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SAMPLE QUANTILES OF STOCHASTIC PROCESSES WITH STATIONARY AND INDEPENDENT INCREMENTS

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The purpose of this note is to obtain a representation of the distribution of the α -quantile of a process with stationary and independent increments as the sum of the supremum and the infimum of two rescaled independent copies of the process. This representation has already been proved for a Brownian motion. The proof is based on already known discrete time results.

1. Introduction. Let $(X(t), t \geq 0)$ be a process with stationary and independent increments with $X(0) = 0$ and consider the version with paths in $D[0, \infty)$ (see [2], page 306). For $0 < \alpha < 1$, define the α -quantile of $(X(s), 0 \leq s \leq t)$ by

$$M(\alpha, t) = \inf \left\{ x : \int_0^t \mathbf{1}(X(s) \leq x) ds > \alpha t \right\}.$$

Our main result is the following theorem

THEOREM 1. *Let $X^{(1)}(t)$ and $X^{(2)}(t)$ be independent copies of $X(t)$. Then,*

$$\begin{pmatrix} M(\alpha, t) \\ X(t) \end{pmatrix} = \begin{pmatrix} \sup_{0 \leq s \leq \alpha t} X^{(1)}(s) + \inf_{0 \leq s \leq (1-\alpha)t} X^{(2)}(s) \\ X^{(1)}(\alpha t) + X^{(2)}((1-\alpha)t) \end{pmatrix} \quad (\text{in law}).$$

This result was obtained for the special case when $X(t)$ is a Brownian motion by Dassios [4] and Embrechts, Rogers and Yor [5]. Using this result, one could calculate an expression for the joint probability density of $M(\alpha, t)$ and $X(t)$. The motivation for these calculations was a problem in mathematical finance, the pricing of the so-called α -quantile (α -percentile) options. For the pricing of these options, see [1, 4, 6].

To prove our theorem, we will use a similar discrete time result, obtained some time ago by Wendel [9] and Port [7].

2. Discrete time results. Consider the sequence $\mathbf{x} = (x_0, x_1, x_2, \dots)$. For integers $0 \leq j \leq n$, define the (j, n) th quantile of \mathbf{x} for $j = 0, 1, 2, \dots, n$ by $M_{j,n}(\mathbf{x}) = \inf\{z : \sum_{i=0}^n \mathbf{1}(x_i \leq z) > j\}$.

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In particular, $M_{0,n}(\mathbf{x}) = \min_{i=0,1,\dots,n}(x_i)$ and $M_{n,n}(\mathbf{x}) = \max_{i=0,1,\dots,n}\{x_i\}$. This definition coincides with the one in [7].

The following result is due to Wendel [9].

PROPOSITION 1. *Let Y_1, Y_2, \dots be a sequence of i.i.d. random variables. Define $\mathbf{X} = (X_0, X_1, \dots)$ by $X_0 = 0$ and $X_n = \sum_{i=1}^n Y_i$ for $n = 1, 2, \dots$. Let $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ be two independent copies of \mathbf{X} . Then,*

$$(2.1) \quad \begin{pmatrix} M_{j,n}(\mathbf{X}) \\ X_n \end{pmatrix} = \begin{pmatrix} M_{j,j}(\mathbf{X}^{(1)}) + M_{0,n-j}(\mathbf{X}^{(2)}) \\ X_j^{(1)} + X_{n-j}^{(2)} \end{pmatrix} \quad (\text{in law}).$$

Wendel actually states this result in characteristic function form. An extension of this result, involving the time the quantile is achieved, was obtained by Port [7]. A careful reading of Port's proof can persuade the reader that the result is actually true where Y_1, Y_2, \dots, Y_n are exchangeable random variables, with $\mathbf{X}^{(1)}$ replaced by $(0, X_1, \dots, X_j)$ and $\mathbf{X}^{(2)}$ by $(0, X_{j+1} - X_j, \dots, X_n - X_j)$.

3. Proof of Theorem 1. Define the occupation time $L(x, t) = \int_0^t \mathbf{1}(X(s) \leq x) ds$. We have that

$$(3.1) \quad \Pr(M(\alpha, t) \leq x, X(t) \leq a) = \Pr(L(x, t) > \alpha t, X(t) \leq a).$$

Similarly for the discrete time process \mathbf{X} define $L_n(x) = \sum_{i=0}^n \mathbf{1}(x_i \leq x)$. We then have

$$(3.2) \quad \Pr(M_{j,n}(\mathbf{X}) \leq x, X_n \leq a) = \Pr(L_n(x) > j, X_n \leq a).$$

Without loss of generality, we will prove Theorem 1 for $t = 1$. Let $(X(s), 0 \leq s \leq 1)$ be as in the Introduction, and for $r = 0, 1, \dots, n$ set $X_r = X(r/n)$. The process $(X_{[ns]}, 0 \leq s \leq 1)$ converges weakly to $(X(s), 0 \leq s \leq 1)$. Since $\int_0^1 \mathbf{1}(X(s) \leq x) ds$ is a continuous functional of $(X(s), 0 \leq s \leq 1)$, we conclude that for all $0 < \alpha < 1$ such that $\Pr(L(x, t) > \alpha, X(t) \leq a)$ is continuous,

$$(3.3) \quad \lim_{n \rightarrow \infty} \Pr(L_n(x) > [n\alpha], X_n \leq a) = \Pr(L(x, 1) > \alpha, X(1) \leq a).$$

Take $X_r^{(1)} = X(r/n)$ for $r = 0, 1, \dots, [n\alpha]$, and $X_r^{(2)} = X(r + [n\alpha]/n) - X([n\alpha]/n)$ for $r = 0, 1, \dots, n - [n\alpha]$. Then $(M_{[n\alpha], [n\alpha]}(\mathbf{X}^{(1)}), X_{[n\alpha]})$ converges in distribution to $(\sup_{0 \leq s \leq \alpha} X(s), X(\alpha))$ and $(M_{0, n-[n\alpha]}(\mathbf{X}^{(2)}), X_n - X_{[n\alpha]})$ converges in distribution to $(\inf_{0 \leq s \leq 1-\alpha} (X(s + \alpha) - X(\alpha)), X(1) - X(\alpha))$. We note that $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ are independent and so from (2.1), (3.1), (3.2) and (3.3) we conclude

$$(3.4) \quad \begin{pmatrix} M(\alpha, 1) \\ X(1) \end{pmatrix} = \begin{pmatrix} \sup_{0 \leq s \leq \alpha} X(s) + \inf_{0 \leq s \leq 1-\alpha} (X(\alpha + s) - X(\alpha)) \\ X(\alpha) + X(1) - X(\alpha) \end{pmatrix} \quad (\text{in law}).$$

Taking $X^{(1)}(s) = X(s)$ and $X^{(2)}(s) = X(\alpha + s) - X(\alpha)$ completes the proof of the theorem.

Equation (3.4) is also true if $(X(t), t \geq 0)$ is a process with exchangeable increments. If $(X(t), t \geq 0)$ is defined on a probability space $(\Omega, \mathcal{F}, \Pr)$, according to [8], Theorem 1, page 202, there exists a nontrivial σ -algebra $\mathcal{F}_1 \subset \mathcal{F}$ such that $(X(t), t \geq 0)$ has stationary and independent increments with respect to \mathcal{F}_1 . This was first proved by Bühlman [3]. Conditioning on \mathcal{F}_1 and taking expectations, we obtain (3.4).

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REFERENCES

- [1] AKAHORI, J. (1995). Some formulae for a new type of path-dependent option. *Ann. Appl. Probab.* **5** 383–388.
- [2] BREIMAN, L. (1968). *Probability*. Addison-Wesley, Reading, MA.
- [3] BÜHLMANN, H. (1960). Austauschbare stochastische variabeln und ihre gremzwertsatze. *Univ. California Publications in Statistics* **3** 1–36.
- [4] DASSIOS, A. (1995). The distribution of the quantiles of a Brownian motion with drift and the pricing of related path-dependent options. *Ann. Appl. Probab.* **5** 389–398.
- [5] EMBRECHTS, P., ROGERS, L. C. G. and YOR, M. (1995). A proof of Dassios' representation of the α -quantile of Brownian motion with drift. *Ann. Appl. Probab.* **5** 757–767.
- [6] MIURA, R. (1992). A note on a look-back option on order statistics. *Hitotsubashi Journal of Commerce and Management* **27** 15–28.
- [7] PORT, S. C. (1963). An elementary probability approach to fluctuation theory. *J. Math. Anal. Appl.* **6** 109–151.
- [8] TAKÁCS, L. (1967). *Combinatorial Methods in the Theory of Stochastic Processes*. Wiley, New York.
- [9] WENDEL, J. G. (1960). Order statistics of partial sums. *Ann. Math. Statist.* **31** 1034–1044.

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References

¹ **Some Formulae for a New Type of Path-Dependent Option**

Jirô Akahori

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⁴ **The Distribution of the Quantile of a Brownian Motion with Drift and the Pricing of Related Path-Dependent Options**

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⁵ **A Proof of Dassios' Representation of the α -Quantile of Brownian Motion with Drift**

P. Embrechts; L. C. G. Rogers; M. Yor

The Annals of Applied Probability, Vol. 5, No. 3. (Aug., 1995), pp. 757-767.

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⁹ **Order Statistics of Partial Sums**

J. G. Wendel

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