A Cox Process with Log - Normal intensity

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Abstract

In this paper we look at pricing stop - loss reinsurance contracts using an approximation technique similar to Basu(1999) and Rogers and Shi (1995) for processes with constant claims and the underlying stochastic intensity following a log - normal distribution. In particular, we look at the Cox process with the underlying stochastic intensity being log - normal.

Keywords : Cox process, Stop-loss reinsurance, Ornstein Uhlenbeck process

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1 Introduction

In this paper, we use an approximation technique to price financial instruments in which credit risk is very significant and the credit risk can be modeled by a Cox process. For details about Cox processes - also known as the doubly stochastic Poisson processes, see Daley and Vere - Jones (1988) and Kallenberg (1997). The approximation technique used is the same as used by Basu (1999) to price bonds and options. The Cox process provides us with a very useful framework for modeling prices of financial instruments in which credit risk is a significant factor. Examples of such instruments are bonds, insurance policies, reinsurance policies among other. Work in this area has been done by a number of people; notable

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among them are Lando (1998), Dassios (1987), Jang (1998) and Dassios and Jang (2002). Most of Dassios' and Jang's work has been to look at the application of the Cox process in valuing insurance and reinsurance claims. On the other hand, Lando has looked at the applications of the Cox process in pricing of bonds and valuing contingent payments to be made on bonds.

Claims arising from catastrophic events depend on the intensity of such natural disasters. Therefore the intensity means the frequency of claims arising from the natural disaster.

In order to calculate the price for catastrophe reinsurance contracts and insurance derivatives, the *claim arrival process* needs to determined. A homogeneous Poisson process can be used as a claim arrival process. Under this approach, the *claim intensity function* is assumed to be constant. Another approach is to use a non-homogeneous Poisson process where the claim intensity is assumed to be a non-random function of time. However, both these processes do not adequately explain the phenomena of catastrophes.

Under a *doubly stochastic Poisson process,* or a *Cox process,* the claim intensity function is assumed to be stochastic. The Cox process is more appropriately used as a claim arrival process as it can allow for the assumption that catastrophic events occur periodically.

A doubly stochastic Poisson process can be viewed as a two step randomization procedure. A process λ_t is used to generate another process N_t by acting as its intensity. This means that N_t is a Poisson process conditional on λ_t (if λ_t is deterministic, then N_t is simply a Poisson process). The term "doubly stochastic" was introduced by Cox (1955).

Many alternative definitions of a doubly stochastic Poisson process can be given. We will offer the one adopted by Brémaud (1981).

Definition : Let N_t be a point process adapted to a history $\mathcal{F}_t$ and let λ_t be a non-negative process. Suppose that λ_t is $\mathcal{F}_t$-measurable, $t \geq 0$ and that

$$\int_0^t \lambda_s ds < \infty \quad \text{almost surely (no explosions).}$$

If for all $0 \leq t_1 \leq t_2$ and $u \in \mathcal{R}$

$$E\left\{e^{iu(N_{t_2}-N_{t_1})}|\mathcal{F}_{t_1}\right\} = \exp\left((e^{iu}-1)\int_{t_1}^{t_2} \lambda_s ds\right) \tag{1}$$

then N_t is called a $\mathcal{F}_t$-doubly stochastic Poisson process with intensity λ_t.

In this paper, we will take $\mathcal{F}_t$ to be the natural filtration of the probability space.

Equation (1) gives us

$$Pr\left\{N_{t_2} - N_{t_1} = k|\lambda_s; t_1 \le s \le t_2\right\} = \frac{e^{-\int_{t_1}^{t_2}\lambda_s ds}\left(\int_{t_1}^{t_2}\lambda_s ds\right)^k}{k!} \tag{2}$$

and

$$E\left\{\theta^{N_{t_2}-N_{t_1}}|\lambda_s; t_1 \le s \le t_2\right\} = \exp\left(-(1-\theta)\int_{t_1}^{t_2}\lambda_s ds\right) \tag{3}$$

so

$$E\left(\theta^{N_{t_2}-N_{t_1}}\right) = E\left\{E\left(\theta^{N_{t_2}-N_{t_1}}|\lambda_s; t_1 \le s \le t_2\right)\right\} = E\left\{e^{-(1-\theta)\int_{t_1}^{t_2}\lambda_s ds}\right\} \tag{4}$$

$$\Rightarrow E\left(\theta^{N_{t_2}-N_{t_1}}\right) = E\left\{e^{-(1-\theta)(X_{t_2}-X_{t_1})}\right\} \tag{5}$$

where

$$X_t = \int_0^t \lambda_s ds \quad \text{the aggregated process.}$$

Thus, it is easy to note that the problem of finding the distribution of N_t, the point process, is equivalent to the problem of finding the distribution of X_t, the aggregated process.

The log-normal Cox process, rather the log-Gaussian Cox process, has also been used in the past in studying spatial data by Møller, Syversveen and Waagepetersen (1998) as well Rathbun and Cressie (1994).

2 Calculations

Here, we are interested in finding the value of a stop-loss reinsurance contract. We assume $t = 1$. Thus, the value of the stop - loss reinsurance contract is given by

$$E(N_1 - k)^+, \tag{6}$$

where, N_1 is conditionally a Poisson random variable with a random parameter M and k is the strike price at which the contract is calculated. Let us assume

$$\lambda_t = ce^{\sigma Y_t}$$

where $\{Y_t, 0 \leq t \leq 1\}$ is a Gaussian process. Also, c is a constant and $c = \lambda_0$, where λ_0 is the initial value of the process λ_t. Now, in this case, define

$$M = c \int_0^1 e^{\sigma Y_s} ds,$$

$\{Y_t, 0 \leq t \leq 1\}$ could represent any stochastic process; later in the paper we give an example where $\{Y_t, 0 \leq t \leq 1\}$ is assumed to follow an Ornstein - Uhlenbeck process with a known initial value. In this case the initial value is assumed to be zero.

Let us first prove the following *Lemma.*

Lemma : *Let N be a Poisson random variable with parameter t. Then,*

$$E(N-k)^+ = tG(t,k) - kG(t,k+1).$$

Proof : Suppose $\{\tilde{N}_t, t \geq 0\}$ is a Poisson process with parameter 1. Then, $\tilde{N}_t$ is a Poisson random variable with parameter t. Further, we have,

$$E(N-k)^+ = E(\tilde{N}_t - k)^+ = \sum_{j=k+1}^{\infty} (j-k) Pr(\tilde{N}_t = j) = \sum_{j=k+1}^{\infty} \sum_{i=k+1}^{j} Pr(\tilde{N}_t = j)$$

$$= \sum_{i=k+1}^{\infty} \sum_{j=i}^{\infty} Pr(\tilde{N}_t = j) = \sum_{i=k+1}^{\infty} Pr(\tilde{N}_t \geq i) = \sum_{i=k}^{\infty} Pr(\tilde{N}_t \geq i+1). \tag{7}$$

Now, $Pr(\tilde{N}_t \geq i+1) = Pr(T_{i+1} \leq t) = \int_0^t \frac{v^i e^{-v}}{i!} dv$, where T_i is the time of the i^{th} jump. Thus, we have using equation (7),

$$E(\tilde{N}_t - k)^+ = \sum_{i=k}^{\infty} Pr(T_{i+1} \leq t) = \sum_{i=k}^{\infty} \int_0^t \frac{v^i e^{-v}}{i!} dv$$

$$= \int_0^t \sum_{i=k}^{\infty} \frac{v^i e^{-v}}{i!} dv = \int_0^t Pr(\tilde{N}_v \geq k) dv$$

$$= \int_0^t \int_0^v \frac{u^{k-1} e^{-u}}{(k-1)!} du dv = \int_0^t (t-u) \frac{u^{k-1} e^{-u}}{(k-1)!} du$$

$$= tG(t,k) - kG(t,k+1). \tag{8}$$

■

Here $G(a, b)$ is the distribution function of a Gamma distribution with parameters *(a, b)*, $a > 0$, $b > 0$ and is given as

$$G(a, b) = \int_0^x \frac{a^b}{\Gamma(b)} e^{-ax} x^{b-1} dx.$$

Further, for convenience, we assume k to be an integer.

Now, as we can see from the *Lemma*

$$E[(N_1 - k)^+ | M] = MG(M, k) - kG(M, k+1) = f(M) \quad \text{say;} \tag{9}$$

f is convex; this is obvious from the fact that *f* can be written as

$$\int_0^t \int_0^v \frac{u^{k-1} e^{-u}}{(k-1)!} du dv.$$

Further, the second derivative of this expression with respect to t is positive and hence the function *f* is convex.

As stated earlier, we are interested in obtaining

$$E[(N_1 - k)^+] = E[E(N_1 - k)^+ | M] = E[f(M)].$$

Now, since *f* is convex, we have using a suitable conditioning factor Z and Jensen's inequality,

$$E[f(M)] = E(E[f(M)|Z]) \geq E(f(E(M|Z))).$$

The conditioning factor Z is exactly the same as used by Rogers and Shi (1995) and Basu (1999) (for a detailed justification of the choice of the conditioning factor, see chapter 3, Basu (1999)) and is given by

$$Z = \frac{\int_0^1 Y_s ds}{\sqrt{\mathrm{Var}(\int_0^1 Y_s ds)}}. \tag{10}$$

Conditionally on Z, Y_t has a Gaussian distribution. Furthermore, Z, itself has a standard normal distribution. Also,

$$E(Y_u | Z) = k_u Z,$$

$$\text{where} \quad k_u = \mathrm{Cov}(Y_u, Z)$$

$$\text{and} \quad \mathrm{Cov}(Y_u, Y_v | Z) = \mathrm{Cov}(Y_u, Y_v) - k_u k_v = s_{uv} \quad \text{say.}$$

Thus,

$$E(M|Z=z) = E(\lambda_0 \int_0^1 e^{\sigma Y_s} ds = \lambda_0 \int_0^1 e^{\sigma k_u z + \frac{\sigma^2}{2} s_{uu}} du = h(z) \quad \text{say.} \tag{11}$$

Now, once we have obtained the value of $h(z)$, we then obtain the lower bound to the value of the stop-loss reinsurance contact, conditionally on the conditioning factor Z. This is obtained by using equation (9) and the previous lemma and is given by

$$\int_0^{h(z)} \int_0^v \frac{u^k e^{-u}}{k!} du dv = \int_0^{h(z)} \int_u^{h(z)} dv \frac{u^k e^{-u}}{k!} du = \int_0^{h(z)} (h(z) - u) \frac{u^k e^{-u}}{k!} du \tag{12}$$

$$= h(z) G(h(z), k) - k G(h(z), k+1) = \Omega(z). \tag{13}$$

Finally, the lower bound to the unconditional price of the stop-loss reinsurance contract is obtained by taking the expectation of $\Omega(z)$ with respect to Z, where Z has a standard Normal distribution. Thus, we finally calculate

$$\int_{-\infty}^{\infty} \Omega(z) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \tag{14}$$

to obtain the unconditional price of the stop-loss reinsurance contract.

Example :

We assume that the process $\{Y_s, 0 \leq s \leq 1\}$ follows an Ornstein - Uhlenbeck process. We give the explicit forms of Z, k_u and s_{uv} in that case. Having these values, using equation (11) it is easy to obtain $h(z)$ and having obtained $h(z)$, we can easily find the lower bound to the value of the stop-loss reinsurance contract, conditionally on Z, by using equation (13). Once we have that, we then use equation (14) to obtain the unconditional value of the lower bound of the stop-loss reinsurance contract.

Thus, here we have

$$dY_t = -aY_t dt + dB_t$$

$$\text{i.e.} \quad Y_t = \int_0^t e^{-a(t-u)} dB_u.$$

Here, Y_0, the initial value is assumed to be zero. The conditioning factor, Z, is then given by

$$Z = \frac{\int_0^1 Y_s ds}{\sqrt{\mathrm{Var}(\int_0^1 Y_s ds)}}.$$

We observe that

$$\mathrm{Var}(\int_0^1 Y_s ds) = \int_0^1 \int_0^s \left(e^{-a(s-u)} dB_u\right)^2 ds = \frac{1}{2a} \frac{2a + 4e^{-a} - e^{-2a} - 3}{a^2} = V, \quad \text{say.}$$

Thus,

$$k_u = \mathrm{Cov}\,(Y_u, Z) = \frac{1}{2a\sqrt{V}} \left\{ \int_0^u \left(e^{a(s+u)} - e^{-a(s+u)}\right) ds + \int_u^1 \left(e^{a(u-s)} - e^{-a(u+s)}\right) ds \right\}$$

$$= \frac{1}{\sqrt{V}} \frac{\sigma^2}{2a} \left\{ \frac{1 - e^{-au}}{a} + \frac{1 - e^{-a(1-u)}}{a} - \frac{e^{-au} - e^{-a(1+u)}}{a} \right\}.$$

Also,

$$\mathrm{Cov}(Y_u, Y_v | Z) = \frac{1}{2a} \left[e^{a|u-v|} - e^{-a(u+v)} \right] - k_u k_v = s_{uv}.$$

Once we have this, then using equations (11), (13) and (14), we can easily find the lower bound to the value of the stop-loss reinsurance contract. The numerical results (Calculated Value) based on these calculations are given in tables 1 and 2. For comparison purposes, we also include the set of simulated values along with the standard errors of simulation.

3 Conclusion and Remarks

Using the conditioning factor in the Cox process situation, we can thus very easily calculate the price of the option. Once M, rather $E(M|Z)$, is evaluated, given the strike price, k, the calculation of the price of the option is just looking up the Gamma distribution tables - in fact, all statistical software would return the values. It is time saving as well as very efficient. Furthermore, the use of the conditioning factor approach means that we can account for all values of the instantaneous variance of the stochastic process driving λ, the parameter. Note that in quite a few cases the simulated value is lower than the calculated lower bound, thus demonstrating the accuracy of the approximation.

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Table 1 : $c = \lambda_0$ = 10

σ	Strike Price	Calculated Value	Simulated Value	Standard Error
0.5	8	2.924	2.911	0.0144
	10	1.706	1.698	0.0117
	12	0.901	0.898	0.0088
	15	0.292	0.292	0.005
	20	0.031	0.03	0.0015
0.75	8	3.49	3.507	0.0184
	10	2.268	2.285	0.0158
	12	1.401	1.416	0.013
	15	0.631	0.642	0.009
	20	0.147	0.152	0.0044
1	8	4.293	4.278	0.0244
	10	3.067	3.065	0.0219
	12	2.143	2.147	0.0192
	15	1.22	1.229	0.0152
	20	0.466	0.47	0.0099

Table 2 : $c = \lambda_0$ = 100

σ	Strike Price	Calculated Price	Simulated Value	Standard Error
0.5	80	25.053	25.001	0.0986
	100	11.198	11.162	0.0754
	120	3.948	3.925	0.0473
0.75	80	31.238	31.194	0.1491
	100	18.053	18.047	0.1258
	110	9.678	9.706	0.0982
	120	3.496	3.549	0.0625
1	80	39.771	39.767	0.2173
	100	26.909	29.956	0.1947
	120	17.783	17.856	0.1684
	150	9.385	9.545	0.1303
	200	3.251	3.41	0.0818