Brownian excursions outside a corridor and two-sided Parisian options

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Abstract

In this paper, we study the excursion time of a Brownian motion with drift outside a corridor by using a four states semi-Markov model. In mathematical finance, these results have an important application in the valuation of double barrier Parisian options. In this paper, we obtain an explicit expression for the Laplace transform of its price.

Keywords: excursion time, four states Semi-Markov model, double barrier Parisian options, Laplace transform.

1 Introduction

The concept of Parisian options was first introduced by Chesney, Jeanblanc-Picque and Yor [6]. It is a special case of path dependent options. The owner of a Parisian option will either gain the right or lose the right to exercise the option upon the price reaching a predetermined barrier level L and staying above or below the level for a predetermined time d before the maturity date T.

More precisely, the owner of a *Parisian down-and-out option* loses the option if the underlying asset price S reaches the level L and remains constantly below this level for a time interval longer than d. For a *Parisian down-and-in option* the same event gives the owner the right to exercise the option. For details on the pricing of Parisian options see [6], [13], [15] and [12].

The double barrier Parisian options are a version with two barriers of the standard Parisian options introduced by Chesney, Jeanblanc-Picque and Yor [6]. In contrast to the Parisian options mentioned above, we consider the excursions both below the lower barrier and above the upper barrier, i.e. outside a corridor formed by these two barriers. Let us look at two examples, depending on whether the condition is that the required excursions above the upper barrier and below the lower barrier have to both happen before the maturity date or that either one of them happens before the maturity. In one example, the owner of a *double barrier Parisian max-out option* loses the option if the underlying asset process S has both an excursion above the upper barrier for longer than a continuous period d_1 and below lower the barrier for longer than d_2 before the maturity of the option. In the other example, the owner of a *double barrier Parisian min-out option* loses the right to exercise the option if either one of these two events happens before the maturity. Later on, we will derive the Laplace transforms which can be used to price this type of options.

In this paper, we are going to use the same definition for the excursion as in [6] and [7]. Let S be a stochastic process and l_1, l_2, $l_1 > l_2$ be the levels of these two barriers. As in [6], we define

$$g^S_{l_i,t} = \sup\{s \le t \mid S_s = l_i\}, \qquad d^S_{l_i,t} = \inf\{s \ge t \mid S_s = l_i\}, \qquad i = 1, 2, \quad (1)$$

with the usual conventions, $\sup\{\emptyset\} = 0$ and $\inf\{\emptyset\} = \infty$. Assuming $d_1 > 0$, $d_2 > 0$,we now define

$$\tau^S_1 = \inf\left\{t > 0 \mid \mathbf{1}_{\{S_t > l_1\}}(t - g^S_{l_1,t}) \ge d_1\right\}, \quad (2)$$

$$\tau^S_2 = \inf\left\{t > 0 \mid \mathbf{1}_{\{l_2 < S_t < l_1\}}\mathbf{1}_{\left\{g^S_{l_1,t} > g^S_{l_2,t}\right\}}(t - g^S_{l_1,t}) \ge d_2\right\}, \quad (3)$$

$$\tau^S_3 = \inf\left\{t > 0 \mid \mathbf{1}_{\{l_2 < S_t < l_1\}}\mathbf{1}_{\left\{g^S_{l_1,t} < g^S_{l_2,t}\right\}}(t - g^S_{l_2,t}) \ge d_3\right\}, \quad (4)$$

$$\tau^S_4 = \inf\{t > 0 \mid \mathbf{1}_{\{S_t < l_2\}}(t - g^S_{l_2,t}) \ge d_4\}, \quad (5)$$

$$\tau^S = \tau^S_1 \wedge \tau^S_4. \quad (6)$$

We can see that τ^S_1 is the first time that the length of the excursion of process S above the barrier l_1 reaches a given level d_1; τ^S_4 corresponds to the one below l_2 with required length d_4; and τ^S is the smaller of τ^S_1 and τ^S_4. We also see that τ^S_2 is the first time that the length of the excursion in the corridor reaches given level d_2, given that the excursion starts from the upper barrier l_1; τ^S_3 corresponds to the one in the corridor starting from the lower barrier l_2. Our aim is to study the excursion outside the corridor, therefore τ^S_2 and τ^S_3 are not of interest here. However we need to use these two stopping times to define our four states semi-Markov model that will be the main tool used for calculation.

Now assume r is the risk-free rate, T is the term of the option, S_t is the price of its underlying asset, K is the strike price and Q is the risk neutral measure. If we have a double barrier Parisian min-out call option with the barrier l_1 and l_2, its price can be expressed as:

$$DP_{min-out-call} = e^{-rT}E_Q\left(\mathbf{1}_{\{\tau^S > T\}}\left(S_T - K\right)^+\right);$$

and the price of a double barrier Parisian min-in put option is:

$$DP_{min-in-put} = e^{-rT}E_Q\left(\mathbf{1}_{\{\tau^S < T\}}\left(K - S_T\right)^+\right).$$

In this paper, we are going to study the excursion time outside the corridor using a semi-Markov model consisting of four states. By applying the model to a Brownian motion, we can get the explicit form of the Laplace transform for the price of double barrier options. One can then invert using techniques as in [13].

In Section 2 we introduce the four states semi-Markov model as well as a new process, the doubly perturbed Brownian motion, which has the same behavior as a Brownian motion except that each time it hits one of the two barriers, it moves towards the other side of the barrier by a jump of size ϵ. In Section 3 we obtain the martingale to which we can apply the optional sampling theorem and get the Laplace transform that we can use for pricing later. We give our main results applied to Brownian motion in Section 4, including the Laplace transforms for the stopping times we defined by(2)-(6) for both a Brownian motion with drift, i.e. $S = W^{\mu}$, and a standard Brownian motion, i.e. $S = W$. In Section 5, we focus on pricing the double barrier Parisian options.

2 Definitions

From the description above, it is clear that we are actually considering four states, the state when the stochastic process is above the barrier l_1 the state when it is below l_2 and two states when it is between l_1 and l_2 depending on whether it comes into the corridor through l_1 or l_2. For each state, we are interested in the time the process spends in it. We introduce a new process

$$Z_t^S = \begin{cases} 1, & \text{if } S_t > l_1 \\ 2, & \text{if } l_1 > S_t > l_2 \text{ and } g^S_{l_1,t} > g^S_{l_2,t} \\ 3, & \text{if } l_1 > S_t > l_2 \text{ and } g^S_{l_1,t} < g^S_{l_2,t} \\ 4, & \text{if } S_t < l_2 \end{cases} .$$

We can now express the variables defined above (see definitions (1)-(5)) in terms of Z_t:

$$g^S_{l_i,t} = \sup\left\{s \le t \mid Z^S_s \ne Z_t\right\}, \tag{7}$$

$$d^S_{l_i,t} = \inf\left\{s \ge t \mid Z^S_s \ne Z_t\right\}, \tag{8}$$

$$\tau_1^S = \inf\left\{t > 0 \mid \mathbf{1}_{\{Z^S_t=1\}}\left(t - g^S_{l_1,t}\right) \ge d_1\right\}, \tag{9}$$

$$\tau_2^S = \inf\left\{t > 0 \mid \mathbf{1}_{\{Z^S_t=2\}}\left(t - g^S_{l_1,t}\right) \ge d_2\right\}, \tag{10}$$

$$\tau_3^S = \inf\left\{t > 0 \mid \mathbf{1}_{\{Z^S_t=3\}}\left(t - g^S_{l_2,t}\right) \ge d_3\right\}, \tag{11}$$

$$\tau_4^S = \inf\left\{t > 0 \mid \mathbf{1}_{\{Z^S_t=4\}}\left(t - g^S_{l_2,t}\right) \ge d_4\right\}. \tag{12}$$

We then define

$$V_t^S = t - \max\left(g^S_{l_1,t}, g^S_{l_2,t}\right), \tag{13}$$

the time Z_t^S has spent in the current state. It is easy to see that (Z_t^S, V_t^S) is a Markov process. Z_t^S is therefore a semi-Markov process with the state space $\{1, 2, 3, 4\}$, where 1 stands for the state when the stochastic process S is above the barrier l_1; 4 corresponds to the state below the barrier l_2; 2 and 3 represent the state when S is in the corridor given that it comes into it through l_1 and l_2 respectively.

For Z_t^S, define the transition intensities $\lambda_{ij}(u)$ by

$$P\left(Z_{t+\Delta t}^S = j, i \neq j \mid Z_t^S = i, V_t^S = u\right) = \lambda_{ij}(u)\Delta t + o\left(\Delta t\right), \tag{14}$$

$$P\left(Z_{t+\Delta t}^S = i \mid Z_t^S = i, V_t^S = u\right) = 1 - \sum_{i \neq j} \lambda_{ij}(u)\Delta t + o\left(\Delta t\right). \tag{15}$$

Define

$$\bar{P}_i(\mu) = \exp\left\{-\int_0^\mu \sum_{i\neq j} \lambda_{ij}(v)\mathrm{d}v\right\}, \quad p_{ij}(\mu) = \lambda_{ij}(\mu)\bar{P}_i(\mu).$$

Notice that

$$P_i(\mu) = 1 - \bar{P}_i(\mu)$$

is the distribution function of the excursion time in state i, which is a random variable U_i defined as

$$U_i = \inf_{s>0}\left\{Z_s^S \neq i \mid Z_0^S = i,\ V_0^S = 0\right\}.$$

Note that because the process is time homogeneous this has the same distribution as

$$\inf_{s>0}\left\{Z_{t+s}^S \neq i \mid Z_t^S = i,\ V_t^S = 0\right\}.$$

for any time t. We have therefore

$$p_{ij}(\mu) = \lim_{\triangle\mu\to 0} \frac{P\left(U_i \in (\mu, \mu + \triangle\mu), Z_{U_i}^S = j\right)}{\triangle\mu}.$$

Moreover, in the definition of Z^S, we deliberately ignored the situation when $S_t = l_i$, $i = 1, 2$. The reason is that we only consider the processes, which

$$\int_0^t \mathbf{1}_{\{S_u = l_i\}}\mathrm{d}u = 0,\ i = 1, 2,\ a.s.$$

Also, when l_1 and l_2 are the regular points of the process (see [5] for definition), we have to deal with the degeneration of p_{ij}. Let us take a Brownian Motion as an example. Assume $W_t^\mu = \mu t + W_t$ with $\mu \geq 0$, where W_t is a standard Brownian Motion. Setting x_0 to be its starting point, we know its density for the first hitting time of level l_i, $i = 1, 2$ is

$$p_{x_0} = \frac{|l_i - x_0|}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(l_i - x_0 - \mu t)^2}{2t}\right\}$$

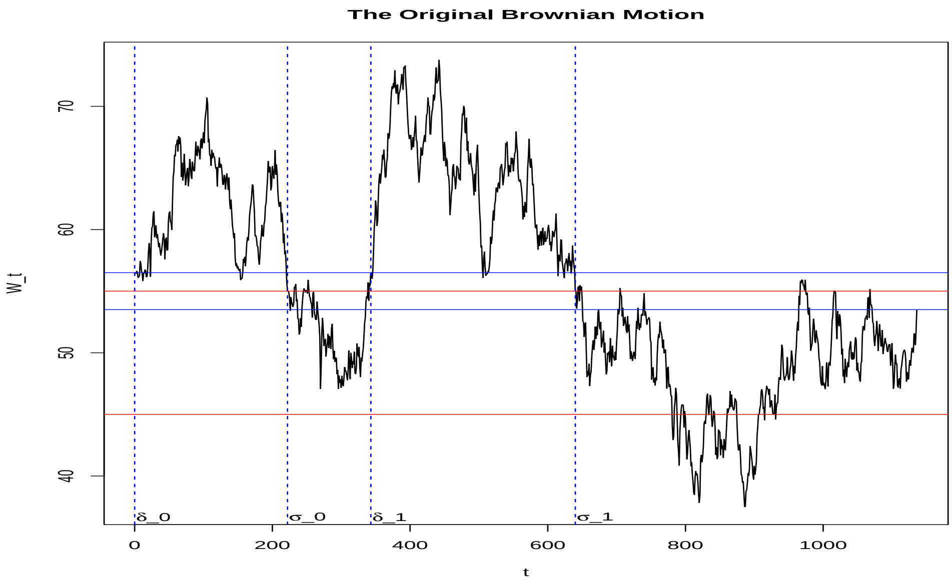

Figure 1: A Sample Path of W_t

(see [4]). According to the definition of the transition density, $p_{12}(t) = p_{21}(t) = p_{l_1}(t) = 0$ and $p_{34}(t) = p_{43}(t) = p_{l_2}(t) = 0$, for $t > 0$.

In [9] in order to solve the similar problem, we introduced the perturbed Brownian motion $X_t^{(\epsilon)}$ with the respect to the barrier we are interested in. We apply the same idea here, and construct a new process the *doubly perturbed Brownian motion*, $Y_t^{(\epsilon)}$, $l_1 - l_2 > \epsilon > 0$, with the respect to barriers l_1 and l_2. Assume $W_0^\mu = l_1 + \epsilon$. Define a sequence of stopping times

$$\begin{aligned}
\delta_0 &= 0,\\
\sigma_n &= \inf\{t > \delta_n \mid W_t^\mu = l_1\},\\
\delta_{n+1} &= \inf\{t > \sigma_n \mid W_t^\mu = l_1 + \epsilon\},
\end{aligned}$$

where $n = 0, 1, \cdots$ (see Figure 1). Now define

$$\begin{cases} X_t^{(\epsilon)} = W_t^\mu & \text{if} \quad \delta_n \le t < \sigma_n \\ X_t^{(\epsilon)} = W_t^\mu - \epsilon & \text{if} \quad \sigma_n \le t < \delta_{n+1} \end{cases}.$$

Similarly, we then define another sequence of stopping times with the respect to process $X_t^{(\epsilon)}$ and barrier l_2

$$\begin{aligned}
\zeta_0 &= 0,\\
\eta_n &= \inf\{t > \zeta_n \,|\, X_t^{(\epsilon)} = l_2\},\\
\zeta_{n+1} &= \inf\{t > \eta_n \,|\, X_t^{(\epsilon)} = l_2 + \epsilon\},
\end{aligned}$$

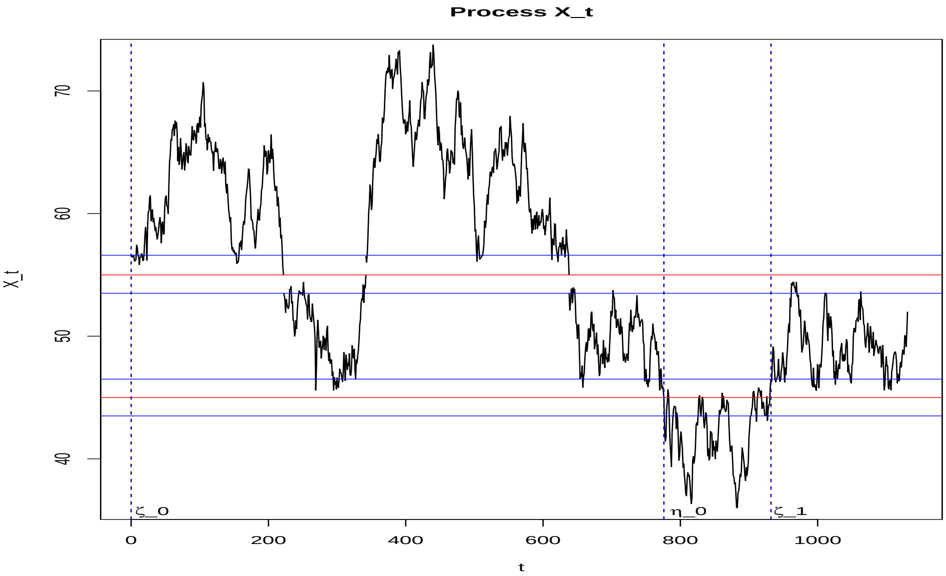

Figure 2: A Sample Path of $X_t^{(\epsilon)}$

where $n = 0, 1, \cdots$ (see Figure 2). Then define

$$\begin{cases} Y_t^{(\epsilon)} = X_t^{(\epsilon)} & \text{if} \quad \zeta_n \leq t < \eta_n \\ Y_t^{(\epsilon)} = X_t^{(\epsilon)} - \epsilon & \text{if} \quad \eta_n \leq t < \zeta_{n+1} \end{cases}.$$

It is actually a process which starts from $l_1 + \epsilon$ and has the same behavior as the related Brownian Motion expect that each time it hits the barrier l_1 or l_2, it will jump towards the opposite side of the barrier with size ϵ (see Figure 3).

From the definition, it is clear that l_1 and l_2 become irregular points for $Y_t^{(\epsilon)}$. Also $Y_t^{(\epsilon)}$ converges to W_t^μ with $W_0^\mu = l_1$ almost surely for all t. Therefore as we saw in [9], the Laplace transforms of the variables defined based on $Y_t^{(\epsilon)}$ converge to those based on W_t^μ. As a result, we can obtain the results for the Brownian Motion by carrying out the calculation for $Y_t^{(\epsilon)}$ and take the limit as $\epsilon \to 0$.

For $Y_t^{(\epsilon)}$, we can define Z^Y, τ_1^Y, τ_2^Y and τ^Y as above (we suppress (ϵ) on the

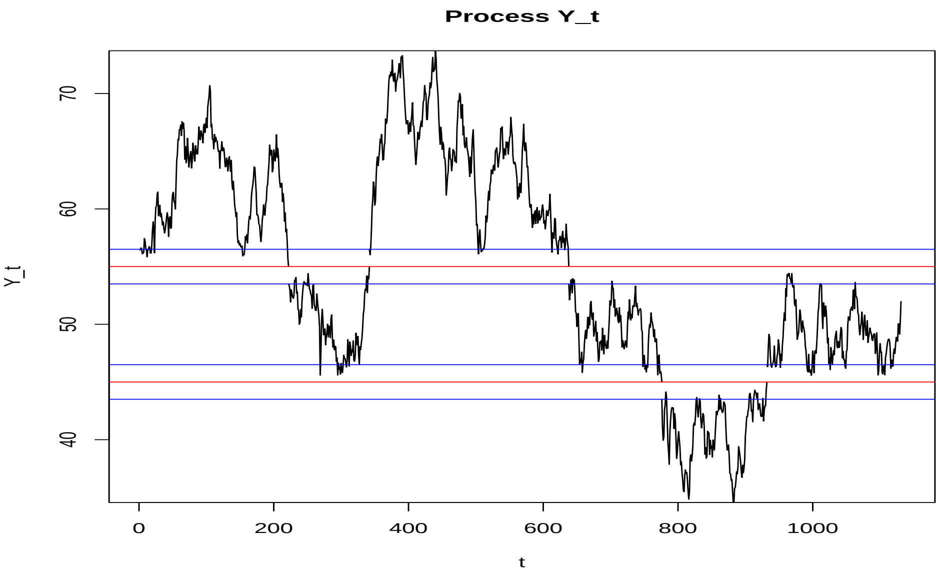

Figure 3: A Sample Path of $Y_t^{(\epsilon)}$

superscript). For Z^Y, we have the transition densities (see [4])

$$\begin{aligned}
p_{12}(t) &= \frac{\epsilon}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(\epsilon+\mu t)^2}{2t}\right\}, & (16)\\
p_{21}(t) &= \exp\left\{\mu\epsilon - \frac{\mu^2 t}{2}\right\} ss_t\left(l_1 - l_2 - \epsilon, l_1 - l_2\right), & (17)\\
p_{24}(t) &= \exp\left\{-\mu\left(l_1 - l_2 - \epsilon\right) - \frac{\mu^2 t}{2}\right\} ss_t\left(\epsilon, l_1 - l_2\right), & (18)\\
p_{31}(t) &= \exp\left\{\mu\left(l_1 - l_2 - \epsilon\right) - \frac{\mu^2 t}{2}\right\} ss_t\left(\epsilon, l_1 - l_2\right), & (19)\\
p_{34}(t) &= \exp\left\{-\mu\epsilon - \frac{\mu^2 t}{2}\right\} ss_t\left(l_1 - l_2 - \epsilon, l_1 - l_2\right), & (20)\\
p_{43}(t) &= \frac{\epsilon}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(\epsilon-\mu t)^2}{2t}\right\}, & (21)
\end{aligned}$$

where

$$ss_t(x,y) = \sum_{k=-\infty}^{\infty} \frac{(2k+1)y - x}{\sqrt{2\pi t^3}} \exp\left\{-\frac{\left((2k+1)y - x\right)^2}{2t}\right\}.$$

Also we know that

$$p_{23}(t) = p_{32}(t) = p_{14}(t) = p_{41}(t) = 0. \tag{22}$$

Clearly, all the arguments above apply to the standard Brownian motion, which is a special case of W_t^μ when $\mu = 0$.

3 Results for the semi-Markov model

In §2 we have introduced the Markov process (Z_t^S, V_t^S). Now we apply the same definition to the doubly perturbed Brownian motion Y_t; therefore we have (Z_t^Y, V_t^Y), where Z_t^Y is the current state of Y_t, taking value from state space $\{1, 2, 3, 4\}$ and V_t^Y is the time Y_t has spent in current state. V_t^Y is also a stochastic process. Now we consider a function of the form

$$f\left(V_t^Y, Z_t^Y, t\right) = f_{Z_t^Y}\left(V_t^Y, t\right),$$

where f_i, $i = 1, 2, 3, 4$ are functions from $\mathbb{R}^2$ to $\mathbb{R}$. The generator $\mathscr{A}$ is defined as an operator such that

$$f\left(V_t^Y, Z_t^Y, t\right) - \int_0^s \mathscr{A} f\left(V_s^Y, Z_s^Y, s\right) \mathrm{d}s$$

is a martingale (see [10], chapter 2). Therefore solving

$$\mathscr{A} f = 0$$

subject to certain conditions will provide us with martingales of the form $f\left(V_t^Y, Z_t^Y, t\right)$ to which we can apply the optional stopping theorem to obtain the Laplace transform we are interested in. More precisely, we will have

$$\begin{cases} \mathscr{A} f_1(u,t) &= \dfrac{\partial f_1(u,t)}{\partial t} + \dfrac{\partial f_1(u,t)}{\partial u} + \lambda_{12}(u)(f_2(0,t) - f_1(u,t)) \\ \mathscr{A} f_2(u,t) &= \dfrac{\partial f_2(u,t)}{\partial t} + \dfrac{\partial f_2(u,t)}{\partial u} + \lambda_{21}(u)(f_1(0,t) - f_2(u,t)) + \lambda_{24}(u)(f_4(0,t) - f_2(u,t)) \\ \mathscr{A} f_3(u,t) &= \dfrac{\partial f_3(u,t)}{\partial t} + \dfrac{\partial f_3(u,t)}{\partial u} + \lambda_{31}(u)(f_1(0,t) - f_3(u,t)) + \lambda_{34}(u)(f_4(0,t) - f_3(u,t)) \\ \mathscr{A} f_4(u,t) &= \dfrac{\partial f_4(u,t)}{\partial t} + \dfrac{\partial f_4(u,t)}{\partial u} + \lambda_{43}(u)(f_4(0,t) - f_3(u,t)) \end{cases},$$

Assume f_i has the form

$$f_i(u,t) = e^{-\beta t} g_i(u).$$

By solving the equation $\mathscr{A} f = 0$, i.e. $\left\{\begin{array}{lll} \mathscr{A} f_1 & = & 0 \\ \mathscr{A} f_2 & = & 0 \\ \mathscr{A} f_3 & = & 0 \\ \mathscr{A} f_4 & = & 0 \end{array}\right.$ subject to $\left\{\begin{array}{lll} g_1(d_1) & = & \alpha_1 \\ g_2(d_2) & = & \alpha_2 \\ g_3(d_2) & = & \alpha_3 \\ g_4(d_2) & = & \alpha_4 \end{array}\right.$

we can get

$$\begin{aligned} g_i(u) \quad = \quad & \alpha_i \exp\left\{-\int_u^{d_i}\left(\beta + \sum_{j\neq i}\lambda_{ij}(v)\right)\mathrm{d}v\right\} && (23) \\ & + \sum_{j\neq i} g_j(0)\int_u^{d_i}\lambda_{ij}(s)\exp\left\{-\int_u^s\left(\beta+\sum_{j\neq i}\lambda_{ij}(v)\right)\mathrm{d}v\right\}\mathrm{d}s. \end{aligned}$$

In our case, we are only interested in the excursion outside the corridor. Hence, we set d_2 and d_3 to be ∞. Also $\lim_{d_2\to\infty} g_2\left(d_2\right) = \lim_{d_3\to\infty} g_3\left(d_3\right) = 0$ gives $\alpha_2 = \alpha_3 = 0$. Therefore, we have

$$g_1(0) \quad = \quad \alpha_1 e^{-\beta d_1}\bar{P}_1(d_1) + \left\{g_1(0)\hat{P}_{21}(\beta) + g_4(0)\hat{P}_{24}(\beta)\right\}\tilde{P}_{12}(\beta), \qquad (24)$$

$$g_4(0) \quad = \quad \alpha_4 e^{-\beta d_4}\bar{P}_4(d_4) + \left\{g_1(0)\hat{P}_{31}(\beta) + g_4(0)\hat{P}_{34}(\beta)\right\}\tilde{P}_{43}(\beta). \qquad (25)$$

Solving (24) and (25) gives

$$\begin{aligned} & g_1(0) && (26) \\ = \quad & \frac{\alpha_1 e^{-\beta d_1}\bar{P}_1(d_1)\left(1-\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta)\right) + \alpha_4 e^{-\beta d_4}\bar{P}_4(d_4)\hat{P}_{24}(\beta)\tilde{P}_{12}(\beta)}{1-\hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)-\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta)+\hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta)-\hat{P}_{31}(\beta)\tilde{P}_{43}(\beta)\hat{P}_{24}(\beta)\tilde{P}_{12}(\beta)}, \\ & g_4(0) && (27) \\ = \quad & \frac{\alpha_4 e^{-\beta d_4}\bar{P}_4(d_4)\left(1-\hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)\right) + \alpha_1 e^{-\beta d_1}\bar{P}_1(d_1)\hat{P}_{31}(\beta)\tilde{P}_{43}(\beta)}{1-\hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)-\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta)+\hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta)-\hat{P}_{31}(\beta)\tilde{P}_{43}(\beta)\hat{P}_{24}(\beta)\tilde{P}_{12}(\beta)}. \end{aligned}$$

where

$$\hat{P}_{ij}(\beta) = \int_0^\infty e^{-\beta s} p_{ij}(s)\mathrm{d}s, \qquad (28)$$

$$\tilde{P}_{ij}(\beta) = \int_0^{d_i} e^{-\beta s} p_{ij}(s)\mathrm{d}s. \qquad (29)$$

As a result, we have obtained the martingale

$$M_t = f\left(V_t^Y, t\right) = e^{-\beta t} g_{Z_t^Y}\left(V_t^Y\right), \quad i = 1,2,3,4. \qquad (30)$$

We now can apply the optional stopping theorem to M_t with the stopping time $\tau^Y \wedge t$, where τ^Y is the stopping time defined by (6):

$$E\left(M_{\tau^Y\wedge t}\right) = E\left(M_0\right). \qquad (31)$$

The right hand side of (31) is

$$E\left(M_{\tau^Y\wedge t}\right)=E\left(M_{\tau^Y}\mathbf{1}_{\{\tau^Y<t\}}\right)+E\left(M_t\mathbf{1}_{\{\tau^Y>t\}}\right).$$

Furthermore,

$$\begin{aligned}
& E\left(M_{\tau^Y}\mathbf{1}_{\{\tau^Y<t\}}\right)\\
= & E\left(M_{\tau^Y}\mathbf{1}_{\left\{\tau_1^Y<\tau_4^Y\right\}}\mathbf{1}_{\left\{\tau_1^Y<t\right\}}\right)+E\left(M_{\tau^Y}\mathbf{1}_{\left\{\tau_1^Y>\tau_4^Y\right\}}\mathbf{1}_{\left\{\tau_4^Y<t\right\}}\right)\\
= & E\left(e^{-\beta\tau^Y}g_1\left(d_1\right)\mathbf{1}_{\left\{\tau_1^Y<\tau_4^Y\right\}}\mathbf{1}_{\left\{\tau_1^Y<t\right\}}\right)+E\left(e^{-\beta\tau^Y}g_4\left(d_4\right)\mathbf{1}_{\left\{\tau_1^Y>\tau_4^Y\right\}}\mathbf{1}_{\left\{\tau_4^Y<t\right\}}\right)\\
= & \alpha_1 E\left(e^{-\beta\tau^Y}\mathbf{1}_{\left\{\tau_1^Y<\tau_4^Y\right\}}\mathbf{1}_{\left\{\tau_1^Y<t\right\}}\right)+\alpha_4 E\left(e^{-\beta\tau^Y}\mathbf{1}_{\left\{\tau_1^Y>\tau_4^Y\right\}}\mathbf{1}_{\left\{\tau_4^Y<t\right\}}\right).
\end{aligned}$$

We also have

$$E\left(M_t\mathbf{1}_{\{\tau^Y>t\}}\right)=e^{-\beta t}E\left(g_{Z_t^Y}\left(V_t^Y\right)\mathbf{1}_{\{\tau^Y>t\}}\right),$$

where Z_t^Y can take values $1,2,3$ or 4.

When $Z_t^Y=1$ or 4, since $\tau^Y>t$, we have $0\le V_t^Y<d_1\wedge d_4$. According to the definition of $g_i(\mu)$ in (23), we have $g_1\left(V_t^Y\right)$ and $g_4\left(V_t^Y\right)$ are bounded.

When $Z_t^Y=2$ or 3, since $\lim_{d_2\to\infty}g_2\left(d_2\right)=\lim_{d_3\to\infty}g_3\left(d_3\right)=0$ and looking at (23) with d_2 and d_3 replaced by ∞ we have that $g_2\left(V_t^Y\right)$ and $g_3\left(V_t^Y\right)$ are bounded.

Therefore

$$\lim_{t\to\infty}E\left(M_t\mathbf{1}_{\{\tau^Y>t\}}\right)=0.$$

Hence we have

$$\lim_{t\to\infty}E\left(M_{\tau^Y\wedge t}\right)=\alpha_1 E\left(e^{-\beta\tau^Y}\mathbf{1}_{\left\{\tau_1^Y<\tau_4^Y\right\}}\right)+\alpha_4 E\left(e^{-\beta\tau^Y}\mathbf{1}_{\left\{\tau_1^Y>\tau_4^Y\right\}}\right). \tag{32}$$

The left hand side of (31) gives

$$\lim_{t\to\infty}E\left(M_0\right)=E\left(M_0\right)=\begin{cases} g_1(0), & Y_0^{(\epsilon)}=l_1+\epsilon\\ g_4(0), & Y_0^{(\epsilon)}=l_2-\epsilon\end{cases}.$$

By taking the proper α_1 and α_4, we will have when $Y_0^{(\epsilon)}=l_1+\epsilon$

$$\begin{aligned}
& E\left(e^{-\beta\tau^Y}\mathbf{1}_{\left\{\tau_1^Y<\tau_4^Y\right\}}\right) \qquad (33)\\
= & \frac{e^{-\beta d_1}\bar{P}_{12}(d_1)\left(1-\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta)\right)}{1-\hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)-\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta)+\hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta)-\hat{P}_{31}(\beta)\tilde{P}_{43}(\beta)\hat{P}_{24}(\beta)\tilde{P}_{12}(\beta)},\\
& E\left(e^{-\beta\tau^Y}\mathbf{1}_{\left\{\tau_1^Y>\tau_4^Y\right\}}\right) \qquad (34)\\
= & \frac{e^{-\beta d_4}\bar{P}_{43}(d_4)\hat{P}_{24}(\beta)\tilde{P}_{12}(\beta)}{1-\hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)-\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta)+\hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta)-\hat{P}_{31}(\beta)\tilde{P}_{43}(\beta)\hat{P}_{24}(\beta)\tilde{P}_{12}(\beta)};
\end{aligned}$$

when $Y_0^{(\epsilon)} = l_2 - \epsilon$

$$
\begin{aligned}
& E\left(e^{-\beta\tau^Y}\mathbf{1}_{\{\tau_1^Y<\tau_4^Y\}}\right) & (35)\\
= \; & \frac{e^{-\beta d_1}\bar{P}_{12}(d_1)\hat{P}_{31}(\beta)\tilde{P}_{43}(\beta)}{1-\hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)-\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta)+\hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta)-\hat{P}_{31}(\beta)\tilde{P}_{43}(\beta)\hat{P}_{24}(\beta)\tilde{P}_{12}(\beta)}, \\
& E\left(e^{-\beta\tau^Y}\mathbf{1}_{\{\tau_1^Y>\tau_4^Y\}}\right) & (36)\\
= \; & \frac{e^{-\beta d_4}\bar{P}_{43}(d_4)\left(1-\hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)\right)}{1-\hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)-\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta)+\hat{P}_{21}(\beta)\tilde{P}_{12}(\beta)\hat{P}_{34}(\beta)\tilde{P}_{43}(\beta)-\hat{P}_{31}(\beta)\tilde{P}_{43}(\beta)\hat{P}_{24}(\beta)\tilde{P}_{12}(\beta)}.
\end{aligned}
$$

4 Main Results

In §2 we have stated that the main difficulty with the Brownian Motion is that its origin point is regular, i.e. the probability that W_t^μ will return to the origin at arbitrarily small time is 1. We have therefore introduced the new processes $Y_t^{(\epsilon)}$ and (Z_t^Y, V_t^Y) with transition densities for Z_t^Y defined in (16) to (22).

In order to simplify the expressions, we define

$$\Psi(x) = 2\sqrt{\pi}x\mathscr{N}\left(\sqrt{2}x\right) - \sqrt{\pi}x + e^{-x^2},$$

where $\mathscr{N}(.)$ is the cumulative distribution function for the standard Normal Distribution.

Theorem 1 *For a Brownian Motion W_t^μ, $\tau_1^{W^\mu}$, $\tau_4^{W^\mu}$, τ^{W^μ} defined as in (2), (5) and (6) with $S_t = W_t^\mu$, we have the following Laplace transforms:*

when $W_0^\mu = l_1$,

$$
\begin{aligned}
E\left(e^{-\beta\tau^{W^\mu}}\mathbf{1}_{\{\tau_1^{W^\mu}<\tau_4^{W^\mu}\}}\right) &= \frac{G_1(d_1,d_4,\mu)}{G(d_1,d_4,\mu)}; & (37)\\
E\left(e^{-\beta\tau^{W^\mu}}\mathbf{1}_{\{\tau_1^{W^\mu}>\tau_4^{W^\mu}\}}\right) &= \frac{G_2(d_4,d_1,-\mu)}{G(d_1,d_4,\mu)}; & (38)\\
E\left(e^{-\beta\tau^{W^\mu}}\right) &= \frac{G_1(d_1,d_4,\mu)+G_2(d_4,d_1,-\mu)}{G(d_1,d_4,\mu)}; & (39)
\end{aligned}
$$

when $W_0^\mu = l_2$,

$$
\begin{aligned}
E\left(e^{-\beta\tau^{W^\mu}}\mathbf{1}_{\{\tau_1^{W^\mu}<\tau_4^{W^\mu}\}}\right) &= \frac{G_2(d_1,d_4,\mu)}{G(d_1,d_4,\mu)}; & (40)\\
E\left(e^{-\beta\tau^{W^\mu}}\mathbf{1}_{\{\tau_1^{W^\mu}>\tau_4^{W^\mu}\}}\right) &= \frac{G_1(d_4,d_1,-\mu)}{G(d_1,d_4,\mu)}; & (41)\\
E\left(e^{-\beta\tau^{W^\mu}}\right) &= \frac{G_1(d_4,d_1,-\mu)+G_2(d_1,d_4,\mu)}{G(d_1,d_4,\mu)}; & (42)
\end{aligned}
$$

where

$$\begin{aligned} G_1(x,y,z) &= e^{-2(l_1-l_2)\sqrt{2\beta+z^2}-\beta x}\left\{\sqrt{y}\Psi\left(|z|\sqrt{\frac{x}{2}}\right)+z\sqrt{\frac{\pi xy}{2}}\right\} \\ &+\frac{\left(1-e^{-2(l_1-l_2)\sqrt{2\beta+z^2}}\right)e^{-\beta x}}{2\sqrt{2\beta+z^2}}\left\{\Psi\left(|z|\sqrt{\frac{x}{2}}\right)+z\sqrt{\frac{\pi x}{2}}\right\} \\ &\left\{\sqrt{\frac{2}{\pi}}\Psi\left(\sqrt{\frac{(2\beta+z^2)\,y}{2}}\right)+\sqrt{(2\beta+z^2)\,y}\right\}, \end{aligned} \tag{43}$$

$$G_2(x,y,z)=e^{-(l_1-l_2)\left(\sqrt{2\beta+z^2}-z\right)-\beta x}\left\{\sqrt{y}\Psi\left(|z|\sqrt{\frac{x}{2}}\right)+z\sqrt{\frac{\pi xy}{2}}\right\}, \tag{44}$$

$$\begin{aligned} G(x,y,z) &= e^{-2(l_1-l_2)\sqrt{2\beta+z^2}}\left\{\sqrt{y}\Psi\left(\sqrt{\frac{(2\beta+z^2)\,x}{2}}\right)+\sqrt{x}\Psi\left(\sqrt{\frac{(2\beta+z^2)\,y}{2}}\right)\right\} \\ &+\frac{\left(1-e^{-2(l_1-l_2)\sqrt{2\beta+z^2}}\right)}{2\sqrt{2\beta+z^2}}\left\{\Psi\left(\sqrt{\frac{(2\beta+z^2)\,x}{2}}\right)+\sqrt{\frac{(2\beta+z^2)\,\pi x}{2}}\right\} \\ &\left\{\sqrt{\frac{2}{\pi}}\Psi\left(\sqrt{\frac{(2\beta+z^2)\,y}{2}}\right)+\sqrt{(2\beta+z^2)\,y}\right\}. \end{aligned} \tag{45}$$

Proof: We apply the transition densities in (16) to (22) to the results in (33) to (36) and take the limit as $\epsilon \to 0$. According to the definition of $Y^{(\epsilon)}$, we know that

$$Y_t^{(\epsilon)} \xrightarrow{a.s} W_t^{\mu}, \quad \text{for all } t.$$

As we saw in [9], since $Y_t^{(\epsilon)} \xrightarrow{a.s} W_t^{\mu}$, for all t, by taking the limit $\epsilon \to 0$, the quantities defined based on $Y_t^{(\epsilon)}$ converge to those based on Brownian motion with drift. Therefore we will get the results shown by (37), (38), (40) and (41). We can therefore get (39) and (42) by

$$E\left(e^{-\beta\tau^{W^\mu}}\right)=E\left(e^{-\beta\tau^{W^\mu}}\mathbf{1}_{\{\tau_1^{W^\mu}<\tau_4^{W^\mu}\}}\right)+E\left(e^{-\beta\tau^{W^\mu}}\mathbf{1}_{\{\tau_1^{W^\mu}>\tau_4^{W^\mu}\}}\right).$$

□

Corollary 1.1 *For a standard Brownian Motion ($\mu = 0$), we have when $W_0 = l_1$,*

$$E\left(e^{-\beta\tau^W}\mathbf{1}_{\{\tau_1^W<\tau_4^W\}}\right) = \frac{G_1(d_1,d_4,0)}{G(d_1,d_4,0)}; \tag{46}$$

$$E\left(e^{-\beta\tau^W}\mathbf{1}_{\{\tau_1^W>\tau_4^W\}}\right) = \frac{G_2(d_4,d_1,0)}{G(d_1,d_4,0)}; \tag{47}$$

$$E\left(e^{-\beta\tau^W}\right) = \frac{G_1(d_1,d_4,0)+G_2(d_4,d_1,0)}{G(d_1,d_4,0)}; \tag{48}$$

when $W_0 = l_2$,

$$E\left(e^{-\beta\tau^W}\mathbf{1}_{\{\tau_1^W<\tau_4^W\}}\right) = \frac{G_2(d_1,d_4,0)}{G(d_1,d_4,0)}; \tag{49}$$

$$E\left(e^{-\beta\tau^W}\mathbf{1}_{\{\tau_1^W>\tau_4^W\}}\right) = \frac{G_1(d_4,d_1,0)}{G(d_1,d_4,0)}; \tag{50}$$

$$E\left(e^{-\beta\tau^W}\right) = \frac{G_1(d_4,d_1,0)+G_2(d_1,d_4,0)}{G(d_1,d_4,0)}; \tag{51}$$

where

$$\begin{aligned} G_1(x,y,0) &= e^{-2(l_1-l_2)\sqrt{2\beta}-\beta x}\sqrt{y} \\ &\quad + \frac{\left(1-e^{-2(l_1-l_2)\sqrt{2\beta}}\right)e^{-\beta x}}{2\sqrt{2\beta}}\left\{\sqrt{\frac{2}{\pi}}\Psi\left(\sqrt{\beta y}\right)+\sqrt{2\beta y}\right\}, \end{aligned} \tag{52}$$

$$G_2(x,y,0) = e^{-(l_1-l_2)\sqrt{2\beta}-\beta x}\sqrt{y}, \tag{53}$$

$$\begin{aligned} G(x,y,0) &= e^{-2(l_1-l_2)\sqrt{2\beta}}\left\{\sqrt{y}\Psi\left(\sqrt{\beta x}\right)+\sqrt{x}\Psi\left(\sqrt{\beta y}\right)\right\} \\ &\quad + \frac{\left(1-e^{-2(l_1-l_2)\sqrt{2\beta}}\right)}{2\sqrt{2\beta}}\left\{\Psi\left(\sqrt{\beta x}\right)+\sqrt{\beta\pi x}\right\}\left\{\sqrt{\frac{2}{\pi}}\Psi\left(\sqrt{\beta y}\right)+\sqrt{2\beta y}\right\}. \end{aligned} \tag{54}$$

Remark 1: By taking the limit $l_1 - l_2 \to 0$, we can get the result for the single barrier two-sided excursion case as in [9].

Remark 2: If we only want to consider the excursion above a barrier, we can let $l_2 \to -\infty$. Similarly, for the one below a barrier, we can let $l_1 \to +\infty$. These results have been shown in [9].

Corollary 1.2 *For a Brownian Motion* W_t^μ, τ^{W^μ} *defined as in (6) with* $S_t = W_t^\mu$, *we have the following Laplace transforms:*

when $W_0^\mu = x_0$, $x_0 > l_1$,

$$\begin{aligned} &E\left(e^{-\beta\tau^{W^\mu}}\right) \\ =\ &\left\{e^{-\left(\mu+\sqrt{2\beta+\mu^2}\right)(x_0-l_1)}\mathcal{N}\left(\sqrt{(2\beta+\mu^2)\,d_1}-\frac{x_0-l_1}{\sqrt{d_1}}\right)\right. \\ &\left.+e^{-\left(\mu-\sqrt{2\beta+\mu^2}\right)(x_0-l_1)}\mathcal{N}\left(-\sqrt{(2\beta+\mu^2)\,d_1}-\frac{x_0-l_1}{\sqrt{d_1}}\right)\right\}\frac{G_1\left(d_1,d_4,\mu\right)+G_2\left(d_4,d_1,-\mu\right)}{G\left(d_1,d_4,\mu\right)} \\ &+e^{-\beta d_1}\left\{1-e^{-(\mu+|\mu|)(x_0-l_1)}\mathcal{N}\left(|\mu|\sqrt{d_1}-\frac{x_0-l_1}{\sqrt{d_1}}\right)\right. \\ &\left.-e^{-(\mu-|\mu|)(x_0-l_1)}\mathcal{N}\left(-|\mu|\sqrt{d_1}-\frac{x_0-l_1}{\sqrt{d_1}}\right)\right\}; \end{aligned} \tag{55}$$

when $W_0^\mu = x_0,\ l_2 \le x_0 \le l_1$,

$$
\begin{aligned}
& E\left(e^{-\beta\tau^{W^\mu}}\right) & (56)\\
= & \frac{e^{(l_1-x_0)\mu}\left\{e^{\sqrt{2\beta+\mu^2}(x_0-l_2)}-e^{-\sqrt{2\beta+\mu^2}(x_0-l_2)}\right\}\{G_1(d_1,d_4,\mu)+G_2(d_4,d_1,-\mu)\}}{\left\{e^{\sqrt{2\beta+\mu^2}(l_1-l_2)}-e^{-\sqrt{2\beta+\mu^2}(l_1-l_2)}\right\}G\left(d_1,d_2,\mu\right)} \\
& + \frac{e^{(l_2-x_0)\mu}\left\{e^{\sqrt{2\beta+\mu^2}(l_1-x_0)}-e^{-\sqrt{2\beta+\mu^2}(l_1-x_0)}\right\}\{G_2\left(d_1,d_4,\mu\right)+G_1(d_4,d_1,-\mu)\}}{\left\{e^{\sqrt{2\beta+\mu^2}(l_1-l_2)}-e^{-\sqrt{2\beta+\mu^2}(l_1-l_2)}\right\}G\left(d_1,d_2,\mu\right)};
\end{aligned}
$$

when $W_0^\mu = x_0,\ x_0 < l_2$,

$$
\begin{aligned}
& E\left(e^{-\beta\tau^{W^\mu}}\right) & (57)\\
= & \left\{e^{\left(\mu-\sqrt{2\beta+\mu^2}\right)(l_2-x)}\mathcal{N}\left(\sqrt{\left(2\beta+\mu^2\right)d_4}-\frac{l_2-x}{\sqrt{d_4}}\right)\right. \\
& \left. +e^{\left(\mu+\sqrt{2\beta+\mu^2}\right)(l_2-x)}\mathcal{N}\left(-\sqrt{\left(2\beta+\mu^2\right)d_4}-\frac{l_2-x}{\sqrt{d_4}}\right)\right\}\frac{G_1\left(d_4,d_1,-\mu\right)+G_2\left(d_1,d_4,\mu\right)}{G\left(d_1,d_4,\mu\right)} \\
& +e^{-\beta d_4}\left\{1-e^{(\mu-|\mu|)(l_2-x)}\mathcal{N}\left(|\mu|\sqrt{d_4}-\frac{l_2-x}{\sqrt{d_4}}\right)\right. \\
& \left. -e^{(\mu+|\mu|)(l_2-x)}\mathcal{N}\left(-|\mu|\sqrt{d_4}-\frac{l_2-x}{\sqrt{d_4}}\right)\right\}. & (58)
\end{aligned}
$$

Proof: We will first prove the case when $x_0 > l_1$. Define $T = \inf\{t \mid W_t^\mu = l_1\}$, i.e the first time W_t^μ hits l_1. By definition, we have $\tau^{W^\mu} = d_1$, if $T \ge d_1$; $\tau^{W^\mu} = T + \tau^{\tilde{W}^\mu}$, if $T < d_1$, where $\tilde{W}^\mu$ here stands for a Brownian motion with drift started from l_1. As a result

$$
\begin{aligned}
& E\left(e^{-\beta\tau^{W^\mu}}\right) \\
= & E\left(e^{-\beta\tau^{W^\mu}}\mathbf{1}_{\{T\ge d_1\}}\right)+E\left(e^{-\beta\tau^{W^\mu}}\mathbf{1}_{\{T<d_1\}}\right) \\
= & e^{-\beta d_1}P\left(T\ge d_1\right)+E\left(e^{-\beta T}\mathbf{1}_{\{T<d_1\}}\right)E\left(e^{-\beta\tau^{\widetilde{W}^\mu}}\right)
\end{aligned}
$$

$E\left(e^{-\beta\tau^{\widetilde{W}^\mu}}\right)$ has been calculated in Theorem 1 (see (39)). The density for T is given in [4] as

$$
p_{x_0} = \frac{|l_1-x_0|}{\sqrt{2\pi t^3}}\exp\left\{-\frac{(l_i-x_0-\mu t)^2}{2t}\right\}.
$$

We can therefore calculate

$$
\begin{aligned}
P\left(T\ge d_1\right) = & 1-e^{-(\mu+|\mu|)(x_0-l_1)}\mathcal{N}\left(|\mu|\sqrt{d_1}-\frac{x_0-l_1}{\sqrt{d_1}}\right) \\
& -e^{-(\mu-|\mu|)(x_0-l_1)}\mathcal{N}\left(-|\mu|\sqrt{d_1}-\frac{x_0-l_1}{\sqrt{d_1}}\right),
\end{aligned}
$$

$$
\begin{aligned}
E\left(e^{-\beta T}\mathbf{1}_{\{T<d_1\}}\right) &= e^{-\left(\mu+\sqrt{2\beta+\mu^2}\right)(x_0-l_1)}\mathscr{N}\left(\sqrt{(2\beta+\mu^2)\,d_1}-\frac{x_0-l_1}{\sqrt{d_1}}\right)\\
&\quad +e^{-\left(\mu-\sqrt{2\beta+\mu^2}\right)(x_0-l_1)}\mathscr{N}\left(-\sqrt{(2\beta+\mu^2)\,d_1}-\frac{x_0-l_1}{\sqrt{d_1}}\right).
\end{aligned}
$$

We therefore get the result in (55). For the case when $x_0 < l_2$, we can apply the same argument.

When $l_2 \le x_0 \le l_1$, we define $\widetilde{T} = \inf\left(t \mid W_t^\mu \notin (l_2, l_1)\right)$. By definition, we have $\tau^{W^\mu} = T + \tau^{\widetilde{W}^\mu}$, if $W_T^\mu = l_1$; $\tau^{W^\mu} = T + \tau^{\underline{W}^\mu}$, if $W_T^\mu = l_2$, where $\underline{W}^\mu$ stands for a Brownian motion with drift started from l_2. Consequently,

$$
\begin{aligned}
& E\left(e^{-\beta\tau^{W^\mu}}\right)\\
=\;& E\left(e^{-\beta T}e^{-\beta\tau^{\widetilde{W}^\mu}}\mathbf{1}_{\{T=l_1\}}\right)+E\left(e^{-\beta T}e^{-\beta\tau^{\underline{W}^\mu}}\mathbf{1}_{\{T=l_2\}}\right)\\
=\;& E\left(e^{-\beta T}\mathbf{1}_{\{T=l_1\}}\right)E\left(e^{-\beta\tau^{\widetilde{W}^\mu}}\right)+E\left(e^{-\beta T}\mathbf{1}_{\{T=l_2\}}\right)E\left(e^{-\beta\tau^{\underline{W}^\mu}}\right)
\end{aligned}
$$

$E\left(e^{-\beta\tau^{\widetilde{W}^\mu}}\right)$ and $E\left(e^{-\beta\tau^{\underline{W}^\mu}}\right)$ have been obtained by Theorem 1, (39) and (42). According to [4], we have

$$
E\left(e^{-\beta T}\mathbf{1}_{\{T=l_1\}}\right)=\frac{e^{(l_1-x_0)\mu}\left\{e^{\sqrt{2\beta+\mu^2}(x_0-l_2)}-e^{-\sqrt{2\beta+\mu^2}(x_0-l_2)}\right\}}{e^{\sqrt{2\beta+\mu^2}(l_1-l_2)}-e^{-\sqrt{2\beta+\mu^2}(l_1-l_2)}},
$$

$$
E\left(e^{-\beta T}\mathbf{1}_{\{T=l_2\}}\right)=\frac{e^{(l_2-x_0)\mu}\left\{e^{\sqrt{2\beta+\mu^2}(l_1-x_0)}-e^{-\sqrt{2\beta+\mu^2}(l_1-x_0)}\right\}}{e^{\sqrt{2\beta+\mu^2}(l_1-l_2)}-e^{-\sqrt{2\beta+\mu^2}(l_1-l_2)}}.
$$

We have therefore obtained (56). □

Theorem 2 *The probability that W_t^μ with $W_0^\mu = x_0$, $l_2 \le x_0 \le l_1$, achieves an excursion above l_1 with length as least d_1 before it achieves an excursion below l_2 with length at least d_4 is*

$$
\begin{aligned}
P\left(\tau_1^{W^\mu}<\tau_4^{W^\mu}\right) &= \frac{e^{(l_1-x_0)\mu}\left\{e^{|\mu|(x_0-l_2)}-e^{-|\mu|(x_0-l_2)}\right\}F_1(d_1,d_4,\mu)}{\left\{e^{|\mu|(l_1-l_2)}-e^{-|\mu|(l_1-l_2)}\right\}F(d_1,d_4,\mu)} \qquad (59)\\
&\quad+\frac{e^{(l_2-x_0)\mu}\left\{e^{|\mu|(l_1-x_0)}-e^{-|\mu|(l_1-x_0)}\right\}F_2(d_1,d_4,\mu)}{\left\{e^{|\mu|(l_1-l_2)}-e^{-|\mu|(l_1-l_2)}\right\}F(d_1,d_4,\mu)},
\end{aligned}
$$

$$
\begin{aligned}
P\left(\tau_1^{W^\mu}>\tau_4^{W^\mu}\right) &= \frac{e^{(l_1-x_0)\mu}\left\{e^{|\mu|(x_0-l_2)}-e^{-|\mu|(x_0-l_2)}\right\}F_2(d_4,d_1,-\mu)}{\left\{e^{|\mu|(l_1-l_2)}-e^{-|\mu|(l_1-l_2)}\right\}F(d_1,d_4,\mu)} \qquad (60)\\
&\quad+\frac{e^{(l_2-x_0)\mu}\left\{e^{|\mu|(l_1-x_0)}-e^{-|\mu|(l_1-x_0)}\right\}F_1(d_4,d_1,-\mu)}{\left\{e^{|\mu|(l_1-l_2)}-e^{-|\mu|(l_1-l_2)}\right\}F(d_1,d_4,\mu)};
\end{aligned}
$$

where

$$F_1(x,y,z) = e^{-2(l_1-l_2)|z|}\left\{\sqrt{y}\Psi\left(|z|\sqrt{\frac{x}{2}}\right)+z\sqrt{\frac{\pi xy}{2}}\right\} \quad (61)$$
$$+\frac{\left(1-e^{-2(l_1-l_2)|z|}\right)}{2|z|}\left\{\Psi\left(|z|\sqrt{\frac{x}{2}}\right)+z\sqrt{\frac{\pi x}{2}}\right\}\left\{\sqrt{\frac{2}{\pi}}\Psi\left(|z|\sqrt{\frac{y}{2}}\right)+|z|\sqrt{y}\right\},$$

$$F_2(x,y,z) = e^{-(l_1-l_2)(|z|-z)}\left\{\sqrt{y}\Psi\left(|z|\sqrt{\frac{x}{2}}\right)+z\sqrt{\frac{\pi xy}{2}}\right\}, \quad (62)$$

$$F(x,y,z) = e^{-2(l_1-l_2)|z|}\left\{\sqrt{y}\Psi\left(|z|\sqrt{\frac{x}{2}}\right)+\sqrt{x}\Psi\left(|z|\sqrt{\frac{y}{2}}\right)\right\} \quad (63)$$
$$+\frac{\left(1-e^{-2(l_1-l_2)|z|}\right)}{2|z|}\left\{\Psi\left(|z|\sqrt{\frac{x}{2}}\right)+|z|\sqrt{\frac{\pi x}{2}}\right\}\left\{\sqrt{\frac{2}{\pi}}\Psi\left(|z|\sqrt{\frac{y}{2}}\right)+|z|\sqrt{y}\right\}.$$

Proof: From Theorem 1 and (56) in Corollary 1.2, we actually know that, when $W_0^\mu = x_0$, $l_2 \le x_0 \le l_1$,

$$E\left(e^{-\beta\tau^{W^\mu}}\mathbf{1}_{\{\tau_1^{W^\mu}<\tau_4^{W^\mu}\}}\right) = \frac{e^{(l_1-x_0)\mu}\left\{e^{|\mu|(x_0-l_2)}-e^{-|\mu|(x_0-l_2)}\right\}G_1(d_1,d_4,\mu)}{\left\{e^{|\mu|(l_1-l_2)}-e^{-|\mu|(l_1-l_2)}\right\}G(d_1,d_4,\mu)} \quad (64)$$
$$+\frac{e^{(l_2-x_0)\mu}\left\{e^{|\mu|(l_1-x_0)}-e^{-|\mu|(l_1-x_0)}\right\}G_2(d_1,d_4,\mu)}{\left\{e^{|\mu|(l_1-l_2)}-e^{-|\mu|(l_1-l_2)}\right\}G(d_1,d_4,\mu)},$$

$$E\left(e^{-\beta\tau^{W^\mu}}\mathbf{1}_{\{\tau_1^{W^\mu}>\tau_4^{W^\mu}\}}\right) = \frac{e^{(l_1-x_0)\mu}\left\{e^{|\mu|(x_0-l_2)}-e^{-|\mu|(x_0-l_2)}\right\}G_2(d_4,d_1,-\mu)}{\left\{e^{|\mu|(l_1-l_2)}-e^{-|\mu|(l_1-l_2)}\right\}G(d_1,d_4,\mu)} \quad (65)$$
$$+\frac{e^{(l_2-x_0)\mu}\left\{e^{|\mu|(l_1-x_0)}-e^{-|\mu|(l_1-x_0)}\right\}G_1(d_4,d_1,-\mu)}{\left\{e^{|\mu|(l_1-l_2)}-e^{-|\mu|(l_1-l_2)}\right\}G(d_1,d_4,\mu)}.$$

Setting $\beta = 0$ in (64) and (65) yields the results. □

Theorem 2 leads to the following remarkable result.

Corollary 2.1 *The probability that a standard Brownian motion W_t with $W_0 = x_0$, $l_2 \le x_0 \le l_1$, we have*

$$P\left(\tau_1^W < \tau_4^W\right) = \frac{\sqrt{d_4}+(x_0-l_2)\sqrt{\frac{2}{\pi}}}{\sqrt{d_1}+\sqrt{d_4}+(l_1-l_2)\sqrt{\frac{2}{\pi}}}, \quad (66)$$

$$P\left(\tau_1^W > \tau_4^W\right) = \frac{\sqrt{d_1}+(l_1-x_0)\sqrt{\frac{2}{\pi}}}{\sqrt{d_1}+\sqrt{d_4}+(l_1-l_2)\sqrt{\frac{2}{\pi}}}. \quad (67)$$

Remark: When we take $l_1 \to 0$, $l_2 \to 0$, $x_0 \to 0$, we can get the results for the one barrier case as in [9].

We will now extent Corollary 1.2 to obtain the joint distribution of W_t and τ^W at an exponential time. This is an application of (56) and Girsanov's theorem.

Theorem 3 *For a standard Brownian Motion W_t with $W_0 = x_0$, $l_2 \leq x_0 \leq l_1$ and τ^W defined as in (4) with $S_t = W_t$, we have the following result:*

For the case $x \geq l_1$,

$$P\left(W_{\tilde{T}} \in \mathrm{d}x, \tau^W < \tilde{T}\right) = a_1(x_0) f(x - l_1, d_1) + a_2(x_0) f(x - l_2, d_4) + a_1(x_0) h(x - l_1, d_1); \quad (68)$$

For the case $l_2 \leq x < l_1$,

$$P\left(W_{\tilde{T}} \in \mathrm{d}x, \tau^W < \tilde{T}\right) = a_1(x_0) f(x - l_1, d_1) + a_2(x_0) f(x - l_2, d_4); \quad (69)$$

For the case $x < l_2$,

$$P\left(W_{\tilde{T}} \in \mathrm{d}x, \tau^W < \tilde{T}\right) = a_1(x_0) f(x - l_1, d_1) + a_2(x_0) f(x - l_2, d_4) + a_2(x_0) h(x - l_2, d_4); \quad (70)$$

where $\tilde{T}$ is a random variable with an exponential distribution of parameter γ that is independent of W_t and

$$f(x, y) = \frac{e^{-\sqrt{2\gamma}|x|}}{\sqrt{2\gamma}} - e^{\gamma y - \sqrt{2\gamma}|x|}\sqrt{2\pi y}\mathcal{N}\left(-\sqrt{2\gamma y}\right), \quad (71)$$

$$h(x, y) = \sqrt{2\pi y}e^{\gamma y}\left\{e^{-\sqrt{2\gamma}|x|}\mathcal{N}\left(\frac{|x|}{\sqrt{y}} - \sqrt{2\gamma y}\right) - e^{\sqrt{2\gamma}|x|}\mathcal{N}\left(-\frac{|x|}{\sqrt{y}} - \sqrt{2\gamma y}\right)\right\}, \quad (72)$$

$$a_1(x_0) = \frac{\gamma\left\{e^{\sqrt{2\gamma}(x_0 - l_2)} - e^{-\sqrt{2\gamma}(x_0 - l_2)}\right\} b_1(d_1, d_4) + \gamma\left\{e^{\sqrt{2\gamma}(l_1 - x_0)} - e^{-\sqrt{2\gamma}(l_1 - x_0)}\right\} b_2(d_1, d_4)}{G\left\{e^{\sqrt{2\gamma}(l_1 - l_2)} - e^{-\sqrt{2\gamma}(l_1 - l_2)}\right\}}, \quad (73)$$

$$a_2(x_0) = \frac{\gamma\left\{e^{\sqrt{2\gamma}(x_0 - l_2)} - e^{-\sqrt{2\gamma}(x_0 - l_2)}\right\} b_2(d_4, d_1) + \gamma\left\{e^{\sqrt{2\gamma}(l_1 - x_0)} - e^{-\sqrt{2\gamma}(l_1 - x_0)}\right\} b_1(d_4, d_1)}{G\left\{e^{\sqrt{2\gamma}(l_1 - l_2)} - e^{-\sqrt{2\gamma}(l_1 - l_2)}\right\}}, \quad (74)$$

$$b_1(x, y) = e^{-2(l_1 - l_2)\sqrt{2\gamma} - \gamma x}\sqrt{y} + \frac{1 - e^{-2\gamma\sqrt{2\gamma}}}{2\sqrt{2\gamma}}e^{-\gamma x}\left\{\sqrt{\frac{2}{\pi}}\Psi\left(\sqrt{\gamma y}\right) + \sqrt{2\gamma y}\right\}, \quad (75)$$

$$b_2(x, y) = e^{-(l_1 - l_2)\sqrt{2\gamma} - \gamma x}\sqrt{y}, \quad (76)$$

$$\begin{aligned} G \;=\; & e^{-2(l_1 - l_2)\sqrt{2\gamma}}\left\{\sqrt{d_4}\Psi\left(\sqrt{\gamma d_1}\right) + \sqrt{d_1}\Psi\left(\sqrt{\gamma d_4}\right)\right\} \quad (77) \\ & + \frac{\left(1 - e^{-2(l_1 - l_2)\sqrt{2\gamma}}\right)}{2\sqrt{2\gamma}}\left\{\Psi\left(\sqrt{\gamma d_1}\right) + \sqrt{\gamma\pi d_1}\right\}\left\{\sqrt{\frac{2}{\pi}}\Psi\left(\sqrt{\gamma d_4}\right) + \sqrt{2\gamma d_4}\right\}. \end{aligned}$$

Proof: see appendix. □

5 Pricing double barrier Parisian Options

We want to price a double barrier Parisian call option with the current price of its underlying asset to be x, $L_1 < x < L_2$, the owner of which will obtain the right to exercise it when either the length of the excursion above the barrier L_1 reaches d_1, or the length of the excursion below the barrier L_2 reaches d_2 before T. Its price formula is given by

$$P_{min-in-call} = e^{-rT} E_Q \left((S_T - K)^+ \mathbf{1}_{\{\tau^S < T\}} \right),$$

where S is the underlying stock price, Q denotes the risk neutral measure. The subscript *min-in-call* means it is a call option which will be triggered when the minimum of two stopping times, τ_1^S and τ_4^S, is less than T, i.e. $\tau^S < T$. We assume S is a geometric Brownian motion:

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t \mathrm{d}W_t, \quad S_0 = x,$$

where $L_1 < x < L_2$, r is the risk free rate, W_t with $W_0 = 0$ is a standard Brownian motion under Q. Set

$$m = \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right), \quad b = \frac{1}{\sigma}\ln\left(\frac{K}{x}\right), \quad B_t = mt + W_t,$$

$$l_1 = \frac{1}{\sigma}\ln\left(\frac{L_1}{x}\right), \quad l_2 = \frac{1}{\sigma}\ln\left(\frac{L_2}{x}\right).$$

We have

$$S_t = x\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\} = x\exp\left\{\sigma(mt + W_t)\right\} = xe^{\sigma B_t}.$$

By applying Girsanov's Theorem, we have

$$P_{min-in-call} = e^{-\left(r + \frac{1}{2}m^2\right)T} E_P\left[\left(xe^{\sigma B_T} - K\right)^+ e^{mB_T}\mathbf{1}_{\{\tau^B < T\}}\right],$$

where P is a new measure, under which B_t is a standard Brownian motion with $B_0 = 0$, and τ^B is the stopping time defined with the respect to barrier l_1, l_2. And we define

$$P^*_{min-in-call} = e^{\left(r + \frac{1}{2}m^2\right)T} P_{min-in-call}.$$

We are going to show that we can obtain the Laplace transform of $P^*_{min-in-call}$ w.r.t T, denoted by $\mathscr{L}_T$.

Firstly, assuming $\tilde{T}$ is a random variable with an exponential distribution with parameter γ that is independent of W_t, we have

$$\begin{aligned}
& E_P\left[\left(xe^{\sigma B_{\tilde{T}}}-K\right)^+ e^{mB_{\tilde{T}}}\mathbf{1}_{\left\{\tau^B<\tilde{T}\right\}}\right] \\
= & \int_b^\infty \left(xe^{\sigma y}-K\right)e^{my}P\left(B_{\tilde{T}}\in \mathrm{d}y,\tau^B<\tilde{T}\right) \\
= & \int_0^\infty \gamma e^{-\gamma T}\int_b^\infty \left(xe^{\sigma y}-K\right)e^{my}P\left(B_T\in \mathrm{d}y,\tau^B<T\right)\mathrm{d}T \\
= & \gamma\int_0^\infty e^{-\gamma T}E_P\left[\left(xe^{\sigma B_T}-K\right)^+ e^{mB_T}\mathbf{1}_{\{\tau^B<T\}}\right]\mathrm{d}T \\
= & \gamma\mathscr{L}_T
\end{aligned}$$

Hence we have

$$\mathscr{L}_T=\frac{1}{\gamma}\int_b^\infty \left(xe^{\sigma y}-K\right)e^{my}P\left(B_{\tilde{T}}\in \mathrm{d}y,\tau^B<\tilde{T}\right).$$

By using the results in Theorem 3, this Laplace transform can be calculated explicitly.

When $b\ge l_1$, i.e. $K\ge L_1$, we have

$$\mathscr{L}_T=\frac{x}{\gamma}F_1(\sigma+m)-\frac{K}{\gamma}F_1(m),$$

where

$$\begin{aligned}
F_1(x) \quad = \quad & a_1(0)\left\{\frac{1}{\sqrt{2\gamma}}-e^{\gamma d_1}\sqrt{2\pi d_1}\mathscr{N}\left(-\sqrt{2\gamma d_1}\right)\right\}\frac{e^{\sqrt{2\gamma}l_1+\left(x-\sqrt{2\gamma}\right)b}}{\sqrt{2\gamma}-x} \\
& +a_2(0)\left\{\frac{1}{\sqrt{2\gamma}}-e^{\gamma d_4}\sqrt{2\pi d_4}\mathscr{N}\left(-\sqrt{2\gamma d_4}\right)\right\}\frac{e^{\sqrt{2\gamma}l_2+\left(x-\sqrt{2\gamma}\right)b}}{\sqrt{2\gamma}-x} \\
& +a_1(0)\sqrt{2\pi d_1}e^{\gamma d_1}\left\{\frac{2xe^{xl_1-rd_1+\frac{d_1x^2}{2}}\mathscr{N}\left(x\sqrt{d_1}-\frac{b-l_1}{\sqrt{d_1}}\right)}{2\gamma-x^2}\right. \\
& \left.+\frac{e^{\sqrt{2\gamma}l_1+\left(x-\sqrt{2\gamma}\right)b}\mathscr{N}\left(\frac{b-l_1}{\sqrt{d_1}}-\sqrt{2\gamma d_1}\right)}{\sqrt{2\gamma}-x}+\frac{e^{-\sqrt{2\gamma}l_1+\left(x+\sqrt{2\gamma}\right)b}\mathscr{N}\left(-\frac{b-l_1}{\sqrt{d_1}}-\sqrt{2\gamma d_1}\right)}{\sqrt{2\gamma}+x}\right\};
\end{aligned}$$

when $l_2<b<l_1$, i.e. $L_2<K<L_1$, we have

$$\mathscr{L}_T=\frac{x}{\gamma}F_2(\sigma+m)-\frac{K}{\gamma}F_2(m),$$

where

$$\begin{aligned}
F_2(x) &= \frac{2a_1(0)e^{l_1 x}}{2\gamma - x^2}\left\{1 + x\sqrt{2\pi d_1}e^{\frac{d_1 x^2}{2}}\mathscr{N}\left(x\sqrt{d_1}\right)\right\} \\
&\quad -a_1(0)\left\{\frac{1}{\sqrt{2\gamma}} - e^{\gamma d_1}\sqrt{2\pi d_1}\mathscr{N}\left(-\sqrt{2\gamma d_1}\right)\right\}\frac{e^{-\sqrt{2\gamma}l_1 + \left(x+\sqrt{2\gamma}\right)b}}{\sqrt{2\gamma} - x} \\
&\quad +a_2(0)\left\{\frac{1}{\sqrt{2\gamma}} - e^{\gamma d_4}\sqrt{2\pi d_4}\mathscr{N}\left(-\sqrt{2\gamma d_4}\right)\right\}\frac{e^{\sqrt{2\gamma}l_2 + \left(x-\sqrt{2\gamma}\right)b}}{\sqrt{2\gamma} - x};
\end{aligned}$$

when $b \le l_2$, i.e. $K \le L_2$, we have

$$\mathscr{L}_T = \frac{x}{\gamma}F_3(\sigma + m) - \frac{K}{\gamma}F_3(m),$$

where

$$\begin{aligned}
F_2(x) &= \frac{2a_1(0)e^{l_1 x}}{2\gamma - x^2}\left\{1 + x\sqrt{2\pi d_1}e^{\frac{d_1 x^2}{2}}\mathscr{N}\left(x\sqrt{d_1}\right)\right\} \\
&\quad -a_1(0)\left\{\frac{1}{\sqrt{2\gamma}} - e^{\gamma d_1}\sqrt{2\pi d_1}\mathscr{N}\left(-\sqrt{2\gamma d_1}\right)\right\}\frac{e^{-\sqrt{2\gamma}l_1 + \left(x+\sqrt{2\gamma}\right)b}}{\sqrt{2\gamma} - x} \\
&\quad +\frac{2a_2(0)e^{l_2 x}}{2\gamma - x^2}\left\{1 - 2\sqrt{\pi d_4 \gamma}e^{\frac{d_4 x^2}{2}}\mathscr{N}\left(x\sqrt{d_4}\right)\right\} \\
&\quad -a_2(0)\left\{\frac{1}{\sqrt{2\gamma}} - e^{\gamma d_4}\sqrt{2\pi d_4}\mathscr{N}\left(-\sqrt{2\gamma d_4}\right)\right\}\frac{e^{-\sqrt{2\gamma}l_2 + \left(x+\sqrt{2\gamma}\right)b}}{\sqrt{2\gamma} - x} \\
&\quad +a_2(0)\sqrt{2\pi d_4}e^{\gamma d_4}\left\{\frac{2\sqrt{2\gamma}e^{xl_2 - rd_4 + \frac{d_4 x^2}{2}}\mathscr{N}\left(x\sqrt{d_4} - \frac{b-l_2}{\sqrt{d_4}}\right)}{2\gamma - x^2}\right. \\
&\quad \left. -\frac{e^{\sqrt{2\gamma}l_2 + \left(x-\sqrt{2\gamma}\right)b}\mathscr{N}\left(\frac{b-l_2}{\sqrt{d_4}} - \sqrt{2\gamma d_4}\right)}{\sqrt{2\gamma} - x} - \frac{e^{-\sqrt{2\gamma}l_2 + \left(x+\sqrt{2\gamma}\right)b}\mathscr{N}\left(-\frac{b-l_2}{\sqrt{d_4}} - \sqrt{2\gamma d_4}\right)}{\sqrt{2\gamma} + x}\right\}.
\end{aligned}$$

Remark: The price can be calculated by numerical inversion of the Laplace transform.

So far, we have shown how to obtain the Laplace transform of

$$P^*_{min-call-in} = e^{\left(r+\frac{1}{2}m^2\right)T}P_{min-call-in}.$$

For

$$P_{min-call-out} = e^{-rT}E_Q\left((S_T - K)^+ \mathbf{1}_{\{\tau^S > T\}}\right),$$

we can get the result from the relationship that

$$P_{min-call-out} = e^{-rT}E_Q\left\{(S_T - K)^+\right\} - P_{min-call-in}.$$

Furthermore, if we set

$$\tilde{\tau}_L^Y = \tau_{1,L}^Y \vee \tau_{2,L}^Y,$$

we can define another type of Parisian options by $\tilde{\tau}_L^Y$:

$$P_{max-call-in} = e^{-rT} E_Q \left((S_T - K)^+ \mathbf{1}_{\left\{\tilde{\tau}_L^S < T\right\}} \right).$$

In order to get its pricing formula, we should use the following relationship:

$$\mathbf{1}_{\left\{\tilde{\tau}_L^S < T\right\}} = \mathbf{1}_{\left\{\tau_{1,L}^S < T\right\}} + \mathbf{1}_{\left\{\tau_{2,L}^S < T\right\}} - \mathbf{1}_{\left\{\tau_L^S < T\right\}}.$$

We have therefore

$$P_{max-call-in} = P_{up-in-call} + P_{down-in-call} - P_{min-call-in}.$$

Similarly, from

$$P_{max-call-out} = e^{-rT} E_Q \left\{ (S_T - K)^+ \right\} - P_{max-call-in},$$

we can work out $P_{max-call-out}$.

6 Appendix: Proof of Theorem 3

Let T be the final time. According to the definition of $\Psi(x)$, we have

$$\Psi(x) = 2\sqrt{\pi} x \mathcal{N}\left(\sqrt{2}x\right) - \sqrt{\pi}x + e^{-x^2} = \sqrt{\pi}x - \sqrt{\pi}x \mathrm{Erfc}\,(x) + e^{-x^2}.$$

It is not difficult to show that

$$E\left(e^{-\beta \tau^{W^\mu}}\right) = E\left(\int_0^\infty \beta e^{-\beta T} \mathbf{1}_{\left\{\tau^{W^\mu} < T\right\}} \mathrm{d}T\right).$$

By Girsanov's theorem, this is equal to

$$\int_0^\infty \beta e^{-(\beta + \frac{1}{2}\mu^2)T - \mu x_0} E\left(e^{\mu W_T} \mathbf{1}_{\{\tau^W < T\}}\right) \mathrm{d}T.$$

Setting $\gamma = \beta + \frac{1}{2}\mu^2$ gives

$$\begin{aligned} E\left(e^{-\beta \tau^{W^\mu}}\right) &= \int_0^\infty (\gamma - \frac{1}{2}\mu^2) e^{-\gamma T - \mu x_0} E\left(e^{\mu W_T} \mathbf{1}_{\{\tau^W < T\}}\right) \mathrm{d}T \\ &= \frac{\gamma - \frac{1}{2}\mu^2}{\gamma} e^{-\mu x_0} E\left(e^{\mu W_{\tilde{T}}} \mathbf{1}_{\left\{\tau^W < \tilde{T}\right\}}\right), \end{aligned}$$

where $\tilde{T}$ is a random variable with an exponential distribution of parameter γ that is independent of W_t. Therefore we have

$$E\left(e^{\mu W_{\tilde{T}}} \mathbf{1}_{\{\tau^W < \tilde{T}\}}\right) = \frac{\gamma e^{\mu x_0}}{\gamma - \frac{1}{2}\mu^2} E\left(e^{-\beta \tau^{W^\mu}}\right)$$

In order to inverse the above moment generating function, we just need to inverse the following expressions:

$$\frac{\mu}{\gamma-\frac{\mu^2}{2}} = \int_0^\infty e^{\mu x} e^{-\sqrt{2\gamma}x}\mathrm{d}x - \int_{-\infty}^0 e^{\mu x} e^{\sqrt{2\gamma}x}\mathrm{d}x,$$

$$\frac{1}{\gamma-\frac{\mu^2}{2}} = \int_0^\infty e^{\mu x} \frac{1}{\sqrt{2\gamma}} e^{-\sqrt{2\gamma}x}\mathrm{d}x + \int_{-\infty}^0 e^{\mu x} \frac{1}{\sqrt{2\gamma}} e^{\sqrt{2\gamma}x}\mathrm{d}x,$$

$$e^{\frac{d_1}{2}\mu^2} = \int_{-\infty}^\infty e^{\mu x} \frac{1}{\sqrt{2\pi d_1}} \exp\left\{-\frac{x^2}{2d_1}\right\}\mathrm{d}x,$$

$$1-\sqrt{\frac{d_i}{2}}\pi\mu e^{\frac{d_i}{2}\mu^2}\mathrm{Erfc}\left(\sqrt{\frac{d_i}{2}}\mu\right) = \int_{-\infty}^0 e^{\mu x}\frac{-x}{d_i} e^{-\frac{x^2}{2d_i}}\mathrm{d}x.$$

The inversion of $\frac{\mu e^{\frac{d_1}{2}\mu^2}}{\gamma-\frac{\mu^2}{2}}$ is

$$\begin{aligned} & \int_0^\infty e^{-\sqrt{2\gamma}y}\frac{1}{\sqrt{2\pi d_1}} e^{-\frac{(x-y)^2}{2d_1}}\mathrm{d}y - \int_{-\infty}^0 e^{\sqrt{2\gamma}y}\frac{1}{\sqrt{2\pi d_1}} e^{-\frac{(x-y)^2}{2d_1}}\mathrm{d}y \\ = \; & e^{\gamma d_1}\left\{e^{-\sqrt{2\gamma}x}\mathscr{N}\left(\frac{x}{\sqrt{d_1}}-\sqrt{2\gamma d_1}\right) - e^{\sqrt{2\gamma}x}\mathscr{N}\left(-\frac{x}{\sqrt{d_1}}-\sqrt{2\gamma d_1}\right)\right\}. \end{aligned}$$

The inversion of $\frac{1-\sqrt{\frac{d_i}{2}}\pi\mu e^{\frac{d_i}{2}\mu^2}\mathrm{Erfc}\left(\sqrt{\frac{d_i}{2}}\mu\right)}{\gamma-\frac{\mu^2}{2}}$ is given below.

For $x \geqslant 0$,

$$\int_{-\infty}^0 \frac{-y}{d_i} e^{-\frac{y^2}{2d_i}} \frac{1}{\sqrt{2\gamma}} e^{-\sqrt{2\gamma}(x-y)}\mathrm{d}y = \frac{e^{-\sqrt{2\gamma}x}}{\sqrt{2\gamma}} - e^{\gamma d_i-\sqrt{2\gamma}x}\sqrt{2\pi d_i}\mathscr{N}\left(-\sqrt{2\gamma d_i}\right);$$

For $x < 0$,

$$\begin{aligned} & \int_{-\infty}^x \frac{-y}{d_i} e^{-\frac{y^2}{2d_i}} \frac{1}{\sqrt{2\gamma}} e^{-\sqrt{2\gamma}(x-y)}\mathrm{d}y + \int_x^0 \frac{-y}{d_i} e^{-\frac{y^2}{2d_i}} \frac{1}{\sqrt{2\gamma}} e^{\sqrt{2\gamma}(x-y)}\mathrm{d}y \\ = \; & \frac{e^{\sqrt{2\gamma}x}}{\sqrt{2\gamma}} - e^{\gamma d_i-\sqrt{2\gamma}x}\sqrt{2\pi d_i}\mathscr{N}\left(\frac{x}{\sqrt{d_i}}-\sqrt{2\gamma d_i}\right) \\ & + e^{\gamma d_i+\sqrt{2\gamma}x}\sqrt{2\pi d_i}\left\{\mathscr{N}\left(\sqrt{2\gamma d_i}\right)-\mathscr{N}\left(\frac{x}{\sqrt{d_i}}+\sqrt{2\gamma d_i}\right)\right\}. \end{aligned}$$

Consequently, we can get Theorem 3.

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