tag{2.1}$$

then N_t *is call a* $\Im_t$*-doubly stochastic Poisson process with intensity* λ_t *where* $\Im_t^{\lambda} = \sigma\left\{\lambda_s;\ s \leq t\right\}$.

Equation (2.1) gives us

$$\Pr\left\{N_{t_2}-N_{t_1}=k|\lambda_s; t_1 \leq s \leq t_2\right\} = \frac{\exp\left(-\int_{t_1}^{t_2}\lambda_s ds\right)\left(\int_{t_1}^{t_2}\lambda_s ds\right)^k}{k!}. \tag{2.2}$$

Now consider the process $X_t = \int_0^t \lambda_s ds$ (*the aggregated process*), then from (2.2) we can easily find that

$$E\left(\theta^{N_{t_2}-N_{t_1}}\right) = E\left\{e^{-(1-\theta)\left(X_{t_2}-X_{t_1}\right)}\right\}. \tag{2.3}$$

Equation (2.3) suggests that the problem of finding the distribution of N_t, the point process, is equivalent to the problem of finding the distribution of X_t, the aggregated process. It means that we just have to find the p.g.f. (probability generating function) of N_t to retrieve the m.g.f. (moment generating function) of X_t and vice versa.

Claims arising from catastrophic events depend on the intensity of natural disasters (e.g. flood, windstorm, hail, and earthquake). One of the processes that can be used to measure the impact of catastrophic events is the shot noise process (Cox & Isham, 1980, 1986 and Klüppelberg & Mikosch, 1995). The shot noise process is particularly useful in the claim arrival process as it measures the frequency, magnitude and time period needed to determine the effect of catastrophic events. As time passes, the shot noise process decreases as more and more claims are settled. This decrease continues until another catastrophe occurs which will result in a positive jump in the shot noise process. Therefore the shot noise process can be used as the parameter of the doubly stochastic Poisson process to measure the number of claims due to catastrophic event, i.e. we will use it as a claim intensity function to generate the Cox process. We will adopt the shot noise process used by Cox & Isham (1980):

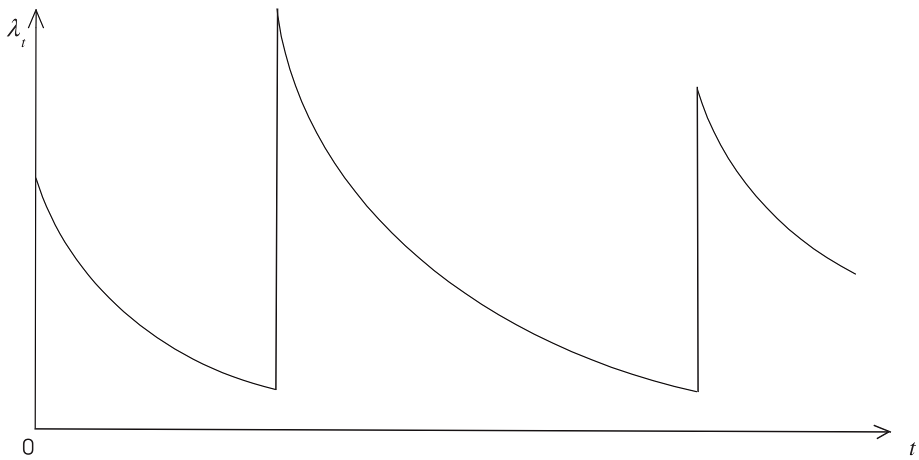
$$\lambda_t = \lambda_0 e^{-\delta t} + \sum_{\substack{\text{all } i \\ s_i \leq t}} y_i e^{-\delta(t-s_i)}$$

where:

λ_0	initial value of λ
y_i	jump size of catastrophe i where $E\left(y_i\right) < \infty$
	(i.e. magnitude of contribution of catastrophe i to intensity)
s_i	time at which catastrophe i occurs, where $s_i < t < \infty$
δ	exponential decay
ρ	the rate of catastrophe jump arrival.

This is illustrated in Figure 1.

Figure 1: Graph illustrating a shot noise process



The piecewise deterministic Markov processes theory developed by Davis (1984) is a powerful mathematical tool for examining non-diffusion models. From now on, we present definitions and important properties of the Cox and shot noise processes with the aid of piecewise deterministic processes theory (Dassios, 1987 and Dassios & Embrechts, 1989). This theory is used to calculate the distribution of the number of claims and the mean of the number of claims. These are important factors in the pricing of any reinsurance product.

The three parameters of the shot noise process described are homogeneous in time. We are now going to generalise the shot noise process by allowing the parameters to depend on time. The rate of jump arrivals, $\rho(t)$, is bounded on all intervals $[0, t)$ (no explosions). $\delta(t)$ is the rate of decay and the distribution function of jump sizes at any time t is $G(y;t)$ $(y > 0)$ with $E(y;t) = \mu_1(t) = \int_0^\infty y dG(y;t)$. We assume that $\delta(t)$, $\rho(t)$ and $G(y;t)$ are all Riemann integrable functions of t and are all positive.

The generator of the process (X_t, N_t, λ_t, t) acting on a function $f(x, n, \lambda, t)$ belonging to its domain is given by

$$\begin{aligned} A\ f(x,n,\lambda,t) &= \frac{\partial f}{\partial t} + \lambda \frac{\partial f}{\partial x} + \lambda \left[f(x,n+1,\lambda,t) - f(x,n,\lambda,t) \right] - \delta(t) \lambda \frac{\partial f}{\partial \lambda} \\ &+ \rho(t) \left[\int_0^\infty f(x,n,\lambda+y,t)\, dG(y;t) - f(x,n,\lambda,t) \right]. \end{aligned} \tag{2.4}$$

For $f(x,n,\lambda,t)$ to belong to the domain of the generator A, it is sufficient that $f(x,n,\lambda,t)$ is differentiable w.r.t. x, λ, t for all x, n, λ, t and that $\left| \int_0^\infty f(\cdot, \lambda + y, \cdot)\, dG(y;t) - f(\cdot, \lambda, \cdot) \right| < \infty$.

Let us find a suitable martingale in order to derive the Laplace transforms of the distribution of λ_t, X_t and the p.g.f. (probability generating function) of N_t at time t.

Theorem 2.2 *Considering constants k and v such that $k \geq 0$ and $v \geq 0$,*

$$\exp\left(-vX_t\right) \cdot \exp\left[-\left\{ke^{\triangle(t)} - ve^{\triangle(t)} \int_0^t e^{-\triangle(r)} dr\right\} \lambda_t\right]$$

$$\cdot \exp\left[\int_0^t \rho(s) \left[1 - \hat{g}\left\{ke^{\triangle(s)} - ve^{\triangle(t)} \int_0^s e^{-\triangle(r)} dr; s\right\}\right] ds\right] \tag{2.5}$$

is a martingale where $\hat{g}(u;s) = \int_0^\infty e^{-u\, y} dG(y;s)$ and $\triangle(t) = \int_0^t \delta(s)\, ds$.

Proof. From (2.4) $f(x, \lambda, t)$ has to satisfy $Af = 0$ for it to be a martingale. Setting $f = e^{-vx} e^{-A(t)\lambda} e^{R(t)}$ we get the equation

$$-\lambda A'(t) + R'(t) - \lambda v + \delta(t) \lambda A(t) + \rho(t) \left[\hat{g}\left\{A(t); t\right\} - 1\right] = 0 \tag{2.6}$$

and solving (2.6) we get

$$A(t) = ke^{\triangle(t)} - ve^{\triangle(t)} \int_0^t e^{-\triangle(r)} dr$$

and

$$R(t) = \int_0^t \rho(s) \left[1 - \hat{g}\left\{ke^{\triangle(s)} - ve^{\triangle(s)} \int_0^s e^{-\triangle(r)} dr; s\right\}\right] ds.$$

Put $\triangle(t) = \int_0^t \delta(s)\, ds$ and the result follows. ■

Let us assume that $\delta(t) = \delta$ throughout the rest of this paper.

Corollary 2.3 *Let $v_1 \geq 0$, $v_2 \geq 0$, $v \geq 0$, $0 \leq \theta \leq 1$. Then*

$$E\left\{e^{-v_1\left(X_{t_2} - X_{t_1}\right)} e^{-v_2 \lambda_{t_2}} | X_{t_1}, \lambda_{t_1}\right\}$$

$$= \exp\left[-\left\{\frac{v_1}{\delta} + \left(v_2 - \frac{v_1}{\delta}\right) e^{-\delta(t_2 - t_1)}\right\} \lambda_{t_1}\right]$$

$$\cdot \exp\left[-\int_{t_1}^{t_2} \rho(s) \left[1 - \hat{g}\left\{\frac{v_1}{\delta} + \left(v_2 - \frac{v_1}{\delta}\right) e^{-\delta(t_2 - s)}; s\right\}\right] ds\right] \tag{2.7}$$

and

$$E\left\{\theta^{\left(N_{t_2} - N_{t_1}\right)} e^{-v\lambda_{t_2}} | N_{t_1}, \lambda_{t_1}\right\}$$

$$= \exp\left[-\left\{\frac{1-\theta}{\delta} + \left(v - \frac{1-\theta}{\delta}\right) e^{-\delta(t_2 - t_1)}\right\} \lambda_{t_1}\right]$$

$$\cdot \exp\left[-\int_{t_1}^{t_2} \rho(s) \left[1 - \hat{g}\left\{\frac{1-\theta}{\delta} + \left(v - \frac{1-\theta}{\delta}\right) e^{-\delta(t_2 - s)}; s\right\}\right] ds\right]. \tag{2.8}$$

Proof. (2.7) follows immediately where we set $v = v_1$, $k = \frac{v_1}{\delta} + \left(v_2 - \frac{v_1}{\delta}\right) e^{-\delta t_2}$ in theorem 2.2. (2.8) follows from (2.7) and (2.3). ■

Now we can easily obtain the Laplace transforms of the distribution of λ_t, X_t and the p.g.f. (probability generating function) of N_t at time t.

Corollary 2.4 *The Laplace transforms of the distribution of λ_t and X_t are given by*

$$E\left\{e^{-v\lambda_{t_2}}|\lambda_{t_1}\right\} = \exp\left[-ve^{-\delta(t_2-t_1)}\lambda_{t_1}\right] \cdot \exp\left[-\int_{t_1}^{t_2}\rho(s)\left[1-\hat{g}\left\{ve^{-\delta(t_2-s)};s\right\}\right]ds\right], \tag{2.9}$$

$$E\left\{e^{-v\left(X_{t_2}-X_{t_1}\right)}|\lambda_{t_1}\right\} = \exp\left[-\frac{v}{\delta}\left\{1-e^{-\delta(t_2-t_1)}\right\}\lambda_{t_1}\right] \cdot \exp\left[-\int_{t_1}^{t_2}\rho(s)\left[1-\hat{g}\left\{\frac{v}{\delta}\left(1-e^{-\delta(t_2-s)}\right);s\right\}\right]ds\right] \tag{2.10}$$

and the probability generating function of N_t is given by

$$E\left\{\theta^{\left(N_{t_2}-N_{t_1}\right)}|\lambda_{t_1}\right\} = \exp\left[-\frac{1-\theta}{\delta}\left\{1-e^{-\delta(t_2-t_1)}\right\}\lambda_{t_1}\right] \cdot \exp\left[-\int_{t_1}^{t_2}\rho(s)\left[1-\hat{g}\left\{\frac{1-\theta}{\delta}\left(1-e^{-\delta(t_2-s)}\right);s\right\}\right]ds\right]. \tag{2.11}$$

Proof. If we set $v_1 = 0$ in (2.7) then (2.9) follows. If we also set $v_2 = 0$, $v = 0$ in (2.7) and (2.8) then (2.10) and (2.11) follow. ■

Let us obtain the asymptotic distributions of λ_t at time t from (2.9), provided that the process started sufficiently far in the past. In this context we interpret it as the limit when $t \to -\infty$. In other words, if we know λ at $'-\infty'$ and no information between $'-\infty'$ to present time t, $'-\infty'$ asymptotic distribution of λ_t can be used as the distribution of λ_t.

Lemma 2.5 *Assume that $\lim_{t\to-\infty}\rho(t) = \rho$ and $\lim_{t\to-\infty}\mu_1(t) = \mu_1$. Then the $'-\infty'$ asymptotic distribution of λ_t has Laplace transform*

$$E\left(e^{-v\lambda_{t_1}}\right) = \exp\left[-\int_{-\infty}^{t_1}\rho(s)\left[1-\hat{g}\left\{ve^{-\delta(t_1-s)};s\right\}\right]ds\right]. \tag{2.12}$$

Proof. From (2.9), it is easy to check that if $\lim_{t\to-\infty}\rho(t) = \rho$ and $\lim_{t\to-\infty}\mu_1(t) = \mu_1$ then $\int_{-\infty}^{t}\rho(s)\left[1-\hat{g}\left\{ve^{-\delta(t-s)};s\right\}\right]ds < \infty$. Therefore the result follows immediately. ■

It will be interesting to find the Laplace transforms of the distribution of λ_t, X_t and the p.g.f. (probability generating function) N_t at time t, using a specific jump size distribution of $G(y;t)$ $(y > 0)$. We use an exponential jump size distribution, i.e. $g(y;t) = \left(\alpha + \gamma e^{\delta t}\right) e^{-\left(\alpha+\gamma e^{\delta t}\right)y}$, $y > 0$, $-\alpha e^{-\delta t} < \gamma \leq 0$. In practice, other thick-tail distributions such as log-normal, gamma and Pareto, etc. can also be applied for jump size distribution of $G(y;t)$ $(y > 0)$. Examining the effect on stop-loss reinsurance premiums and prices for catastrophe insurance derivatives caused by changes in the jump size distribution will be also of interest.

Let us assume that $\rho(t) = \rho\frac{\alpha}{\alpha+\gamma e^{\delta t}}$. The reason for this particular assumption will become apparent later when we change the probability measure.

Theorem 2.6 *Let the jump size distribution be exponential, i.e.* $g(y;t) = \left(\alpha + \gamma e^{\delta t}\right)\exp\left\{-\left(\alpha + \gamma e^{\delta t}\right)y\right\}$, $y > 0$, $\alpha e^{-\delta t} < \gamma \leq 0$, *and assume that* $\rho(t) = \rho\frac{\alpha}{\alpha+\gamma e^{\delta t}}$. *Then*

$$E\left\{e^{-v\lambda_{t_1}}|\lambda_{t_0}\right\} = \exp\left\{-v\lambda_{t_0}e^{-\delta(t_1-t_0)}\right\}\cdot\left(\frac{\gamma e^{\delta t_0}+\alpha e^{-\delta(t_1-t_0)}}{\gamma e^{\delta t_0}+\alpha}\right)^{\frac{\rho}{\delta}} \cdot\left(\frac{\gamma e^{\delta t_0}+ve^{-\delta(t_1-t_0)}+\alpha}{\gamma e^{\delta t_0}+(v+\alpha)e^{-\delta(t_1-t_0)}}\right)^{\frac{\rho}{\delta}}, \tag{2.13}$$

$$E\left\{e^{-v\left(X_{t_2}-X_{t_1}\right)}|\lambda_{t_1}\right\} = \exp\left[-\frac{v}{\delta}\left\{1-e^{-\delta(t_2-t_1)}\right\}\lambda_{t_1}\right]\left(\frac{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}{\gamma e^{\delta t_1}+\alpha}\right)^{\frac{\rho}{\delta}} \cdot\left(\frac{\gamma e^{\delta t_1}+\alpha+\frac{v}{\delta}\left(1-e^{-\delta(t_2-t_1)}\right)}{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}\right)^{\frac{\alpha\rho}{\delta\alpha+v}} \tag{2.14}$$

and

$$E\left\{\theta^{\left(N_{t_2}-N_{t_1}\right)}|\lambda_{t_1}\right\} = \exp\left[-\frac{1-\theta}{\delta}\left\{1-e^{-\delta(t_2-t_1)}\right\}\lambda_{t_1}\right]\left(\frac{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}{\gamma e^{\delta t_1}+\alpha}\right)^{\frac{\rho}{\delta}} \cdot\left(\frac{\gamma e^{\delta t_1}+\alpha+\frac{1-\theta}{\delta}\left(1-e^{-\delta(t_2-t_1)}\right)}{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}\right)^{\frac{\alpha\rho}{\delta\alpha+(1-\theta)}}. \tag{2.15}$$

If λ_t *is* '$-\infty$' *asymptotic,*

$$E\left(e^{-v\lambda_{t_1}}\right) = \left(\frac{\gamma+\alpha e^{-\delta t_1}}{\gamma+(v+\alpha)e^{-\delta t_1}}\right)^{\frac{\rho}{\delta}}, \tag{2.16}$$

$$E\left\{e^{-v\left(X_{t_2}-X_{t_1}\right)}\right\} = \left(\frac{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}{\gamma e^{\delta t_1}+\alpha+\frac{v}{\delta}\left(1-e^{-\delta(t_2-t_1)}\right)}\right)^{\frac{\rho}{\delta}} \cdot\left(\frac{\gamma e^{\delta t_1}+\alpha+\frac{v}{\delta}\left(1-e^{-\delta(t_2-t_1)}\right)}{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}\right)^{\frac{\alpha\rho}{\delta\alpha+v}} \tag{2.17}$$

and

$$E\left\{\theta^{\left(N_{t_2}-N_{t_1}\right)}\right\}=\left(\frac{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}{\gamma e^{\delta t_1}+\alpha+\frac{1-\theta}{\delta}\left(1-e^{-\delta(t_2-t_1)}\right)}\right)^{\frac{\rho}{\delta}}$$
$$\cdot\left(\frac{\gamma e^{\delta t_1}+\alpha+\frac{1-\theta}{\delta}\left(1-e^{-\delta(t_2-t_1)}\right)}{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}\right)^{\frac{\alpha\rho}{\delta\alpha+(1-\theta)}}. \tag{2.18}$$

Proof. If we set $\rho(t)=\rho\frac{\alpha}{\alpha+\gamma e^{\delta t}}$ and $g(y;t)=\left(\alpha+\gamma e^{\delta t}\right)\exp\left\{-\left(\alpha+\gamma e^{\delta t}\right)y\right\}$, $y>0$, $-\alpha e^{-\delta t}<\gamma\leq 0$ in (2.9), (2.10) and (2.11) then (2.13), (2.14) and (2.15) follow. Let $t_0\rightarrow-\infty$ in (2.13) then (2.16) follows immediately, from which (2.17) and (2.18) follow. ■

Now let us derive the expected value of claim number process, N_t.

Theorem 2.7 *The expectation of claim number process,* N_t *is given by*

$$E\left(N_{t_2}-N_{t_1}\right)=\int_{t_1}^{t_2}E\left(\lambda_s\right)ds$$
$$=\left(\frac{1-e^{-\delta(t_2-t_1)}}{\delta}\right)E\left(\lambda_{t_1}\right)+\frac{1}{\delta}\int_{t_1}^{t_2}\left(1-e^{-\delta(t_2-s)}\right)\rho(s)\mu_1(s)\,ds. \tag{2.19}$$

If the jump size distribution is exponential, i.e. $g(y;t)=\left(\alpha+\gamma e^{\delta t}\right)\exp\left\{-\left(\alpha+\gamma e^{\delta t}\right)y\right\}$, $y>0$, $-\alpha e^{-\delta t}<\gamma\leq 0$ *with* $\rho(t)=\rho\frac{\alpha}{\alpha+\gamma e^{\delta t}}$ *and* λ_t *is* '$-\infty$' *asymptotic, then*

$$E\left(N_{t_2}-N_{t_1}\right)=\frac{\rho}{\delta\alpha}\left(t_2-t_1\right)-\frac{\rho}{\delta^2\alpha}\ln\left(\frac{\gamma e^{\delta t_2}+\alpha}{\gamma e^{\delta t_1}+\alpha}\right). \tag{2.20}$$

Proof. Using (2.4), we can obtain

$$E\left(\lambda_{t_1}|\lambda_{t_0}\right)=\lambda_{t_0}e^{-\delta(t_1-t_0)}+e^{-\delta t_1}\int_{t_0}^{t_1}e^{\delta s}\rho(s)\mu_1(s)\,ds \tag{2.21}$$

and by letting $t_0\rightarrow-\infty$ in (2.21), we can obtain the '$-\infty$' asymptotic expected value of λ_t;

$$E\left(\lambda_{t_1}\right)=e^{-\delta t_1}\int_{-\infty}^{t_1}e^{\delta s}\rho(s)\mu_1(s)\,ds. \tag{2.22}$$

From (2.2)

$$E\left(N_{t_2}-N_{t_1}\right)=\int_{t_1}^{t_2}E\left(\lambda_s\right)ds. \tag{2.23}$$

Condition on λ_{t_1} in (2.23) and use (2.21) then (2.19) follows immediately. If the jump size distribution is exponential, i.e. $g(y;t)=\left(\alpha+\gamma e^{\delta t}\right)\exp\left\{-\left(\alpha+\gamma e^{\delta t}\right)y\right\}$,

$y > 0$, $-\alpha e^{-\delta t} < \gamma \leq 0$ and $\rho(t) = \rho \frac{\alpha}{\alpha + \gamma e^{\delta t}}$ the $' - \infty'$ asymptotic expected value of λ_t becomes

$$E(\lambda_{t_1}) = \frac{\rho}{\delta(\alpha + \gamma e^{\delta t_1})}. \tag{2.24}$$

Therefore set $\rho(s) = \rho \frac{\alpha}{\alpha + \gamma e^{\delta s}}$, $\mu_1(s) = \frac{1}{\alpha + \gamma e^{\delta s}}$ in (2.19) and use (2.24), then (2.20) follows immediately. ■

The shot noise process λ_t has been taken to be unobservable. This implies that catastrophes can only be observed on the basis of an observed process N_t of reported claims. However in practical situation, as we observe catastrophes, we can trace back which and how many claims are caused by them. Therefore "the filtering problem" can be applied to obtain the best estimate λ_t on the basis of the observed process N_t of reported claims or observed catastrophes (Dassios & Jang, 1998a and Jang, 1998).

3. No-arbitrage, the Esscher transform and change of probability measure

Harrison & Kreps (1979) and Harrison & Pliska (1981) launched the approach for the pricing and analysis of movements of the financial derivatives whose prices are determined by the price of the underlying assets. Their mathematical framework originates from the idea of risk-neutral, or non-arbitrage valuation of Cox & Ross (1976). Sondermann (1991) introduced the non-arbitrage approach for the pricing of reinsurance contracts. He proved that if there is no arbitrage opportunities in the market, reinsurance premiums are calculated by the expectation of their value at maturity with respect to a new probability measure and not with respect to the original probability measure. This new probability measure is called the equivalent martingale probability measure. The existence of an equivalent martingale probability measure is equivalent to the assumption of no arbitrage opportunities in the market.

Let us assume that there exist a liquid reinsurance market, i.e. at any time $t \leq T$, the insurer can decide to sell any part of the risk of C_u, $t \leq u \leq T$, based on the information available at time t where C_u follows doubly stochastic compound Poisson process with shot noise intensity defined on the probability space (Ω, F, P). Let PR_u denote the total value of premiums received up to time u defined on (Ω, F, P) and define the reinsurance strategy that is adopted from Embrechts & Meister (1995).

Definition 3.1 *Let $s \in [0, T]$, a reinsurance strategy $\{\xi_u;\ t \leq u \leq T\}$ is a predictable stochastic process on (Ω, F, P) with $0 \leq \xi_u \leq 1$ for all $u \in [t, T]$.*

Assuming that interest rates is constant, let us define the specified process R_t, $0 \leq t \leq T$, given by

$$R_t = PR_t - C_t \quad (0 \leq t \leq T)$$

denoting the net surplus from insurance business up to time t. If the insurer choose at time t some reinsurance strategy $\{\xi_u;\ t \leq u \leq T\} \in H_t$ where H_t denotes the set of all reinsurance strategies starting at time t, then the company's final gain at time T is given by

$$G_T(\xi) = \int_t^T \xi_u \, dR_u$$

where it is assumed that the reinsurer receives direct insurer's premiums for his engagement. A strategy $\{\xi_u;\ t \leq u \leq T\}$ allowing for a possible profit without the possibility of a loss is called an arbitrage strategy, i.e. a strategy $\{\xi_u;\ t \leq u \leq T\}$ satisfying

(i) $G_T(\xi) \geq 0$, $P-$ almost surely

(ii) $E_P\left[G_T(\xi)\right] > 0$

is called an arbitrage strategy. Therefore, for the reinsurance market (Ω, F, P), R_t does not allow for arbitrage strategies if there is an equivalent probability measure P^* such that the process R_t is a martingale. A probability measure P^* is called an equivalent martingale probability measure if:

(i) $P^*(A) = 0$ iff $P(A) = 0$, for any $A \in \Im_t$;

(ii) The Radon-Nikodym derivative $\frac{dP^*}{dP}$ belongs to $L^2\left(\Omega, \Im_t, P\right)$;

(iii) R_t is a martingale under P^*, i.e.

$$E^*\left[R_t | \Im_s\right] = R_s, \quad P^* - \text{a.s.}$$

for any $0 \leq s \leq t \leq T$, where E^* denotes the expectation with respect to P^* (Harrison & Kreps, 1979 and Sondermann, 1991).

Cummins & Geman (1995) also employed this non-arbitrage pricing technique for catastrophe insurance derivatives. Alternative pricing for catastrophe insurance derivatives such as general equilibrium and the utility maximisation approach can be found in Aase (1994) and Embrechts & Meister (1995).

We will examine an equivalent martingale probability measure obtained via the Esscher transform (Gerber & Shiu, 1996). In general, the Esscher transform is defined as a change of probability measure for certain stochastic processes. An Esscher transform of such a process induces an equivalent probability measure on the process. The parameters involved for an Esscher transform are determined so that the price of a random future payment is a martingale under the new probability measure. A random payment therefore is calculated as the expectation of that at maturity with respect to the equivalent martingale probability measure (also known as the risk-neutral Esscher measure).

If the market is complete, the fair price of a contingent claim is the expectation with respect to exactly one equivalent martingale probability measure (i.e. by assuming that there is an absence of arbitrage opportunities in the market). For example, when the underlying stochastic process follows geometric Brownian motion or homogeneous Poisson process, we can obtain the fair price with respect to a unique equivalent martingale probability measure. However, as the underlying stochastic process for the claim arrival process is the Cox process, we will have infinitely many equivalent martingale probability measures. In other words, we will have several choices of equivalent martingale probability measures to price a stop-loss reinsurance contract & insurance derivatives as the market is incomplete.

It is not the purpose of this paper to decide which is the appropriate one to use. The insurance companies' attitude towards risk determines which equivalent martingale probability measure should be used. The attractive thing about the Esscher transform is that it provides us with at least one equivalent martingale probability measure in incomplete market situations.

We here offer the definition of the Esscher transform that is adopted from Gerber & Shiu (1996).

Definition 3.2 *Let X_t be a stochastic process and h^* a real number. For a measurable function f, the expectation of the random variable $f(X_t)$ with respect to the equivalent martingale probability measure is*

$$E^*\left[f\left(X_t\right)\right] = E\left[f\left(X_t\right)\frac{e^{h^*X_t}}{E\left(e^{h^*X_t}\right)}\right] = \frac{E\left[f\left(X_t\right)e^{h^*X_t}\right]}{E\left[e^{h^*X_t}\right]}, \tag{3.1}$$

where the process $e^{h^*X_t}$ is a martingale and $E\left(e^{h^*X_t}\right) < \infty$.

From definition 3.2, we need to obtain a martingale that can be used to define a change of probability measure, i.e. it can be used to define the Radon-Nikodym derivative $\frac{dP^*}{dP}$ where P is the original probability measure and P^* is the equivalent martingale probability measure with parameters involved.

Let M_t be the total number of catastrophe jumps up to time t. We will assume that claim points and catastrophe jumps do not occur at the same time.

The generator of the process $(X_t, N_t, C_t, \lambda_t, M_t, t)$ acting on a function $f(x, n, c, \lambda, m, t)$ belonging to its domain is given by

$$\begin{aligned}
&A\ f(x,n,c,\lambda,m,t)\\
&= \frac{\partial f}{\partial t} + \lambda\frac{\partial f}{\partial x} + \lambda\left[\int_0^\infty f\left(x,n+1,c+z,\lambda,m,t\right)dH\left(z\right) - f\left(x,n,c,\lambda,m,t\right)\right]\\
&- \delta\lambda\frac{\partial f}{\partial\lambda} + \rho\left[\int_0^\infty f\left(x,n,c,\lambda+y,m+1,t\right)dG\left(y\right) - f\left(x,n,c,\lambda,m,t\right)\right].
\end{aligned} \tag{3.2}$$

Clearly, for $f\left(x,n,c,\lambda,m,t\right)$ to belong to the domain of the generator A, it is essential that $f\left(x,n,c,\lambda,m,t\right)$ is differentiable w.r.t. x, c, λ, t for all x, n, c, λ, m, t and that $\left|\int_0^\infty f\left(.,\lambda+y,.\right)dG\left(y\right) - f\left(.,\lambda,.\right)\right| < \infty$ and $\left|\int_0^\infty f\left(.,c+z,.\right)dH\left(z\right) - f\left(.,c,.\right)\right| < \infty$.

Theorem 3.3 *Considering constants θ^*, v^*, ψ^* and γ^* such that $\theta^* \geq 1$, $v^* \leq 0$, $\psi^* \geq 1$ and $\gamma^* \leq 0$,*

$$\theta^{*N_t}e^{-v^*C_t}e^{-\left\{\theta^*\hat{h}(v^*)-1\right\}\int_0^t\lambda_s ds}\psi^{*M_t}e^{-\gamma^*\lambda_t e^{\delta t}}\exp[\rho\int_0^t\left\{1-\psi^*\hat{g}\left(\gamma^*e^{\delta s}\right)\right\}ds] \tag{3.3}$$

is a martingale where $\hat{h}\left(v^*\right) = \int_0^\infty e^{-v^*\ z}dH\left(z\right)$.

Proof. From (3.2), $f\left(x,n,c,\lambda,m,t\right)$ has to satisfy $Af = 0$ for $f(X_t, N_t, C_t, \lambda_t, M_t, t)$ to be a martingale. Trying $\theta^{*n}e^{-v^*c}e^{\phi^*x}\psi^{*m}\exp\left(-\gamma^*\lambda e^{\delta t}\right)e^{A(t)}$ we get the equation

$$A'\left(t\right) + \lambda\phi^* + \lambda\left\{\theta^*\hat{h}\left(v^*\right) - 1\right\} + \rho\left\{\psi^*\hat{g}\left(\gamma^*e^{\delta t}\right) - 1\right\} = 0 \tag{3.4}$$

and solving we (3.4) we get

$$\phi^* = -\left\{\theta^* \hat{h}(v^*) - 1\right\} \text{ and } A(t) = \rho \int_0^t \left\{1 - \psi^* \hat{g}\left(\gamma^* e^{\delta s}\right)\right\} ds$$

and the result follows. ■

Now let us look at how the processes λ_t and N_t change after changing probability measure. To do so we start with a technical lemma.

Lemma 3.4 *Assume that $f(n, \lambda, t) = f(\lambda, t)$ for all n and that $e^{-v^* \lambda_t}$ is a martingale. Consider a constant v^* such that $v^* \geq 0$. Then*

$$A^* \ f(\lambda, 0) = \frac{A \ \left\{f(\lambda, 0) \ e^{-v^* \lambda}\right\}}{e^{-v^* \lambda}}. \tag{3.5}$$

Proof. The generator of the process (λ_t, t) acting on a function $f(\lambda, t)$ with respect to the equivalent martingale probability measure is

$$A^* \ f(\lambda, 0) = \lim_{t \downarrow 0} \frac{E^*\left[f(\lambda_t, t) | \lambda_0 = \lambda\right] - f(\lambda, 0)}{t}. \tag{3.6}$$

We will use $\frac{e^{-v^* \lambda_t}}{E\left(e^{-v^* \lambda_t}\right)}$ as the Radon-Nikodym derivative to define equivalent martingale probability measure where $E\left(e^{-v^* \lambda_t}\right) < \infty$. Hence, the expected value of $f(\lambda_t, t)$ given λ with respect to the equivalent martingale probability measure is

$$E^*\left[f(\lambda_t, t) | \lambda_0 = \lambda\right] = \frac{E\left[f(\lambda_t, t) \cdot e^{-v^* \lambda_t} | \lambda_0 = \lambda\right]}{E\left(e^{-v^* \lambda_t} | \lambda_0 = \lambda\right)}. \tag{3.7}$$

Since the denominator in (3.7) is a martingale, it becomes

$$E^*\left\{f(\lambda_t, t) | \lambda_0 = \lambda\right\} = \frac{f(\lambda, 0) \cdot e^{-v^* \lambda} + \int_0^t E\left[A \ f(\lambda_s, s) \cdot e^{-v^* \lambda s} | \lambda_0 = \lambda\right] ds}{e^{-v^* \lambda}}. \tag{3.8}$$

Set (3.8) in (3.6) then

$$A^* \ f(\lambda, 0) = \frac{1}{e^{-v^* \lambda}} \lim_{t \downarrow 0} \int_0^t \frac{E\left[A \ f(\lambda_s, s) \cdot e^{-v^* \lambda_s} | \lambda_0 = \lambda\right] ds}{t}. \tag{3.9}$$

Therefore, from Dynkin's formula (Øksendal, 1992), (3.5) follows immediately. ■

Let us examine the generator A^* of the process $(X_t, N_t, C_t, \lambda_t, M_t, t)$ acting on a function $f(x, n, c, \lambda, m, t)$ with respect to the equivalent martingale probability measure.

Theorem 3.5 *Consider constants θ^*, v^*, ψ^* and γ^* such that $\theta^* \geq 1$,*

$v^* \le 0$, $\psi^* \ge 1$ and $\gamma^* \le 0$. Suppose that $\hat{h}(v^*) < \infty$ and $\hat{g}\left(\gamma^* e^{\delta t}\right) < \infty$. Then

$$
\begin{aligned}
&A\ f(x,n,c,\lambda,m,t) \\
&= \frac{\partial f}{\partial t} + \lambda\frac{\partial f}{\partial x} + \theta^* \hat{h}(v^*)\,\lambda\{\int_0^\infty f(x,n+1,c+z,\lambda,m,t)\,dH^*(z) - f(x,n,c,\lambda,m,t)\} \\
&- \delta\lambda\frac{\partial f}{\partial \lambda} + \rho^*(t)\,\{\int_0^\infty f(x,n,c,\lambda+y,m+1,t)\,dG^*(y;t) - f(x,n,c,\lambda,m,t)\},
\end{aligned}
\tag{3.10}
$$

where $dH^*(z) = \frac{e^{-v^* z}dH(z)}{\hat{h}(v^*)}$, $\rho^*(t) = \rho\psi^*\hat{g}\left(\gamma^* e^{\delta t}\right)$ *and* $dG^*(y;t) = \frac{\exp\left(-\gamma^* e^{\delta t}y\right)dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$.

Proof. From theorem 3.3, we can use

$$
\frac{\theta^* e^{-v^* C_t} e^{-\left\{\theta^* \hat{h}(v^*)-1\right\}\int_0^t \lambda_s ds} \psi^{*M_t} e^{-\gamma^* \lambda_t e^{\delta t}} \exp[\rho\int_0^t \left\{1-\psi^*\hat{g}\left(\gamma^* e^{\delta s}\right)\right\}ds]}{E[\theta^{*N_t} e^{-v^* C_t} e^{-\left\{\theta^* \hat{h}(v^*)-1\right\}\int_0^t \lambda_s ds} \psi^{*M_t} e^{-\gamma^* \lambda_t e^{\delta t}} \exp[\rho\int_0^t \{1-\psi^*\hat{g}(\gamma^* e^{\delta s})\}ds]]}
\tag{3.11}
$$

as the Radon-Nikodym derivative to define an equivalent martingale probability measure. Therefore from lemma 3.4,

$$
\begin{aligned}
&A^*\ f(X_t,N_t,C_t,\lambda_t,M_t,t) \\
&= A\ f(X_t,N_t,C_t,\lambda_t,M_t,t) \\
&\cdot \frac{\theta^* e^{-v^* C_t} e^{-\left\{\theta^* \hat{h}(v^*)-1\right\}\int_0^t \lambda_s ds} \psi^{*M_t} e^{-\gamma^* \lambda_t e^{\delta t}} \exp[\rho\int_0^t \left\{1-\psi^*\hat{g}\left(\gamma^* e^{\delta s}\right)\right\}ds]}{E[\theta^{*N_t} e^{-v^* C_t} e^{-\left\{\theta^* \hat{h}(v^*)-1\right\}\int_0^t \lambda_s ds} \psi^{*M_t} e^{-\gamma^* \lambda_t e^{\delta t}} \exp[\rho\int_0^t \{1-\psi^*\hat{g}(\gamma^* e^{\delta s})\}ds]]}.
\end{aligned}
$$

From (3.2), using the generator with respect to the original probability measure,

$$A\ f(x,n,c,\lambda,m,t)\,\theta^{*N_t}e^{-v^*C_t}\exp[-\{\theta^*\hat{h}(v^*)-1\}\int_0^t\lambda_s ds]$$

$$\cdot\,\psi^{*M_t}\exp\left(-\gamma^*\lambda_t e^{\delta t}\right)\exp[\rho\int_0^t\left\{1-\psi^*\hat{g}\left(\gamma^* e^{\delta s}\right)\right\}ds]$$

$$=[\frac{\partial f}{\partial t}+\lambda\frac{\partial f}{\partial x}-\delta\lambda\frac{\partial f}{\partial \lambda}+\lambda\{\theta^*\int_0^\infty f(x,n+1,c+z,\lambda,m,t)\,e^{-v^*z}dH(z)$$

$$-\,\theta^*\hat{h}(v^*)\,f(x,n,c,\lambda,m,t)\}+\rho\{\psi^*\int_0^\infty f(x,n,c,\lambda+y,m+1,t)\,e^{-\gamma^* e^{\delta t}y}dG(y)$$

$$-\,\psi^*\hat{g}\left(\gamma^* e^{\delta t}\right)f(x,n,c,\lambda,m,t)\}]\cdot\theta^{*N_t}e^{-v^*C_t}e^{-\left\{\theta^*\hat{h}(v^*)-1\right\}\int_0^t\lambda_s ds}$$

$$\cdot\,\psi^{*M_t}\exp\left(-\gamma^*\lambda_t e^{\delta t}\right)\exp[\rho\int_0^t\left\{1-\psi^*\hat{g}\left(\gamma^* e^{\delta s}\right)\right\}ds].$$

Therefore

$$\begin{aligned}
&A^*\ f(x,n,c,\lambda,m,t)\\
&=\frac{\partial f}{\partial t}+\lambda\frac{\partial f}{\partial x}-\delta\lambda\frac{\partial f}{\partial \lambda}\\
&+\theta^*\hat{h}(v^*)\,\lambda\{\int_0^\infty f(x,n+1,c+z,\lambda,m,t)\,dH^*(z)-f(x,n,c,\lambda,m,t)\}\\
&+\rho^*(t)\{\int_0^\infty f(x,n,c,\lambda+y,m+1,t)\,dG^*(y;t)-f(x,n,c,\lambda,m,t)\}
\end{aligned}\tag{3.12}$$

where $dH^*(z)=\frac{e^{-v^*z}dH(z)}{\hat{h}(v^*)}$, $\rho^*(t)=\rho\psi^*\hat{g}\left(\gamma^* e^{\delta t}\right)$ and $dG^*(y;t)=\frac{\exp\left(-\gamma^* e^{\delta t}y\right)dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$.
■

Theorem 3.5 yields the following:

(i) The claim intensity function λ_t has changed to $\lambda_t\theta^*\hat{h}(v^*)$;

(ii) The rate of jump arrival ρ has changed to $\rho^*(t)=\rho\psi^*\hat{g}\left(\gamma^* e^{\delta t}\right)$ (it now depends on time);

(iii) The jump size measure $dG(y)$ has changed to $dG^*(y;t)=\frac{\exp\left(-\gamma^* e^{\delta t}y\right)dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$ (it now depends on time);

(iv) The claim size measure $dH(z)$ has changed to $dH^*(z)=\frac{e^{-v^*z}dH(z)}{\hat{h}(v^*)}$.

In other words, the risk-neutral Esscher measure is the measure with respect to which N_t becomes the Cox process with parameter where three parameters $\lambda_t \theta^* \hat{h}(v^*)$ where three parameters of the shot noise process λ_t are δ, $\rho^*(t) = \rho\psi^* \hat{g}\left(\gamma^* e^{\delta t}\right)$, $dG^*(y;t) = \frac{\exp\left(-\gamma^* e^{\delta t} y\right) dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$ and claim size distribution becomes $dH^*(z) = \frac{e^{-v^* z} dH(z)}{\hat{h}(v^*)}$.

In practice, the reinsurer will calculate the values of a stop-loss contract & insurance derivatives using $\theta^* > 1$, $\psi^* > 1$, $\gamma^* < 0$ and $v^* < 0$. This results in the reinsurer assuming that there will be a higher value of claim intensity itself, a higher value of the damage caused by the catastrophe, more catastrophes occurring in a given period of time and a higher value of claim size. These assumptions are necessary, as the reinsurer wants compensation for the risks involved in operating in incomplete market. The reinsurer also aims to maximise their shareholders' wealth by earning profits rather than operating at breakeven point where premiums are equal to expected claims that is calculated with respect to the original probability measure.

If $\theta^* = 1$, $\psi^* = 1$, $\gamma^* = 0$ and $v^* = 0$ then net premium and non-arbitrage free price are calculated which should cover the expected losses over the period of contract. Therefore we can consider θ^*, ψ^*, γ^* and v^* as security loading factors by which gross premium and non-arbitrage price, that should be finally charged, will be calculated. However, as expected, we have quite a flexible family of equivalent probability measures by the combination of θ^*, ψ^*, γ^* and v^*. It means that insurance companies have various ways of levying the security loading on the net premium and non-arbitrage free price to obtain the gross premium and non-arbitrage price (i.e. by changing equivalent martingale probability measures using the combination of θ^*, ψ^*, γ^* and v^*). One of the interesting results by changing measure (i.e. by assuming that there is an absence of arbitrage opportunities in the market) is that we can justify reinsurers' security loading on the net premium for stop-loss reinsurance contract and on non-arbitrage free prices for insurance derivatives in practice.

Now let us evaluate the '$-\infty$' asymptotic expected value of N_t and the p.g.f. (probability generating function) of the '$-\infty$' asymptotic distribution of N_t with respect to the equivalent martingale probability measure, i.e. $E^*(N_t)$ and $E^*\left(\theta^{N_t}\right)$. We will assume that the jump size distribution is exponential, i.e. $g(y) = \alpha e^{-\alpha y}$, $y > 0$, $\alpha > 0$ and that λ_t is '$-\infty$' asymptotic. Therefore we can obtain that $g^*(y;t) = \left(\alpha + \gamma^* e^{\delta t}\right) \exp\left\{-\left(\alpha + \gamma^* e^{\delta t}\right) y\right\}$, $y > 0$, $-\alpha e^{-\alpha y} < \gamma^* \leq 0$ and $t < \frac{1}{\delta} \ln\left(-\frac{\alpha}{\gamma^*}\right)$ since $dG^*(y;t) = \frac{\exp\left(-\gamma^* e^{\delta t} y\right) dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$. It is clear that such a model is appropriate in the short term only, as it breaks down for $t \geq \frac{1}{\delta} \ln\left(-\frac{\alpha}{\gamma^*}\right)$.

For simplicity, let us assume that $v^* = 0$ and $\psi^* = 1$, i.e. we only consider θ^* and γ^* as security loading factors.

Corollary 3.6 *Let the jump size distribution be exponential. Consider constants* θ, θ^*, v^*, ψ^* *and* γ^* *such that* $0 \leq \theta \leq 1$, $\theta^* \geq 1$, $v^* = 0$, $\psi^* = 1$ *and*

$\gamma^* \leq 0$. *Furthermore if* λ_t *is* '$-\infty$' *asymptotic, then*

$$E^*\left(\theta^{N_{t_2}-N_{t_1}}\right) = \left\{\frac{\gamma^* e^{\delta t_1} + \alpha e^{-\delta(t_2-t_1)}}{\gamma^* e^{\delta t_1} + \alpha + \frac{\theta^*(1-\theta)}{\delta}\left(1 - e^{-\delta(t_2-t_1)}\right)}\right\}^{\frac{\rho}{\delta}} \cdot \left\{\frac{\gamma^* e^{\delta t_1} + \alpha + \frac{\theta^*(1-\theta)}{\delta}\left(1 - e^{-\delta(t_2-t_1)}\right)}{\gamma^* e^{\delta t_1} + \alpha e^{-\delta(t_2-t_1)}}\right\}^{\frac{\alpha\rho}{\delta\alpha+\theta^*(1-\theta)}} \tag{3.13}$$

and

$$E^*\left(N_{t_2} - N_{t_1}\right) = \frac{\theta^*\rho}{\delta\alpha}\left(t_2 - t_1\right) - \frac{\theta^*\rho}{\delta^2\alpha}\ln\left(\frac{\gamma^* e^{\delta t_2} + \alpha}{\gamma^* e^{\delta t_1} + \alpha}\right) \tag{3.14}$$

where $0 < t_1 < t_2 < t$.

Proof. From theorem 3.5 and (2.3), $E^*\left(\theta^{N_{t_2}-N_{t_1}}\right) = E[\exp\{-\theta^* \hat{h}(v^*)(1-\theta)\int_{t_1}^{t_2} \lambda_s ds\}]$ where $dH^*(z) = \frac{e^{-v^* z} dH(z)}{\hat{h}(v^*)}$, $\rho^*(t) = \rho\psi^* \hat{g}\left(\gamma^* e^{\delta t}\right)$ and $dG^*(y;t) = \frac{\exp\left(\gamma^* e^{\delta t} y\right) dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$. Since $v^* = 0$, $\psi^* = 1$ and the jump size distribution is exponential, $\rho^*(s) = \rho\frac{\alpha}{\alpha+\gamma^* e^{\delta s}}$ and $\mu_1^*(s) = \frac{1}{\alpha+\gamma^* e^{\delta s}}$. Therefore if we set $v = \theta^*(1-\theta)$ in (2.21) and multiply θ^* to (2.24), putting $\gamma = \gamma^*$, the results follow immediately. ■

4. Pricing of a stop-loss reinsurance contract for catastrophic event & catastrophe insurance derivatives

Now let us look at the stop-loss reinsurance premium and catastrophe insurance derivatives prices at present time 0, assuming that there is an absence of arbitrage opportunities in the market. This can be achieved by using an equivalent martingale probability measure, P^*, within the pricing model used for calculating premium for reinsurance contract. Therefore, from (1.4), (1.5) and (1.6), the stop-loss reinsurance gross premium at time 0 is

$$E^*\left[(C_t - b)^+\right], \tag{4.1}$$

the arbitrage-free price for the insurance futures contract at time 0 is

$$F_0 = E^*\left[25,000 \times Min\left(\frac{C_t}{\Pi}, 2\right)\right] \tag{4.2}$$

and the arbitrage-free price for the insurance call option on futures at time 0 is

$$P_0 = \frac{25,000}{\Pi} E^*\left[(C_t - B)^+\right] \tag{4.3}$$

where all symbols have previously been defined and for simplicity, we assume interest rates to be constant.

It will be interesting to derive the premium and pricing formulae, using a specific claim size distribution of $H(z)$ $(z > 0)$. We assume that the claim size distribution is gamma, i.e. $h(z) = \frac{\beta^\varphi z^{\varphi-1} e^{-\beta z}}{(\varphi-1)!}$, $z > 0$, $\beta > 0$, $\varphi \geq 1$. Then

$$E^*\left[(C_t-b)^+\right]=\sum_{n=1}^{\infty}a_n^*\left\{\frac{n\varphi}{\beta}\int_b^{\infty}\frac{\beta^{n\varphi+1}c^{n\varphi}e^{-\beta c}}{(n\varphi)!}dc-b\int_b^{\infty}\frac{\beta^{n\varphi}c^{n\varphi-1}e^{-\beta c}}{(n\varphi-1)!}dc\right\}, \tag{4.4}$$

$$F_0=\frac{25,000}{\Pi}\left[\begin{array}{c}\left\{\sum_{n=1}^{\infty}a_n^*\frac{n\varphi}{\beta}\right\}\\ -\left\{\sum_{n=1}^{\infty}a_n^*\left(\frac{n\varphi}{\beta}\int_{2\Pi}^{\infty}\frac{\beta^{n\varphi+1}c^{n\varphi}e^{-\beta c}}{(n\varphi)!}dc-2\Pi\int_{2\Pi}^{\infty}\frac{\beta^{n\varphi}c^{n\varphi-1}e^{-\beta c}}{(n\varphi-1)!}dc\right)\right\}\end{array}\right] \tag{4.5}$$

and

$$P_0=\frac{25,000}{\Pi}\sum_{n=1}^{\infty}a_n^*\left\{\frac{n\varphi}{\beta}\int_B^{\infty}\frac{\beta^{n\varphi+1}c^{n\varphi}e^{-\beta c}}{(n\varphi)!}dc-B\int_B^{\infty}\frac{\beta^{n\varphi}c^{n\varphi-1}e^{-\beta c}}{(n\varphi-1)!}dc\right\} \tag{4.6}$$

where $a_n^*=P^*(N_t=n)$ and $B=\frac{\Pi K}{25,000}$.

In practice, other distributions such as exponential, log-normal and Pareto, etc. can also be applied for claim size distribution of $H(z)$ $(z>0)$. Examining the effect on stop-loss reinsurance premiums and prices for catastrophe insurance derivatives caused by changes in the claim size distribution will be also of interest.

Now let us illustrate the calculation of stop-loss reinsurance gross premium for catastrophic events & the arbitrage-free prices of the catastrophe insurance derivatives using the models derived previously. From (3.13), the p.g.f. of N_t is

$$\begin{aligned}E^*\left(\theta^{N_t}\right)&=\sum_{n=1}^{\infty}\theta^nP^*(N_t=n)=\sum_{n=0}^{\infty}\theta^na_n^*\\&=\left\{\frac{\gamma^*+\alpha e^{-\delta t}}{\gamma^*+\alpha+\frac{\theta^*(1-\theta)}{\delta}\left(1-e^{-\delta t}\right)}\right\}^{\frac{\rho}{\delta}}\left\{\frac{\gamma^*+\alpha+\frac{\theta^*(1-\theta)}{\delta}\left(1-e^{-\delta t}\right)}{\gamma^*+\alpha e^{-\delta t}}\right\}^{\frac{\alpha\rho}{\delta\alpha+\theta^*(1-\theta)}}\end{aligned} \tag{4.7}$$

The parameter values used to expand (4.7) with respect to θ are

$$\theta^*=1.1,\ \gamma^*=-0.1,\ \alpha=1,\ \delta=0.3,\ \rho=4,\ t=1.$$

Using these parameter values we can calculate the mean of the claim number in a unit period of time. From (3.14)

$$E^*(N_t)=\frac{\theta^*\rho}{\delta\alpha}t-\frac{\theta^*\rho}{\delta^2\alpha}\ln\left(\frac{\gamma^*e^{\delta t}+\alpha}{\gamma^*+\alpha}\right)\approx 16.61.$$

By expanding (4.7) using the *MAPLE* algebraic manipulations package we can obtain $a_n^*=P^*(N_t=n)$ which is as follows:

$$
\begin{aligned}
E^*\left(\theta^{N_t}\right) &= \sum_{n=1}^{\infty} \theta^n P^*\left(N_t = n\right) \\
&= \sum_{n=0}^{\infty} \theta^n a_n^* = \left\{ \frac{0.64082}{0.9 + 0.95033(1-\theta)} \right\}^{\frac{4.4(1-\theta)}{0.09+0.33(1-\theta)}} \\
&= 0.000014982 + 0.000116628\theta + 0.000482266\theta^2 + 0.00142225\theta^3 + 0.00333355\theta^4 + \\
&\quad 0.006615\theta^5 + 0.011523\theta^6 + 0.018086\theta^7 + 0.026045\theta^8 + 0.034881\theta^9 + 0.0439\theta^{10} \\
&\quad + 0.052349\theta^{11} + 0.059537\theta^{12} + 0.064932\theta^{13} + 0.068214\theta^{14} + 0.06929\theta^{15} \\
&\quad + 0.068273\theta^{16} + 0.065434\theta^{17} + 0.061148\theta^{18} + 0.055831\theta^{19} + 0.049898\theta^{20} \\
&\quad + 0.043723\theta^{21} + 0.037616\theta^{22} + 0.031815\theta^{23} + 0.026484\theta^{24} + 0.02172\theta^{25} \\
&\quad + 0.017567\theta^{26} + 0.014023\theta^{27} + 0.011056\theta^{28} + 0.0086166\theta^{29} + 0.0066419\theta^{30} \\
&\quad + 0.0050667\theta^{31} + 0.0038272\theta^{32} + 0.0028639\theta^{33} + 0.0021241\theta^{34} + 0.0015621\theta^{35} \\
&\quad + 0.0011396\theta^{36} + 0.00082497\theta^{37} + 0.000592282\theta^{38} + 0.00042301\theta^{39} \\
&\quad + 0.000299981\theta^{40} + 0.00021112\theta^{41} + 0.000147775\theta^{42} + 0.000102790\theta^{43} + \\
&\quad 0.0000711101\theta^{44} + 0.000048911^{45} + 0.0000334690\theta^{46} + 0.0000227850\theta^{47} + \\
&\quad 0.0000154360\theta^{48} + 0.0000104070\theta^{49} + 0.0000069850\theta^{50} + 0.0000046672\theta^{51} + \\
&\quad 0.0000031051\theta^{52} + 0.0000020573\theta^{53} + 0.0000013575\theta^{54} + O\left(\theta^{55}\right).
\end{aligned}
\tag{4.8}
$$

Example 4.1

The parameter values used to calculate (4.4) are

$$n : 1 \sim 41,\ \varphi = 1,\ \beta = 1,\ b = 0,\ 5,\ 10,\ 16.61,\ 20,\ 25,\ 30$$

$$E^*\left(C_t\right) = E^*\left(N_t\right) E\left(Z\right) = 16.61.$$

By computing (4.4) using *S-Plus* the calculation of the stop-loss reinsurance gross premiums for catastrophic events at each retention level b are shown in Table 4.1.

Table 4.1

Retention level b	Reinsurance gross premium
0	16.58403
5	11.61916
10	7.06779
16.61	2.83349
20	1.58701
25	0.59582
30	0.19512

Example 4.2

We will now examine the effect on stop-loss reinsurance gross premiums caused by changes in the value of θ^* and γ^*. By expanding (4.7) using *MAPLE* at each value of θ^* and γ^* respectively and computing (4.4) by *S-Plus*, the calculation of the stop-loss reinsurance gross premiums for catastrophic events at the retention limit $b = 25$ are shown in Table 4.2 and Table 4.3.

Table 4.2

θ^*	$\gamma^* = -0.1$
1.0	0.3544252
1.1	0.595824
1.2	0.9299355
1.3	1.366049
1.4	1.90885
1.5	2.558786

Table 4.3

γ^*	$\theta^* = 1.1$
0.0	0.3029752
-0.1	0.595824
-0.2	1.207256
-0.3	2.512553
-0.4	5.364622
-0.5	11.65184

Example 4.3

The parameter values used to calculate (4.5) are

$$n : 1 \sim 41,\ \varphi = 1,\ \beta = 1,\ \Pi = E^*(C_t) = E^*(N_t)\,E(Z) = 16.61.$$

By computing (4.5) using *S-Plus* the calculation of an arbitrage-free price of catastrophe insurance futures is as follows:

$$F_0 = \$25,000 \times (0.9984363 - 0.005339982) = \$24,827.41.$$

Example 4.4

The parameter values used to calculate (4.6) are

$$n : 1 \sim 41,\ \varphi = 1,\ \beta = 1,\ \Pi = 16.61,\ K = \$25,000.$$

By computing (4.6) using *S-Plus* the calculation of an arbitrage-free price of catastrophe insurance option on futures is as follows:

$$P_0 = \frac{\$25,000}{16.61} \times 2.833487 = \$4,264.73.$$

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