On the quantiles of the Brownian motion and their hitting times.

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May 2003

Abstract

The distribution of the α-quantile of a Brownian motion on an interval $[0,t]$ has been obtained motivated by a problem in financial mathematics. In this paper we generalise these results by calculating an explicit expression for the joint density of the α-quantile of a standard Brownian motion, its first and last hitting times and the value of the process at time t. Our results can be easily generalised for a Brownian motion with drift. It is shown that the first and last hitting times follow a transformed arcsine law.

1 Introduction

Let $(X(s), s \geq 0)$ be a real valued stochastic process on a probability space $(\Omega, \mathcal{F}, \Pr)$. For $0 < \alpha < 1$, define the $\alpha-$quantile of the path of $(X(s), s \geq 0)$ up to a fixed time t by

$$M_X(\alpha, t) = \inf \left\{ x : \int_0^t \mathbf{1}(X(s) \leq x)\, ds > \alpha t \right\}. \tag{1}$$

The study of the quantiles of various stochastic processes has been recently undertaken as a response to a problem arising in the field of mathematical finance, the pricing of a particular path-dependent financial option; see Miura [7], Akahori [1] and Dassios [3]. This involves calculating quantities such as $E(h(M_X(\alpha, t)))$, where $h(x) = (e^x - b)^+$ or some other appropriate function. This requires obtaining the distribution of $X(t)$. In the case where $(X(s), s \geq 0)$ is a Lévy process (having stationary and independent increments) the following result was obtained:

Proposition 1 *Let* $X^{(1)}(s)$ *and* $X^{(2)}(s)$ *be independent copies of* $X(s)$. *Then,*

$$\begin{pmatrix} M_X(\alpha,t) \\ X(t) \end{pmatrix} \overset{(\mathsf{law})}{=} \begin{pmatrix} \sup_{0\le s\le \alpha t} X^{(1)}(s) + \inf_{0\le s\le (1-\alpha)t} X^{(2)}(s) \\ X^{(1)}(\alpha t) + X^{(2)}((1-\alpha)t) \end{pmatrix}. \quad (2)$$

When $(X(s), s\ge 0)$ is a Brownian motion, we can use this result and obtain an explicit formula for the joint density of $M_X(\alpha,t)$ and $X(t)$. This result was first proved for a Brownian motion with drift; see Dassios [3] and Embrechts, Rogers and Yor [5] and for Lévy processes by Dassios [4]. There is also a similar result for discrete time random walks first proved by Wendel [8].

We now let

$$L_X(\alpha,t) = \inf\{s\in[0,t] : X(s) = M_X(\alpha,t)\}$$

be the first, and

$$K_X(\alpha,t) = \sup\{s\in[0,t] : X(s) = M_X(\alpha,t)\},$$

the last time the process hits $M_X(\alpha,t)$. One can now introduce a 'barrier' element to the financial application by making the option worthless if the quantile is hit too early or too late. For example, this can involve calculating quantities such as $E\left(h\left(M_X(\alpha,t)\right)\mathbf{1}\left(L_X(\alpha,t) > v, K_X(\alpha,t) < u\right)\right)$.

The first study of these quantities can be found in Chaumont [2]. By using combinatorial arguments he derives of the same type as Proposition 1 that are extensions to Wendel's results in discrete time. In the case where the random walk steps can only take the values +1 or -1, a represenation for the analogues of $L_X(\alpha,t)$ and $K_X(\alpha,t)$ is obtained. Finally he derives a continuous time representation for the triple law of $M_X(\alpha,t)$, $L_X(\alpha,t)$ and $X(t)$, extending Proposition 1 when $X(t)$ is a Brownian motion. We will adopt a direct approach that seems better suited to obtaining explicit expressions for the densities involved. We will also derive alternative representations and prove a remarkable arc-sine law.

For the rest of the paper we assume that $(X(s), s\ge 0)$ is a standard Brownian motion. We will derive the joint density of $M_X(\alpha,t)$, $L_X(\alpha,t)$, $K_X(\alpha,t)$ and $X(t)$. If we denote this density by $f(y,x,u,v)$, our results can be generalised for a Brownian motion with drift m, using a Cameron-Martin-Girsanov transformation. The corresponding density will be

$$f(y,x,u,v)\exp\left(mx - m^2t/2\right).$$

Before we obtain the density of $(M_X(\alpha,t), L_X(\alpha,t), K_X(\alpha,t), X(t))$, we will first show that the law of $L_X(\alpha,t)$ (and $K_X(\alpha,t)$) is a transformed arcsine law.

2 An arcsine law for $L_X(\alpha, t)$.

Let $S_X(t) = \sup_{0 \le s \le t} \{X(s)\}$ and $\theta_X(t) = \sup\{s \in [0,t] : X(s) = S_X(t)\}$. We prove the following theorem:

Theorem 1 *For $u > 0$,*

$$\Pr(L_X(\alpha,t) > u) = \Pr(u < \theta_X(t) \le \alpha t) + \Pr(u < \theta_X(t) \le (1-\alpha)t) \quad (3)$$

and

$$\Pr(L_X(\alpha,t) \in du) = \frac{\mathbf{1}(u \le \alpha t) + \mathbf{1}(u \le (1-\alpha)t)}{\pi\sqrt{u(t-u)}} du. \quad (4)$$

Furthermore, $K_X(\alpha,t)$ has the same distribution as $t - L_X(\alpha,t)$.

Proof We will first prove that

$$\Pr(M_X(\alpha,t) > 0, L_X(\alpha,t) > u) = \Pr(u < \theta_X(t) \le \alpha t). \quad (5)$$

We observe that

$$\Pr(M_X(\alpha,t) > 0, L_X(\alpha,t) > u) = \Pr(M_X(\alpha,t) > S_X(u)) =$$
$$\Pr\left(\int_0^t \mathbf{1}(X(s) \le S_X(u))\, ds < \alpha t\right) =$$
$$\Pr\left(\int_u^t \mathbf{1}(X(s) - X(u) \le S_X(u) - X(u))\, ds < \alpha t\right). \quad (6)$$

Let $X^*(s) = X(u+s) - X(u)$. $(X^*(s), s \ge 0)$ is a standard Brownian motion which is independent of $(X(s), 0 \le s \le u)$. We condition on $S_X(u) - X(u) = c$, and set $\tau_c = \inf\{s > 0 : X^*(s) = c\}$ and $X^{**}(s) = X^*(\tau_c + s) - c$. $(X^{**}(s), s \ge 0)$ is a standard Brownian motion which is independent of both $(X(s), 0 \le s \le u)$ and $(X^*(s), 0 \le s \le \tau_c)$. We have that

$$\Pr\left(\int_0^{t-u} \mathbf{1}(X^*(s) \le c)\, ds < \alpha t - u\right) =$$
$$\int_0^{\alpha t - u} \Pr(\tau_c \in dr) \Pr\left(\int_0^{t-u-r} \mathbf{1}(X^{**}(s) \le 0)\, ds < \alpha t - u - r\right)$$

and since $\int_0^{t-u-r} \mathbf{1}(X^{**}(s) \le 0)\, ds$ has the same (arcsine) law as $\theta_{X^{**}}(t-u-r)$, this is equal to

$$\int_0^{\alpha t - u} \Pr(\tau_c \in dr) \Pr(\theta_{X^{**}}(t-u-r) < \alpha t - u - r) =$$

$$\int_0^{\alpha t-u} \Pr(\tau_c \in dr) \Pr\left(\sup_{0\le s\le \alpha t-u-r} X^{**}(s) > \sup_{\alpha t-u-r\le s\le t-u-r} X^{**}(s) \right) =$$

$$\Pr\left(\sup_{0\le s\le \alpha t-u} X^{*}(s) > \sup_{\alpha t-u\le s\le t-u} X^{*}(s), \sup_{0\le s\le \alpha t-u} X^{*}(s) > c \right)$$

and so (6) is equal to

$$\Pr\left(\begin{array}{c} \sup_{u\le s\le \alpha t} X(s) - X(u) > \sup_{\alpha t\le s\le t} X(s) - X(u), \\ \sup_{u\le s\le \alpha t} X(s) - X(u) > \sup_{0\le s\le u} X(s) - X(u) \end{array} \right) =$$

$$\Pr(u < \theta_X(t) \le \alpha t).$$

Since $(-X(s), s \ge 0)$ is also a standard Brownian motion and $M_{-X}(\alpha, t) = -M_X(1-\alpha, t)$ almost surely, we use $-X(s)$ instead of $X(s)$ and we get

$$\Pr(M_X(\alpha, t) < 0, L_X(\alpha, t) > u) = \Pr(u < \theta_X(t) \le (1-\alpha)t). \quad (7)$$

Adding (5) and (7) we get (3), and since $\theta_X(t)$ has an arcsine law, (4) follows. To see that $K_X(\alpha, t)$ has the same distribution as $L_X(\alpha, t)$, set $\tilde{X}(s) = X(t-s) - X(t)$. Clearly $\left(\tilde{X}(s), 0 \le s \le t\right)$ is a standard Brownian motion and we can easily see that $M_{\tilde{X}}(\alpha, t) = M_X(\alpha, t) - X(t)$ and $K_{\tilde{X}}(\alpha, t) = t - L_X(\alpha, t)$. □

We can also extend our result and obtain the joint distribution of

$$(M_X(\alpha, t), L_X(\alpha, t))$$

(also of $(M_X(\alpha, t) - X(t), t - K_X(\alpha, t))$.

Theorem 2 *For $b > 0$,*

$$\Pr(M_X(\alpha, t) \in db, L_X(\alpha, t) \in du) = \quad (8)$$

$$\Pr(S_X(t) \in db, \theta_X(t) \in du)\, \mathbf{1}(0 < u < \alpha t), \quad (9)$$

and for $b < 0$,

$$\Pr(M_X(\alpha, t) \in db, L_X(\alpha, t) \in du) = \quad (10)$$

$$\Pr(S_X(t) \in d|b|, \theta_X(t) \in du)\, \mathbf{1}(0 < u < (1-\alpha)t). \quad (11)$$

Furthermore $(M_X(\alpha, t), L_X(\alpha, t))$ and $(M_X(\alpha, t) - X(t), t - K_X(\alpha, t))$ have the same distribution.

Proof Let $b > 0$ and $u < \alpha t$. We then have that

$$\Pr\left(M_X(\alpha,t) > b, L_X(\alpha,t) > u\right) = \Pr\left(S_X(u) < M_X(\alpha,t), M_X(\alpha,t) > b\right) =$$

$$\Pr\left(b < S_X(u) < M_X(\alpha,t)\right) + \Pr\left(S_X(u) < b < M_X(\alpha,t)\right). \tag{12}$$

Let $\tau_b = \inf\{s > 0 : X(s) = b\}$ and $X^*(s) = X(\tau_b + s) - c$. $(X^*(s), s \geq 0)$ is a standard Brownian motion which is independent of $(X(s), 0 \leq s \leq \tau_c)$. Using theorem 1, we have

$$\Pr\left(b < S_X(u) < M_X(\alpha,t)\right) =$$

$$\int_0^u \Pr(\tau_b \in dr) \Pr\left(\int_0^{t-r} \mathbf{1}\left(X^*(s) \leq S_{X^*}(u-r)\right) < \alpha t - r\right) =$$

$$\int_0^u \Pr(\tau_b \in dr) \Pr\left(M_{X^*}\left(\frac{\alpha t - r}{t-r}, t-r\right) > 0, L_{X^*}\left(\frac{\alpha t - r}{t-r}, t-r\right) > u - r\right) =$$

$$\int_0^u \Pr(\tau_b \in dr) \Pr\left(u - r < \theta_{X^*}(t-r) < \alpha t - r\right) = \tag{13}$$

$$\Pr\left(u < \theta_X(t) < \alpha t, S_X(u) > b\right). \tag{14}$$

Furthermore,

$$\Pr\left(S_X(u) < b < M_X(\alpha,t)\right) = \Pr\left(S_X(u) < b, \int_0^t \mathbf{1}\left(X(s) \leq b\right) ds < \alpha t\right) =$$

$$\int_u^{\alpha t} \Pr(\tau_b \in dr) \Pr\left(\int_0^{t-r} \mathbf{1}\left(X^*(s) \leq 0\right) < \alpha t - r\right)$$

$$= \int_u^{\alpha t} \Pr(\tau_b \in dr) \Pr\left(\theta_{X^*}(t-r) < \alpha t - r\right) =$$

$$\Pr\left(u < \theta_X(t) < \alpha t, S_X(u) < b, \sup_{u \leq s \leq \alpha t} X(s) > b\right). \tag{15}$$

Adding (14) and (15) together, we see that (12) is equal to

$$\Pr\left(u < \theta_X(t) < \alpha t, \sup_{u \leq s \leq \alpha t} X(s) > b\right) = \Pr\left(u < \theta_X(t) < \alpha t, S_X(t) > b\right)$$

which leads to (9).

Since $(-X(s), s \geq 0)$ is also a standard Brownian motion and $M_{-X}(\alpha,t) = -M_X(1-\alpha,t)$ almost surely, we use $-X(s)$ instead of $X(s)$ and we get that for $b < 0$,

$$\Pr\left(M_X(\alpha,t) < b, L_X(\alpha,t) > u\right) = \Pr\left(u < \theta_X(t) \leq (1-\alpha)t, S_X(t) > |b|\right),$$

which leads to (11).

To see that $(t - K_X(\alpha,t), M_X(\alpha,t) - X(t))$ has the same distribution as $(L_X(\alpha,t), M_X(\alpha,t))$, set again $\tilde{X}(s) = X(t-s) - X(t)$. Clearly $\left(\tilde{X}(s), 0 \le s \le t\right)$ is a standard Brownian motion and we can easily see that $M_{\tilde{X}}(\alpha,t) = M_X(\alpha,t) - X(t)$, (and so $M_{\tilde{X}}(\alpha,t) - \tilde{X}(t) = M_X(\alpha,t)$) and $K_{\tilde{X}}(\alpha,t) = t - L_X(\alpha,t)$.□

Remarks

1. The distribution of $(\theta_X(t), S_X(t))$ is well known (see for example Karatzas and Shreve [6], page 102. From this and theorem 2, we can deduce the density of $(L_X(\alpha,t), M_X(\alpha,t))$. This is given by

$$\Pr(M_X(\alpha,t) \in db, L_X(\alpha,t) \in du) = \frac{|b|}{\pi\sqrt{u^3(t-u)}} \exp\left(-\frac{b^2}{2u}\right) \cdot$$

$$[\mathbf{1}(0 < u < \alpha t, b > 0) + \mathbf{1}(0 < u < (1-\alpha)t, b < 0)]\, dbdu. \qquad (16)$$

2. Theorem 2 also leads to an alternative expression for the distribution of $M_X(\alpha,t)$; that is

$$\Pr(M_X(\alpha,t) \in db) = \Pr(S_X(t) \in db, 0 < \theta_X(t) < \alpha t),$$

for $b > 0$ and

$$\Pr(M_X(\alpha,t) \in db) = \Pr(S_X(t) \in d|b|, 0 < \theta_X(t) < (1-\alpha)t),$$

for $b < 0$.

3. Using the argument at the end of the proof, we can generalise the last assertion of the theorem and observe that

$$(K_X(\alpha,t), M_X(\alpha,t) - X(t), -X(t))$$

has the same law as

$$(t - L_X(\alpha,t), M_X(\alpha,t), X(t))$$

and so we see that

$$(K_X(\alpha,t), M_X(\alpha,t), X(t))$$

and

$$(t - L_X(\alpha,t), M_X(\alpha,t) - X(t), -X(t)),$$

have the same distribution, a fact we will use in the following section.

3 The joint law of $(L_X(\alpha,t), K_X(\alpha,t), M_X(\alpha,t), X(t))$.

From now on we will denote the density of τ_b by $k(\cdot,\cdot)$; that is for $v>0$,

$$\Pr(\tau_b \in dv) = k(v,b)\,dv = \frac{2|b|}{\sqrt{2\pi v^3}} \exp\left(-\frac{b^2}{2v}\right) dv. \tag{17}$$

We will also denote the joint density of $\left(M_X\left(\frac{v}{t},t\right), X(t)\right)$ by $g(\cdot,\cdot,\cdot,\cdot)$; that is for $0<v<t$,

$$\Pr\left(M_X\left(\frac{v}{t},t\right) \in db, X(t) \in da\right) = g(b,a,v,t)\,dbda.$$

We can calculate $g(\cdot,\cdot,\cdot,\cdot)$ by using the proposition in the introduction. $\left(M_X\left(\frac{v}{t},t\right), X(t)\right)$ has the same distribution as

$$\left(S_{X_1}(v) - S_{X_2}(t-v), X_1(v) - X_2(t-v)\right),$$

where $(X_1(s), 0 \le s \le v)$ and $(X_2(s), 0 \le s \le t-v)$ are independent standard Brownian motions. The density of $(S_X(t), X(t))$ is given by

$$\Pr(S_X(t) \in db, X(t) \in da) = \tag{18}$$

$$\frac{2(2b-a)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2b-a)^2}{2t}\right) \mathbf{1}(b \ge 0, b \ge a)\,dadb \tag{19}$$

(see Karatzas and Shreve [6], p.95). We observe that since (19) is bounded, $g(\cdot,\cdot,\cdot,\cdot)$ is a bounded density. For our results, we need to calculate $g(0,0,v,t)$. This is the same as the value of the density of $\left(M_X\left(\frac{v}{t},t\right), M_X\left(\frac{v}{t},t\right) - X(t)\right)$ at $(0,0)$. From (19) we see that

$$\Pr(S_X(t) \in dy, S_X(t) - X(t) \in dx) = \tag{20}$$

$$\frac{2(y+x)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(y+x)^2}{2t}\right) \mathbf{1}(y \ge 0, x \ge 0)\,dydx \tag{21}$$

and it is a simple exercise to verify that

$$g(0,0,v,t) =$$

$$\int_0^\infty \int_0^\infty \frac{2(y+x)}{\sqrt{2\pi v^3}} \exp\left(-\frac{(y+x)^2}{2v}\right) \frac{2(y+x)}{\sqrt{2\pi(t-v)^3}} \exp\left(-\frac{(y+x)^2}{2(t-v)}\right) dxdy$$

$$= \frac{\sqrt{v(t-v)}}{t^2}. \tag{22}$$

We will now obtain a preliminary result.

Lemma 1 *For any u and v, such that $0 < u < v < t$, we have that*

$$\Pr\left(L_X(\alpha,t) > u, M_X(\alpha,t) \in db, X(t) \in da, K_X(\alpha,t) > v\right) =$$
$$\Pr\left(\tau_b > u, M_X(\alpha,t) \in db, X(t) \in da, K_X(\alpha,t) > v\right). \qquad (23)$$

Proof Since $M_{-X}(\alpha,t) = -M_X(1-\alpha,t)$, it suffices to prove (23) for $b > 0$. We have to prove that

$$\lim_{\delta\to 0,\varepsilon\to 0} \frac{1}{\delta\varepsilon}\cdot$$
$$\{\Pr\left(L_X(\alpha,t) > u, M_X(\alpha,t) \in (b,b+\delta], X(t) \in (a,a+\varepsilon], K_X(\alpha,t) > v\right) -$$
$$\Pr\left(\tau_b > u, M_X(\alpha,t) \in (b,b+\delta], X(t) \in (a,a+\varepsilon], K_X(\alpha,t) > v\right)\} = 0. \qquad (24)$$

Let $X^*(s) = X(s+u) - X(u)$. We then have that

$$\Pr\left(L_X(\alpha,t) > u, M_X(\alpha,t) \in (b,b+\delta], X(t) \in (a,a+\varepsilon], K_X(\alpha,t) > v\right) -$$
$$\Pr\left(\tau_b > u, M_X(\alpha,t) \in (b,b+\delta], X(t) \in (a,a+\varepsilon], K_X(\alpha,t) > v\right) =$$
$$\Pr\left(b < S_X(u) < M_X(\alpha,t) \le b+\delta, X(t) \in (a,a+\varepsilon], K_X(\alpha,t) > v\right) \le$$
$$\Pr\left(b < S_X(u) < M_X(\alpha,t) \le b+\delta, X(t) \in (a,a+\varepsilon]\right) =$$
$$\Pr\begin{pmatrix} b < S_X(u) < b+\delta, \\ S_X(u) < M_{X^*}(\alpha t-u, t-u) + X(u) \le b+\delta, \\ X^*(t-u) + X(u) \in (a,a+\varepsilon] \end{pmatrix}. \qquad (25)$$

Since $(X^*(s), 0 \le s \le t-u)$ is independent of $(X(s), 0 \le s \le u)$, and $g(\cdot,\cdot,\cdot,\cdot)$ is bounded, we condition on $S_X(u) = y$ and $X(u) = x$ and see that there is a constant K, such that

$$\Pr\begin{pmatrix} y < M_{X^*}(\alpha t-u, t-u) + x \le b+\delta, \\ X^*(t-u) + x \in (a,a+\varepsilon] \end{pmatrix} \le K\varepsilon(b+\delta-y).$$

We therefore conclude that (25) is bounded by

$$K\varepsilon E\left((b+\delta-S_X(u))\mathbf{1}(b < S_X(u) < b+\delta)\right) \le K\varepsilon\delta\Pr\left(b < S_X(u) < b+\delta\right)$$

and by the continuity of the distribution of $S_X(u)$, we see that the limit in (24) is zero. □

As a corollary we will obtain the distribution of $(L_X(\alpha,t), M_X(\alpha,t), X(t))$.

Corollary 1 *The law of* $(L_X(\alpha,t), M_X(\alpha,t), X(t))$ *is given by*

$$\Pr\left(L_X(\alpha,t) \in du, M_X(\alpha,t) \in db, X(t) \in da\right) =$$

$$\begin{cases} k(b,u)\, g(0, a-b, \alpha t - u, t-u)\, \mathbf{1}(0 < u < \alpha t)\, dudbda & b > 0 \\ k(b,u)\, g(0, a-b, \alpha t, t-u)\, \mathbf{1}(0 < u < \alpha t)\, dudbda & b < 0 \end{cases} \tag{26}$$

Proof For $b > 0$, since $(X(s+\tau_b) - X(\tau_b), 0 \le s \le t - \tau_b)$ is independent of $(X(s), 0 \le s \le \tau_b)$, we have that

$$\Pr\left(\tau_b > v, M_X(\alpha,t) \in (b, b+\delta], X(t) \in (a, a+\varepsilon)\right) =$$

$$\int_v^{\alpha t} \Pr(\tau_b \in du) \Pr\left(M_X(\alpha t - u, t-u) \in (0,\delta], X(t) \in (a-b, a-b+\varepsilon)\right).$$

For $b < 0$, we use that $M_{-X}(\alpha,t) = M_X(1-\alpha,t)$ and so

$$g(0, b-a, (1-\alpha)t - u, t-u) = g(0, a-b, \alpha t, t-u).$$

□

We can now obtain the law of $(L_X(\alpha,t), K_X(\alpha,t), M_X(\alpha,t), X(t))$.

Theorem 3

$$\Pr\left(L_X(\alpha,t) \in du, K_X(\alpha,t) \in dv, M_X(\alpha,t) \in db, X(t) \in da\right) =$$

$$\frac{2|b||b-a|\, dudvdbda}{\pi^2 (v-u)^2 \sqrt{u^3 (t-v)^3}} \exp\left(-\frac{b^2}{2u} - \frac{(b-a)^2}{2(t-v)}\right) \times$$

$$\begin{cases} \sqrt{(v-u-(1-\alpha)t)(1-\alpha)t}\,\mathbf{1}(u>0, u+(1-\alpha)t < v < t) & b>0, b>a \\ \sqrt{(\alpha t - u)(v - \alpha t)}\,\mathbf{1}(0 < u < \alpha t < v < t) & b>0, b<a \\ \sqrt{(v-u-\alpha t)\alpha t}\,\mathbf{1}(u>0, u+\alpha t < v < t) & b<0, b>a \\ \sqrt{((1-\alpha)t - u)(v-(1-\alpha)t)}\,\mathbf{1}(0<u<(1-\alpha)t<v<t) & b<0, b<a \end{cases}. \tag{27}$$

Proof We start with the case $b > 0, b > a$. Using (23), and choosing ε such that $a + \varepsilon < b$, we need to look at

$$\Pr\left(\tau_b \le r, K_X(\alpha,t) \le v, M_X(\alpha,t) \in (b, b+\delta], X(t) \in (a, a+\varepsilon]\right) =$$

$$\Pr\begin{pmatrix} \tau_b \le r, M_X(\alpha,t) \in (b, b+\delta], \\ X(t) \in (a, a+\varepsilon], M_X(\alpha,t) \le \sup_{v \le s \le t} X(s) \end{pmatrix} =$$

$$\int_0^r \Pr(\tau_b \in du) \Pr\begin{pmatrix} M_X(\alpha t - u, t - u) \in (0, \delta], \\ X(t-u) \in (a-b, a-b+\varepsilon], \\ M_X(\alpha t - u, t-u) \leq \sup_{v-u \leq s \leq t-u} X(s) \end{pmatrix} =$$

$$\int_0^r \Pr(\tau_b \in du) \Pr\begin{pmatrix} M_X(\alpha t - u, t - u) \in (0, \delta], \\ X(t-u) \in (a-b, a-b+\varepsilon], \\ K_X(\alpha t - u, t-u) \leq v - u \end{pmatrix}. \tag{28}$$

Using the last remark of the previous section, we then see that

$$\Pr\begin{pmatrix} M_X(\alpha t - u, t - u) \in (0, \delta], \\ X(t-u) \in (a-b, a-b+\varepsilon], \\ K_X(\alpha t - u, t-u) \leq v - u \end{pmatrix} =$$

$$\Pr\begin{pmatrix} M_X(\alpha t - u, t - u) - X(t-u) \in (0, \delta], \\ -X(t-u) \in (a-b, a-b+\varepsilon], \\ L_X(\alpha t - u, t-u) \geq t - v \end{pmatrix}. \tag{29}$$

From the previous theorem we see that the density of

$$(L_X(\alpha t - u, t-u), M_X(\alpha t - u, t-u) - X(t-u), -X(t-u))$$

at $(t-v, 0, a-b)$ is

$$k(b-a, t-v)\, g(0, 0, v-u-(1-\alpha)t, v-u)\, \mathbf{1}(0 < t-v < \alpha t - u).$$

Combining this with (28) we get that $(L_X(\alpha, t), K_X(\alpha, t), M_X(\alpha, t), X(t))$ has a continuous density at (u, v, b, a) that is given by

$$k(b, u)\, k(b-a, t-v)\, g(0, 0, v-u-(1-\alpha)t, v-u) \cdot \tag{30}$$

$$\mathbf{1}(u > 0, u + (1-\alpha)t < v < t). \tag{31}$$

We now look at the case $b > 0, b < a$. Using (23), and choosing δ such that $b + \delta < a$, we need to look at

$$\Pr(\tau_b \leq r, K_X(\alpha, t) > v, M_X(\alpha, t) \in (b, b+\delta], X(t) \in (a, a+\varepsilon]) =$$

$$\Pr\begin{pmatrix} \tau_b \leq r, M_X(\alpha, t) \in (b, b+\delta], \\ X(t) \in (a, a+\varepsilon], M_X(\alpha, t) < \inf_{v \leq s \leq t} X(s) \end{pmatrix} =$$

$$\int_0^r \Pr(\tau_b \in du) \Pr\begin{pmatrix} M_X(\alpha t - u, t - u) \in (0, \delta], \\ X(t-u) \in (a-b, a-b+\varepsilon], \\ M_X(\alpha t - u, t-u) < \inf_{v-u \leq s \leq t-u} X(s) \end{pmatrix} =$$

$$\int_0^r \Pr(\tau_b \in du) \Pr \begin{pmatrix} M_X(\alpha t - u, t-u) \in (0, \delta], \\ X(t-u) \in (a-b, a-b+\varepsilon], \\ K_X(\alpha t - u, t-u) < v - u \end{pmatrix}. \qquad (32)$$

Using (29) and the previous theorem we see that the density of

$$(L_X(\alpha t - u, t-u), M_X(\alpha t - u, t-u) - X(t-u), -X(t-u))$$

at $(t-v, 0, a-b)$ is

$$k(b-a, t-v)\, g(0, 0, \alpha t - u, v-u)\, \mathbf{1}(\alpha t < v).$$

Combining this with (28) we get that

$$(L_X(\alpha, t), K_X(\alpha, t), M_X(\alpha, t), X(t))$$

has a continuous density at (u, v, b, a) that is given by

$$k(b, u)\, k(b-a, t-v)\, g(0, 0, \alpha t - u, v-u)\, \mathbf{1}(0 < u < \alpha t < v < t). \qquad (33)$$

Substituting (17) and (22) into (31) and (33), we get the first two legs of (27). Considering $(-X(s), 0 \le s \le t)$ and observing that $M_{-X}(\alpha, t) = -M_X(1-\alpha, t)$, $L_{-X}(\alpha, t) = L_X(1-\alpha, t)$ and $K_{-X}(\alpha, t) = K_X(1-\alpha, t)$ yields the rest of (27). □

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