On the quantiles of the Brownian motion and their hitting times.*

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Abstract

The distribution of the α-quantile of a Brownian motion on an interval $[0,t]$ has been obtained motivated by a problem in financial mathematics. In this paper we generalise these results by calculating an explicit expression for the joint density of the α-quantile of a standard Brownian motion, its first and last hitting times and the value of the process at time t. Our results can be easily generalised for a Brownian motion with drift. It is shown that the first and last hitting times follow a transformed arcsine law.

1 Introduction

Let $(X(s), s \geq 0)$ be a real valued stochastic process on a probability space $(\Omega, \mathcal{F}, \Pr)$. For $0 < \alpha < 1$, define the $\alpha-$quantile of the path of $(X(s), s \geq 0)$ up to a fixed time t by

$$M_X(\alpha, t) = \inf \left\{ x : \int_0^t \mathbf{1}(X(s) \leq x)\, ds > \alpha t \right\}. \tag{1}$$

The study of the quantiles of various stochastic processes has been recently undertaken as a response to a problem arising in the field of mathematical finance, the pricing of a particular path-dependent financial option; see Miura [6], Akahori [1] and Dassios [2]. This involves calculating quantities

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such as $E\left(h\left(M_X\left(\alpha,t\right)\right)\right)$, where $h\left(x\right)=\left(e^x-b\right)^+$ or some other appropriate function. This requires obtaining the distribution of $X\left(t\right)$. In the case where $\left(X\left(s\right),s\geq 0\right)$ is a Lévy process (having stationary and independent increments) the following result was obtained:

Let $X^{(1)}\left(s\right)$ and $X^{(2)}\left(s\right)$ be independent copies of $X\left(s\right)$. Then,

$$\begin{pmatrix} M_X\left(\alpha,t\right) \\ X\left(t\right) \end{pmatrix} \overset{\text{(law)}}{=} \begin{pmatrix} 0\leq s\leq \alpha t \sup X^{(1)}\left(s\right)+0\leq s\leq\left(1-\alpha\right)t \inf X^{(2)}\left(s\right) \\ X^{(1)}\left(\alpha t\right)+X^{(2)}\left(\left(1-\alpha\right)t\right) \end{pmatrix}. \tag{2}$$

When $\left(X\left(s\right),s\geq 0\right)$ is a Brownian motion, we can use this result and obtain an explicit formula for the joint density of $M_X\left(\alpha,t\right)$ and $X\left(t\right)$. This result was first proved for a Brownian motion with drift; see Dassios [2] and Embrechts, Rogers and Yor [4] and for Lévy processes by Dassios [3]. There is also a similar result for discrete time random walks first proved by Wendel [7].

We now let

$$L_X\left(\alpha,t\right)=\inf\left\{s\in\left[0,t\right]:X\left(s\right)=M_X\left(\alpha,t\right)\right\}$$

be the first, and

$$K_X\left(\alpha,t\right)=\sup\left\{s\in\left[0,t\right]:X\left(s\right)=M_X\left(\alpha,t\right)\right\},$$

the last time the process hits $M_X\left(\alpha,t\right)$. One can now introduce a 'barrier' element to the financial application by making the option worthless if the quantile is hit too early or too late. For example, this can involve calculating quantities such as $E\left(h\left(M_X\left(\alpha,t\right)\right)\mathbf{1}\left(L_X\left(\alpha,t\right)>v,K_X\left(\alpha,t\right)<u\right)\right)$.

For the rest of the paper we assume that $\left(X\left(s\right),s\geq 0\right)$ is a standard Brownian motion. We will derive the joint density of $M_X\left(\alpha,t\right),L_X\left(\alpha,t\right),K_X\left(\alpha,t\right)$ and $X\left(t\right)$. If we denote this density by $f\left(y,x,u,v\right)$, our results can be generalised for a Brownian motion with drift m, using a Cameron-Martin-Girsanov transformation. The corresponding density will be $f\left(y,x,u,v\right)\exp\left(mx-m^2t/2\right)$.

Before we obtain the density of $\left(M_X\left(\alpha,t\right),L_X\left(\alpha,t\right),K_X\left(\alpha,t\right),X\left(t\right)\right)$, we will first show that the law of $L_X\left(\alpha,t\right)$ (and $K_X\left(\alpha,t\right)$) is a transformed arcsine law.

2 An arcsine law for $L_X\left(\alpha,t\right)$.

Let $S_X\left(t\right)=\sup_{0\leq s\leq t}\left\{X\left(s\right)\right\}$ and $\theta_X\left(t\right)=\sup\left\{s\in\left[0,t\right]:X\left(s\right)=S_X\left(t\right)\right\}$. We prove the following theorem:

For $u > 0$,

$$\Pr\left(L_X(\alpha,t) > u\right) = \Pr\left(u < \theta_X(t) \leq \alpha t\right) + \Pr\left(u < \theta_X(t) \leq (1-\alpha)t\right) \quad (3)$$

and

$$\Pr\left(L_X(\alpha,t) \in du\right) = \frac{\mathbf{1}(u \leq \alpha t) + \mathbf{1}(u \leq (1-\alpha)t)}{\pi\sqrt{u(t-u)}} du. \quad (4)$$

Furthermore, $K_X(\alpha,t)$ has the same distribution as $t - L_X(\alpha,t)$.

Proof We will first prove that

$$\Pr\left(M_X(\alpha,t) > 0, L_X(\alpha,t) > u\right) = \Pr\left(u < \theta_X(t) \leq \alpha t\right). \quad (5)$$

We observe that

$$\Pr\left(M_X(\alpha,t) > 0, L_X(\alpha,t) > u\right) = \Pr\left(M_X(\alpha,t) > S_X(u)\right) =$$

$$\Pr\left(\int_0^t \mathbf{1}\left(X(s) \leq S_X(u)\right) ds < \alpha t\right) =$$

$$\Pr\left(\int_u^t \mathbf{1}\left(X(s) - X(u) \leq S_X(u) - X(u)\right) ds < \alpha t\right). \quad (6)$$

Let $X^*(s) = X(u+s) - X(u)$. $(X^*(s), s \geq 0)$ is a standard Brownian motion which is independent of $(X(s), 0 \leq s \leq u)$. We condition on $S_X(u) - X(u) = c$, and set $\tau_c = \inf\{s > 0 : X^*(s) = c\}$ and $X^{**}(s) = X^*(\tau_c + s) - c$. $(X^{**}(s), s \geq 0)$ is a standard Brownian motion which is independent of both $(X(s), 0 \leq s \leq u)$ and $(X^*(s), 0 \leq s \leq \tau_c)$. We have that

$$\Pr\left(\int_0^{t-u} \mathbf{1}\left(X^*(s) \leq c\right) ds < \alpha t - u\right) =$$

$$\int_0^{\alpha t - u} \Pr\left(\tau_c \in dr\right) \Pr\left(\int_0^{t-u-r} \mathbf{1}\left(X^{**}(s) \leq 0\right) ds < \alpha t - u - r\right)$$

and since $\int_0^{t-u-r} \mathbf{1}\left(X^{**}(s) \leq 0\right) ds$ has the same (arcsine) law as $\theta_{X^{**}}(t-u-r)$, this is equal to

$$\int_0^{\alpha t - u} \Pr\left(\tau_c \in dr\right) \Pr\left(\theta_{X^{**}}(t-u-r) < \alpha t - u - r\right) =$$

$$\int_0^{\alpha t - u} \Pr\left(\tau_c \in dr\right) \Pr\left(\sup_{0 \leq s \leq \alpha t - u - r} X^{**}(s) > \sup_{\alpha t - u - r \leq s \leq t - u - r} X^{**}(s)\right) =$$

$$\Pr\left(\sup_{0\le s\le \alpha t-u} X^*(s) > \sup_{\alpha t-u\le s\le t-u} X^*(s), \sup_{0\le s\le \alpha t-u} X^*(s) > c\right)$$

and so (6) is equal to

$$\Pr\left(\begin{array}{c}\sup_{u\le s\le \alpha t} X(s)-X(u) > \sup_{\alpha t\le s\le t} X(s)-X(u), \\ \sup_{u\le s\le \alpha t} X(s)-X(u) > \sup_{0\le s\le u} X(s)-X(u)\end{array}\right) =$$

$$\Pr\left(u<\theta_X(t)\le \alpha t\right).$$

Since $(-X(s), s\ge 0)$ is also a standard Brownian motion and $M_{-X}(\alpha,t) = -M_X(1-\alpha,t)$ almost surely, we use $-X(s)$ instead of $X(s)$ and we get

$$\Pr\left(M_X(\alpha,t)<0, L_X(\alpha,t)>u\right) = \Pr\left(u<\theta_X(t)\le(1-\alpha)t\right). \quad (7)$$

Adding (5) and (7) we get (3), and since $\theta_X(t)$ has an arcsine law, (4) follows. To see that $K_X(\alpha,t)$ has the same distribution as $L_X(\alpha,t)$, set $\tilde{X}(s) = X(t-s)-X(t)$. Clearly $\left(\tilde{X}(s), 0\le s\le t\right)$ is a standard Brownian motion and we can easily see that $M_{\tilde{X}}(\alpha,t) = M_X(\alpha,t)-X(t)$ and $K_{\tilde{X}}(\alpha,t) = t-L_X(\alpha,t)$.□

We can also extend our result and obtain the joint distribution of $(M_X(\alpha,t), L_X(\alpha,t))$ (also of $(M_X(\alpha,t)-X(t), t-K_X(\alpha,t))$.

For $b>0$,

$$\Pr\left(M_X(\alpha,t)\in db, L_X(\alpha,t)\in du\right) = \Pr\left(S_X(t)\in db, \theta_X(t)\in du\right)\mathbf{1}(0<u<\alpha t), \quad (8)$$

and for $b<0$,

$$\Pr\left(M_X(\alpha,t)\in db, L_X(\alpha,t)\in du\right) = \Pr\left(S_X(t)\in d|b|, \theta_X(t)\in du\right)\mathbf{1}(0<u<(1-\alpha)t). \quad (9)$$

Furthermore $(M_X(\alpha,t), L_X(\alpha,t))$ and $(M_X(\alpha,t)-X(t), t-K_X(\alpha,t))$ have the same distribution.

Proof Let $b>0$ and $u<\alpha t$. We then have that

$$\Pr\left(M_X(\alpha,t)>b, L_X(\alpha,t)>u\right) = \Pr\left(S_X(u)<M_X(\alpha,t), M_X(\alpha,t)>b\right) =$$

$$\Pr\left(b<S_X(u)<M_X(\alpha,t)\right) + \Pr\left(S_X(u)<b<M_X(\alpha,t)\right). \quad (10)$$

Let $\tau_b = \inf\{s>0 : X(s)=b\}$ and $X^*(s) = X(\tau_b+s)-c$. $(X^*(s), s\ge 0)$ is a standard Brownian motion which is independent of $(X(s), 0\le s\le \tau_c)$. Using theorem 1, we have

$$\Pr\left(b<S_X(u)<M_X(\alpha,t)\right) =$$

$$\int_0^u \Pr(\tau_b \in dr)\Pr\left(\int_0^{t-r} \mathbf{1}\left(X^*(s) \le S_{X^*}(u-r)\right) < \alpha t - r\right) =$$

$$\int_0^u \Pr(\tau_b \in dr)\Pr\left(M_{X^*}\left(\frac{\alpha t - r}{t-r}, t-r\right) > 0, L_{X^*}\left(\frac{\alpha t - r}{t-r}, t-r\right) > u-r\right) =$$

$$\int_0^u \Pr(\tau_b \in dr)\Pr\left(u-r < \theta_{X^*}(t-r) < \alpha t - r\right) = \Pr\left(u < \theta_X(t) < \alpha t, S_X(u) > b\right). \tag{11}$$

Furthermore,

$$\Pr\left(S_X(u) < b < M_X(\alpha,t)\right) = \Pr\left(S_X(u) < b, \int_0^t \mathbf{1}\left(X(s) \le b\right) ds < \alpha t\right) =$$

$$\int_u^{\alpha t} \Pr(\tau_b \in dr)\Pr\left(\int_0^{t-r} \mathbf{1}\left(X^*(s) \le 0\right) < \alpha t - r\right)$$

$$= \int_u^{\alpha t} \Pr(\tau_b \in dr)\Pr\left(\theta_{X^*}(t-r) < \alpha t - r\right) =$$

$$\Pr\left(u < \theta_X(t) < \alpha t, S_X(u) < b, \sup_{u \le s \le \alpha t} X(s) > b\right). \tag{12}$$

Adding (11) and (12) together, we see that (10) is equal to

$$\Pr\left(u < \theta_X(t) < \alpha t, \sup_{u \le s \le \alpha t} X(s) > b\right) = \Pr\left(u < \theta_X(t) < \alpha t, S_X(t) > b\right)$$

which leads to (8).

Since $(-X(s), s \ge 0)$ is also a standard Brownian motion and $M_{-X}(\alpha,t) = -M_X(1-\alpha,t)$ almost surely, we use $-X(s)$ instead of $X(s)$ and we get that for $b < 0$,

$$\Pr\left(M_X(\alpha,t) < b, L_X(\alpha,t) > u\right) = \Pr\left(u < \theta_X(t) \le (1-\alpha)t, S_X(t) > |b|\right),$$

which leads to (9).

To see that $(t - K_X(\alpha,t), M_X(\alpha,t) - X(t))$ has the same distribution as $(L_X(\alpha,t), M_X(\alpha,t))$, set again $\tilde{X}(s) = X(t-s) - X(t)$. Clearly $\left(\tilde{X}(s), 0 \le s \le t\right)$ is a standard Brownian motion and we can easily see that $M_{\tilde{X}}(\alpha,t) = M_X(\alpha,t) - X(t)$, (and so $M_{\tilde{X}}(\alpha,t) - \tilde{X}(t) = M_X(\alpha,t)$) and $K_{\tilde{X}}(\alpha,t) = t - L_X(\alpha,t)$.$\square$

Remarks

1. The distribution of $(\theta_X(t), S_X(t))$ is well known (see for example Karatzas and Shreve [5], page 102. From this and theorem 2, we can deduce the density of $(L_X(\alpha,t), M_X(\alpha,t))$. This is given by

$$\Pr(M_X(\alpha,t) \in db, L_X(\alpha,t) \in du) =$$

$$\frac{|b|}{\pi\sqrt{u^3(t-u)}} \exp\left(-\frac{b^2}{2u}\right)\left[\mathbf{1}(0 < u < \alpha t, b > 0) + \mathbf{1}(0 < u < (1-\alpha)t, b < 0)\right] dbdu. \tag{13}$$

2. Theorem 2 also leads to an alternative expression for the distribution of $M_X(\alpha,t)$; that is

$$\Pr(M_X(\alpha,t) \in db) = \Pr(S_X(t) \in db, 0 < \theta_X(t) < \alpha t),$$

for $b > 0$ and

$$\Pr(M_X(\alpha,t) \in db) = \Pr(S_X(t) \in d|b|, 0 < \theta_X(t) < (1-\alpha)t),$$

for $b < 0$.

3. Using the argument at the end of the proof, we can generalise the last assertion of the theorem and observe that $(K_X(\alpha,t), M_X(\alpha,t) - X(t), -X(t))$ has the same law as $(t - L_X(\alpha,t), M_X(\alpha,t), X(t))$ and so we see that $(K_X(\alpha,t), M_X(\alpha,t), X(t))$ and $(t - L_X(\alpha,t), M_X(\alpha,t) - X(t), -X(t))$, have the same distribution, a fact we will use in the following section.

3 The joint law of $(L_X(\alpha,t), K_X(\alpha,t), M_X(\alpha,t), X(t))$.

From now on we will denote the density of τ_b by $k(\cdot,\cdot)$; that is for $v > 0$,

$$\Pr(\tau_b \in dv) = k(v,b)\,dv = \frac{2|b|}{\sqrt{2\pi v^3}} \exp\left(-\frac{b^2}{2v}\right) dv. \tag{14}$$

We will also denote the joint density of $\left(M_X\left(\frac{v}{t},t\right), X(t)\right)$ by $g(\cdot,\cdot,\cdot,\cdot)$; that is for $0 < v < t$,

$$\Pr\left(M_X\left(\frac{v}{t},t\right) \in db, X(t) \in da\right) = g(b,a,v,t)\,dbda.$$

We can calculate $g(\cdot,\cdot,\cdot,\cdot)$ by using the proposition in the introduction. $\left(M_X\left(\frac{v}{t},t\right), X(t)\right)$ has the same distribution as $(S_{X_1}(v) - S_{X_2}(t-v), X_1(v) - X_2(t-v))$, where $(X_1(s), 0 \le s \le v)$ and $(X_2(s), 0 \le s \le t-v)$ are independent standard Brownian motions. The density of $(S_X(t), X(t))$ is given by

$$\Pr(S_X(t) \in db, X(t) \in da) = \frac{2(2b-a)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2b-a)^2}{2t}\right) \mathbf{1}(b \ge 0, b \ge a)\, da db \tag{15}$$

(see Karatzas and Shreve [5], p.95). We observe that since (15) is bounded, $g(\cdot,\cdot,\cdot,\cdot)$ is a bounded density. For our results, we need to calculate $g(0,0,v,t)$. This is the same as the value of the density of $\left(M_X\left(\frac{v}{t},t\right), M_X\left(\frac{v}{t},t\right) - X(t)\right)$ at $(0,0)$. From (15) we see that

$$\Pr(S_X(t) \in dy, S_X(t) - X(t) \in dx) = \frac{2(y+x)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(y+x)^2}{2t}\right) \mathbf{1}(y \ge 0, x \ge 0)\, dy dx \tag{16}$$

and it is a simple exercise to verify that

$$\begin{aligned} g(0,0,v,t) &= \int_0^\infty \int_0^\infty \frac{2(y+x)}{\sqrt{2\pi v^3}} \exp\left(-\frac{(y+x)^2}{2v}\right) \frac{2(y+x)}{\sqrt{2\pi (t-v)^3}} \exp\left(-\frac{(y+x)^2}{2(t-v)}\right) dx dy \\ &= \frac{\sqrt{v(t-v)}}{t^2}. \end{aligned} \tag{17}$$

We will now obtain a preliminary result.

For any u and v, such that $0 < u < v < t$, we have that

$$\begin{aligned} &\Pr(L_X(\alpha,t) > u, M_X(\alpha,t) \in db, X(t) \in da, K_X(\alpha,t) > v) = \\ &\quad \Pr(\tau_b > u, M_X(\alpha,t) \in db, X(t) \in da, K_X(\alpha,t) > v). \end{aligned} \tag{18}$$

Proof Since $M_{-X}(\alpha,t) = -M_X(1-\alpha,t)$, it suffices to prove (18) for $b > 0$. We have to prove that

$$\begin{aligned} &\lim_{\delta\to 0, \varepsilon \to 0} \frac{1}{\delta\varepsilon} \{\Pr(L_X(\alpha,t) > u, M_X(\alpha,t) \in (b, b+\delta], X(t) \in (a, a+\varepsilon], K_X(\alpha,t) > v) - \\ &\quad \Pr(\tau_b > u, M_X(\alpha,t) \in (b, b+\delta], X(t) \in (a, a+\varepsilon], K_X(\alpha,t) > v)\} = 0. \end{aligned} \tag{19}$$

Let $X^*(s) = X(s+u) - X(u)$. We then have that

$$\begin{aligned} &\Pr(L_X(\alpha,t) > u, M_X(\alpha,t) \in (b, b+\delta], X(t) \in (a, a+\varepsilon], K_X(\alpha,t) > v) - \\ &\quad \Pr(\tau_b > u, M_X(\alpha,t) \in (b, b+\delta], X(t) \in (a, a+\varepsilon], K_X(\alpha,t) > v) = \end{aligned}$$

$$\Pr\left(b < S_X(u) < M_X(\alpha,t) \le b+\delta, X(t) \in (a, a+\varepsilon], K_X(\alpha,t) > v\right) \le$$

$$\Pr\left(b < S_X(u) < M_X(\alpha,t) \le b+\delta, X(t) \in (a, a+\varepsilon]\right) =$$

$$\Pr\left(\begin{array}{c} b < S_X(u) < b+\delta, \\ S_X(u) < M_{X^*}(\alpha t - u, t-u) + X(u) \le b+\delta, \\ X^*(t-u) + X(u) \in (a, a+\varepsilon] \end{array}\right). \tag{20}$$

Since $(X^*(s), 0 \le s \le t-u)$ is independent of $(X(s), 0 \le s \le u)$, and $g(\cdot,\cdot,\cdot,\cdot)$ is bounded, we condition on $S_X(u) = y$ and $X(u) = x$ and see that there is a constant K, such that

$$\Pr\left(\begin{array}{c} y < M_{X^*}(\alpha t - u, t-u) + x \le b+\delta, \\ X^*(t-u) + x \in (a, a+\varepsilon] \end{array}\right) \le K\varepsilon(b+\delta-y).$$

We therefore conclude that (20) is bounded by

$$K\varepsilon E\left((b+\delta-S_X(u))\mathbf{1}(b < S_X(u) < b+\delta)\right) \le K\varepsilon\delta\Pr(b < S_X(u) < b+\delta)$$

and by the continuity of the distribution of $S_X(u)$, we see that the limit in (19) is zero. $\Box$

As a corollary we will obtain the distribution of $(L_X(\alpha,t), M_X(\alpha,t), X(t))$.

The law of $(L_X(\alpha,t), M_X(\alpha,t), X(t))$ is given by

$$\Pr\left(L_X(\alpha,t) \in du, M_X(\alpha,t) \in db, X(t) \in da\right) =$$

$$\left\{\begin{array}{ll} k(b,u)\,g(0, a-b, \alpha t - u, t-u)\,\mathbf{1}(0 < u < \alpha t)\,dudbda & b > 0 \\ k(b,u)\,g(0, a-b, \alpha t, t-u)\,\mathbf{1}(0 < u < \alpha t)\,dudbda & b < 0 \end{array}\right. \tag{21}$$

Proof For $b > 0$, since $(X(s+\tau_b) - X(\tau_b), 0 \le s \le t-\tau_b)$ is independent of $(X(s), 0 \le s \le \tau_b)$, we have that

$$\Pr\left(\tau_b > v, M_X(\alpha,t) \in (b, b+\delta], X(t) \in (a, a+\varepsilon)\right) =$$

$$\int_v^{\alpha t} \Pr(\tau_b \in du)\Pr\left(M_X(\alpha t - u, t-u) \in (0,\delta], X(t) \in (a-b, a-b+\varepsilon)\right).$$

For $b < 0$, we use that $M_{-X}(\alpha,t) = M_X(1-\alpha,t)$ and so $g(0, b-a, (1-\alpha)t - u, t-u) = g(0, a-b, \alpha t, t-u)$. $\Box$

We can now obtain the law of $(L_X(\alpha,t), K_X(\alpha,t), M_X(\alpha,t), X(t))$.

$$\Pr\left(L_X(\alpha,t) \in du, K_X(\alpha,t) \in dv, M_X(\alpha,t) \in db, X(t) \in da\right) =$$

$$\frac{2\left|b\right|\left|b-a\right|dudvdbda}{\pi^2\left(v-u\right)^2\sqrt{u^3\left(t-v\right)^3}}\exp\left(-\frac{b^2}{2u}-\frac{\left(b-a\right)^2}{2\left(t-v\right)}\right)\times$$

$$\left\{\begin{array}{cc} \sqrt{\left(v-u-\left(1-\alpha\right)t\right)\left(1-\alpha\right)t}\mathbf{1}\left(u>0,u+\left(1-\alpha\right)t<v<t\right) & b>0,b>a \\ \sqrt{\left(\alpha t-u\right)\left(v-\alpha t\right)}\mathbf{1}\left(0<u<\alpha t<v<t\right) & b>0,b<a \\ \sqrt{\left(v-u-\alpha t\right)\alpha t}\mathbf{1}\left(u>0,u+\alpha t<v<t\right) & b<0,b>a \\ \sqrt{\left(\left(1-\alpha\right)t-u\right)\left(v-\left(1-\alpha\right)t\right)}\mathbf{1}\left(0<u<\left(1-\alpha\right)t<v<t\right) & b<0,b<a \end{array}\right. . \tag{22}$$

Proof We start with the case $b>0,b>a$. Using (18), and choosing ε such that $a+\varepsilon<b$, we need to look at

$$\Pr\left(\tau_b\le r,K_X\left(\alpha,t\right)\le v,M_X\left(\alpha,t\right)\in\left(b,b+\delta\right],X\left(t\right)\in\left(a,a+\varepsilon\right]\right)=$$

$$\Pr\left(\tau_b\le r,M_X\left(\alpha,t\right)\in\left(b,b+\delta\right],X\left(t\right)\in\left(a,a+\varepsilon\right],M_X\left(\alpha,t\right)\le\sup_{v\le s\le t}X\left(s\right)\right)=$$

$$\int_0^r\Pr\left(\tau_b\in du\right)\Pr\left(\begin{array}{c}M_X\left(\alpha t-u,t-u\right)\in\left(0,\delta\right],X\left(t-u\right)\in\left(a-b,a-b+\varepsilon\right],\\ M_X\left(\alpha t-u,t-u\right)\le\sup_{v-u\le s\le t-u}X\left(s\right)\end{array}\right)=$$

$$\int_0^r\Pr\left(\tau_b\in du\right)\Pr\left(\begin{array}{c}M_X\left(\alpha t-u,t-u\right)\in\left(0,\delta\right],X\left(t-u\right)\in\left(a-b,a-b+\varepsilon\right],\\ K_X\left(\alpha t-u,t-u\right)\le v-u\end{array}\right). \tag{23}$$

Using the last remark of the previous section, we then see that

$$\Pr\left(\begin{array}{c}M_X\left(\alpha t-u,t-u\right)\in\left(0,\delta\right],X\left(t-u\right)\in\left(a-b,a-b+\varepsilon\right],\\ K_X\left(\alpha t-u,t-u\right)\le v-u\end{array}\right)=$$

$$\Pr\left(\begin{array}{c}M_X\left(\alpha t-u,t-u\right)-X\left(t-u\right)\in\left(0,\delta\right],-X\left(t-u\right)\in\left(a-b,a-b+\varepsilon\right],\\ L_X\left(\alpha t-u,t-u\right)\ge t-v\end{array}\right). \tag{24}$$

From the previous theorem we see that the density of

$$\left(L_X\left(\alpha t-u,t-u\right),M_X\left(\alpha t-u,t-u\right)-X\left(t-u\right),-X\left(t-u\right)\right)$$

at $\left(t-v,0,a-b\right)$ is

$$k\left(b-a,t-v\right)g\left(0,0,v-u-\left(1-\alpha\right)t,v-u\right)\mathbf{1}\left(0<t-v<\alpha t-u\right).$$

Combining this with (23) we get that $\left(L_X\left(\alpha,t\right),K_X\left(\alpha,t\right),M_X\left(\alpha,t\right),X\left(t\right)\right)$ has a continuous density at $\left(u,v,b,a\right)$ that is given by

$$k\left(b,u\right)k\left(b-a,t-v\right)g\left(0,0,v-u-\left(1-\alpha\right)t,v-u\right)\mathbf{1}\left(u>0,u+\left(1-\alpha\right)t<v<t\right). \tag{25}$$

We now look at the case $b > 0, b < a$. Using (18), and choosing δ such that $b + \delta < a$, we need to look at

$$\Pr\left(\tau_b \le r, K_X(\alpha,t) > v, M_X(\alpha,t) \in (b, b+\delta], X(t) \in (a, a+\varepsilon]\right) =$$

$$\Pr\left(\tau_b \le r, M_X(\alpha,t) \in (b, b+\delta], X(t) \in (a, a+\varepsilon], M_X(\alpha,t) < \inf_{v \le s \le t} X(s)\right) =$$

$$\int_0^r \Pr(\tau_b \in du) \Pr\begin{pmatrix} M_X(\alpha t - u, t-u) \in (0,\delta], X(t-u) \in (a-b, a-b+\varepsilon], \\ M_X(\alpha t - u, t-u) < \inf_{v-u \le s \le t-u} X(s) \end{pmatrix} =$$

$$\int_0^r \Pr(\tau_b \in du) \Pr\begin{pmatrix} M_X(\alpha t - u, t-u) \in (0,\delta], X(t-u) \in (a-b, a-b+\varepsilon], \\ K_X(\alpha t - u, t-u) < v-u \end{pmatrix}. \tag{26}$$

Using (24) and the previous theorem we see that the density of

$$(L_X(\alpha t - u, t-u), M_X(\alpha t - u, t-u) - X(t-u), -X(t-u))$$

at $(t-v, 0, a-b)$ is

$$k(b-a, t-v)\, g(0, 0, \alpha t - u, v-u)\, \mathbf{1}(\alpha t < v).$$

Combining this with (23) we get that $(L_X(\alpha,t), K_X(\alpha,t), M_X(\alpha,t), X(t))$ has a continuous density at (u, v, b, a) that is given by

$$k(b,u)\, k(b-a, t-v)\, g(0, 0, \alpha t - u, v-u)\, \mathbf{1}(0 < u < \alpha t < v < t). \tag{27}$$

Substituting (14) and (17) into (25) and (27), we get the first two legs of (22). Considering $(-X(s), 0 \le s \le t)$ and observing that $M_{-X}(\alpha,t) = -M_X(1-\alpha,t)$, $L_{-X}(\alpha,t) = L_X(1-\alpha,t)$ and $K_{-X}(\alpha,t) = K_X(1-\alpha,t)$ yields the rest of (22). □

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