QUANTILE AND OTHER OCCUPATION TIME OPTIONS

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LOOK-BACK OPTIONS

A statistic which is a functional of the path of the stochastic process $(Y(t), t \geq 0)$ that denotes the price of the underlying asset. Pricing such options involves calculating $E^*(h(V(t_2)) \mid \mathcal{F}_{t_1})$, where the expectation is calculated under a changed measure, h is a known function, $0 \leq t_1 < t_2$ are fixed times, $\mathcal{F}_t$ is the filtration generated by $Y(t)$ and $V(t)$ is an $\mathcal{F}_t$-measurable process.

Call option $e^{-r(t_2-t_1)}E^*\Big((V(t_2) - b)^+ \mid \mathcal{F}_{t_1}\Big) = E^*(\max(V(t_2) - b, 0) \mid \mathcal{F}_{t_1})$

GEOMETRIC AVERAGE OPTIONS

$Y(t) = Y(0)\exp(X(t))$,

$$V(t) = Y(0)\exp\left(\frac{\int_0^t X(s)ds}{t}\right)$$

ASIAN OPTIONS

Arithmetic average options, where

$$V(t) = \frac{\int_0^t Y(s)ds}{t}$$

are also called Asian options.

QUANTILE OPTIONS

Another statistic that can be used is the median or more generally any α-quantile $(0 < \alpha < 1)$ of the underlying stochastic process. This was first introduced by Miura . The α-quantile is going to be the level at which the process spends a proportion of size at least α of its time below that level and a proportion of size at least $1-\alpha$ above. For $0<\alpha<1$, define $M_X(\alpha,t)$ as

$$M_X(\alpha,t)=\inf\left\{x\colon \int_0^t 1(X(s)\le x)d\,s>\alpha t\right\}.$$

Note $Y(t) = Y(0)\exp\ (X(t))$; so $M_Y(\alpha,\ t) = Y(0)\exp\left(M_X(\alpha,t)\right)$.

Also the events $\{M_X(\alpha,\ t) > x\}$ and $\left\{\int_0^t 1(X(s) \le x)d\,s<\alpha t\right\}$ are identical.

Proposition 1. *Let* $X(t) = \sigma B(t) + \mu t$*, where* $\mu \in R$*,* $\sigma \in R^+$ *and* $(B(t),\ t \geq 0)$ *is a standard Brownian motion. Furthermore, let* $X^{(1)}(t)$ *and* $X^{(2)}(t)$ *be independent copies of* $X(t)$*. Then,*

$$M(\alpha,t) \stackrel{(law)}{=} \sup_{0\leq s\leqslant\alpha t} X^{(1)}(s) + \inf_{0\leq s\leqslant(1-\alpha t)} X^{(2)}(s).$$

Proposition 2. *Let* $X(t)$*,* $M(\alpha,\ t)$*,* $X^{(1)}(t)$ *and* $X^{(2)}(t)$ *as in Proposition 1. Then,*

$$\begin{pmatrix} M(\alpha,t) \\ X(t) \end{pmatrix} \stackrel{(law)}{=} \begin{pmatrix} \sup_{0\leq s\leqslant\alpha t} X^{(1)}(s) + \inf_{0\leq s\leqslant(1-\alpha t)} X^{(2)}(s) \\ X^{(1)}(\alpha t) + X^{(2)}((1-\alpha)t) \end{pmatrix}.$$

Using this result one could calculate an expression for the joint probability density of $M(\alpha,t)$ and $X(t)$.

Density of $\sup_{0\leq s\leqslant\alpha t} X^{(1)}(s)$:

$$\frac{\sqrt{2}}{\sigma\sqrt{\pi\alpha t}}\exp\left(\frac{-(x-\mu\alpha t)^2}{\sigma\sqrt{\alpha t}}\right) - \frac{2\mu}{\sigma^2}e^{\frac{2\mu}{\sigma^2}x}\Phi\left(\frac{-x-\mu\alpha t}{\sigma\sqrt{\alpha t}}\right)$$

Propositions 1 and 2 are true when $X(t)$ **is a process with stationary and independent increments, whose paths are right continuous with left limits. (Lévy processes).**

DISCRETE TIME RESULTS.

Consider the sequence $x = (x_0, x_1, x_2, ...)$. For integers $0 \leq j \leq n$, define the$(j,n)^{th}$ quantile of x for $j = 0, 1, 2, ..., n$ by

$$M_{j,n}(x) = \inf\left\{ z : \sum_{i=0}^{n} 1(x_i \leq z) > j \right\}.$$

It should also be remarked that if $x_{(0)}$, $x_{(1)}$, $x_{(2)}$, ..., $x_{(n)}$ is an increasing order permutation of x_0, x_1, x_2, ..., x_n, then $M_{j,n}(x) = x_{(j)}$. So, in particular, $M_{0,n}(x) = \min_{i=0,1,\ldots,n} \{x_i\}$ and $M_{n,n}(x) = \max_{i=0,1,\ldots,n} \{x_i\}$. Also note that in this setup $M_{0,0}(x) = x_0$.

The following result is due to Wendel

Proposition 3. *Let Y_1, Y_2, ..., Y_n be i.i.d. random variables. Define $X = (X_0, X_1, ..., X_n)$ by*

$$X_n = \begin{cases} \sum_{i=1}^{n} Y_i & n = 1, 2, ... \\ 0 & n = 0 \end{cases}.$$

and let $X^{(1)}$ and $X^{(2)}$ be two independent copies of X. Then,

$$\begin{pmatrix} M_{j,n}(X) \\ X_n \end{pmatrix} \overset{(law)}{=} \begin{pmatrix} M_{j,j}\left(X^{(1)}\right) + M_{0,n-j}\left(X^{(2)}\right) \\ X_j^{(1)} + X_{n-j}^{(2)} \end{pmatrix}.$$

An extension of this result, involving the time the quantile is achieved, was obtained by Port. He defined the ordering $\prec$ by $X_i \prec X_j$ if $X_i < X_j$ or $X_i = X_j$ but $i < j$. Then, one could alternatively define $M_{0,n}(X)$, $M_{1,n}(X)$, ..., $M_{n,n}(X)$ as the rearrangement of X_0, X_1, ..., X_n such that $M_{0,n}(X) \prec M_{1,n}(X) \prec \ldots \prec M_{n,n}(X)$. He then defined $L_{k,n}(X)$as the index in $X_0, X_1, \ldots, X_n$ of $M_{k,n}(X)$ and extended the result to

$$\begin{pmatrix} M_{j,n}(X) \\ L_{j,n}(X) \\ X_n \end{pmatrix} \overset{(law)}{=} \begin{pmatrix} M_{j,j}\left(X^{(1)}\right) + M_{0,n-j}\left(X^{(2)}\right) \\ L_{j,j}\left(X^{(1)}\right) + L_{0,n-j}\left(X^{(2)}\right) \\ X_j^{(1)} + X_{n-j}^{(2)} \end{pmatrix}.$$

DOES THE PROPERTY CHARACTERISE LEVY PROCESSES?

Let $((T_i, Y_i), i = 1, 2, ...)$ be a sequence of independent and identically distributed pairs of random variables on a probability space $(\Omega, \mathcal{F}, \Pr)$ taking values in $R^+ \times R$ and having joint distribution function $G(u, y)$. Let

$$S_0 = 0, \;\; S_n = \sum_{i=1}^{n} T_i, \; n = 1, 2, ...$$

and define the renewal process $(N(t), t \geq 0)$ by

$$N(t) = \sup_{n=0,1,2,\ldots} \{n\colon S_n \leq t\}.$$

We define $(X(t), t \geq 0)$ by

$$X(t) = \begin{cases} \sum_{i=1}^{N(t)} Y_i & N(t) = 1, 2, ... \\ 0 & N(t) = 0 \end{cases}.$$

It should be noted that $X(t)$ is semi-Markov, but not a Markov process. However, the pair $(X(t), U(t))$ is a Markov process.

Theorem 4. *Define*

$$M_X(\alpha, t) = \inf\left\{x\colon \int_0^t 1(X(s) \le x)d\,s > \alpha t\right\}.$$

Let $X^{(1)}(t)$, $X^{(2)}(t)$ *be independent copies of* $X(t)$; *then*

$$M(\alpha, t) \stackrel{(law)}{=} \sup_{0 \le s \leqslant \alpha t} X^{(1)}(s) + \inf_{0 \le s \leqslant (1-\alpha t)} X^{(2)}(s).$$

We now let

$$L_X(\alpha,t)=\inf\{s\in[0,t]\colon X(s)=M_X(\alpha,t)\}$$

be the first, and

$$K_X(\alpha,t)=\sup\{s\in[0,t]\colon X(s)=M_X(\alpha,t)\},$$

the last time the process hits $M_X(\alpha,t)$. One can now introduce a 'barrier' element to the financial application by making the option worthless if the quantile is hit too early or too late. For example, this can involve calculating quantities such as $E(h(M_X(\alpha, t))1(L_X(\alpha,t)>v, K_X(\alpha,t)<u))$.
The first study of these quantities can be found in Chaumont (1999).

For the rest of the paper we assume that $(X(s), s \geq 0)$ is a standard Brownian motion, unless otherwisew specified. Without loss of generality, we will restrict our attention to the case $t = 1$ taking advantage of the Brownian scaling. For simplicity we set $M_X(\alpha, t) = M_X(\alpha)$, $L_X(\alpha,\ t) = L_X(\alpha)$ and $K_X(\alpha,\ t) = K_X(\alpha)$.We will derive the joint density of $M_X(\alpha)$, $L_X(\alpha)$, $K_X(\alpha)$ and $X(1)$. If we denote this density by $f(y, x, u, v)$, our results can be generalised for a Brownian motion with drift m, using a Cameron-Martin-Girsanov transformation. The corresponding density will be

$$f(y, x, u, v) \exp\left(m\, x - m^2/2\right).$$

Before we obtain the density of $(M_X(\alpha),\ L_X(\alpha),\ K_X(\alpha),\ X(1))$, we will first show that the law of $L_X(\alpha)$ (and $K_X(\alpha)$) is a transformed arcsine law.

An arcsine law for $L_X(\alpha, t)$.

Let $S_X(t) = \sup_{0 \le s \le t} \{X(s)\}$ and $\theta_X(t) = \sup\{s \in [0, t]: X(s) = S_X(t)\}$. Define also the stopping time $\tau_c = \inf\{s > 0: X(s) = c\}$. We will first obtain the joint distribution of

$$(M_X(\alpha), L_X(\alpha))$$

(also of $(M_X(\alpha) - X(1), 1 - K_X(\alpha))$.

Theorem 5. *For $b > 0$,*

$$\Pr(M_X(\alpha) \in d\,b, L_X(\alpha) \in d\,u) =$$

$$\Pr(S_X(1) \in d\,b, \theta_X(1) \in d\,u) 1(0 < u < \alpha), \tag{1}$$

and for $b < 0$,

$$\Pr(M_X(\alpha) \in d\,b, L_X(\alpha) \in d\,u) =$$

$$\Pr(S_X(1) \in d|b|, \theta_X(1) \in d\,u) 1(0 < u < (1 - \alpha)). \tag{2}$$

Furthermore, $(M_X(\alpha), L_X(\alpha))$ and $(M_X(\alpha) - X(1), 1 - K_X(\alpha))$ have the same distribution.

Proof. Let $b > 0$ and $u < \alpha$. We then have that

$\Pr\,(M_X(\alpha) > b,\, L_X(\alpha) > u) = \Pr\,(S_X(u) < M_X(\alpha),$
$M_X(\alpha) > b) =$

$$\Pr\,(b < S_X(u) < M_X(\alpha)) + \Pr\,(S_X(u) < b < M_X(\alpha)). \tag{3}$$

Let $\tau_b = \inf\{s > 0: X(s) = b\}$ and $X^*(s) = X(\tau_b + s) - b$. $(X^*(s),\ s \geq 0)$ is a standard Brownian motion which is independent of $(X(s), 0 \leq s \leq \tau_b)$. Using theorem 1, we have

$$\Pr\,(b < S_X(u) < M_X(\alpha)) =$$

$$\Pr\left(S_X(u) > b, \int_0^1 1(X(s) \leq S_X(u))d\,s < \alpha\right) =$$

$$\Pr\left(S_X(u) > b, \int_u^1 1(X(s) - X(u) \leq S_X(u) - X(u))d\,s < \alpha - u\right). \tag{4}$$

We now condition on $\sigma\{X(s),\ 0 \leq s \leq u\}$. Let $X^*(s) = X(u+s) - X(u)$. $(X^*(s),\, s \geq 0)$ is a standard Brownian motion which is independent of $(X(s), 0 \leq s \leq u)$. We condition on $S_X(u) - X(u) = c$, and set $\tau_c = \inf\{s > 0\colon X^*(s) = c\}$ and $X^{**}(s) = X^*(\tau_c + s) - c$. $(X^{**}(s),\, s \geq 0)$ is a standard Brownian motion which is independent of both $(X(s), 0 \leq s \leq u)$ and $(X^*(s), 0 \leq s \leq \tau_c)$. We have that

$$\Pr\left(\int_0^{1-u} 1(X^*(s) \leq c) d\, s < \alpha - u\right) =$$

$$\int_0^{\alpha-u} \Pr\,(\tau_c \in d\, r) \Pr\left(\int_0^{1-u-r} 1(X^{**}(s) \leq 0) d\, s < \alpha 1 - u - r\right)$$

and since $\int_0^{1-u-r} 1(X^{**}(s) \leq 0) d\, s$ has the same (arcsine) law as

$\theta_{X^{**}}(1-u-r)$, this is equal to

$$\int_0^{\alpha-u} \Pr\,(\tau_c \in d\, r) \Pr\,(\theta_{X^{**}}(1-u-r) < \alpha - u - r) =$$

$$\int_0^{\alpha-u} \Pr\,(\tau_c \in d\, r) \Pr\left(\sup_{0 \leq s \leq \alpha-u-r} X^{**}(s) > \sup_{\alpha-u-r \leq s \leq 1-u-r} X^{**}(s)\right) =$$

$$\Pr\left(\sup_{0 \leq s \leq \alpha-u} X^*(s) > \sup_{\alpha-u \leq s \leq t-u} X^*(s),\ \sup_{0 \leq s \leq \alpha-u} X^*(s) > c\right)$$

and so (4) is equal to

$$\Pr\left(\begin{array}{c}\sup_{u\le s\le\alpha} X(s)-X(u) > \sup_{\alpha\le s\le 1} X(s)-X(u),\\ \sup_{u\le s\le\alpha} X(s)-X(u) > \sup_{0\le s\le u} X(s)-X(u),\\ \sup_{0\le s\le u} X(s) > b\end{array}\right) =$$

$$\Pr\left(S_X(u) > b, u < \theta_X(1) \le \alpha\right). \tag{5}$$

Furthermore,

$$\Pr\ \left(S_X(u)\ <\ b\ <\ M_X(\alpha)\right)\ =\ \Pr\ \left(S_X(u)\ <\ b,\ \int_0^1 1(X(s)\le b)d\,s < \alpha\right) =$$

$$\int_u^\alpha \Pr\left(\tau_b \in d\,r\right)\Pr\left(\int_0^{1-r} 1(X^*(s)\le 0) < \alpha - r\right)$$

$$= \int_u^\alpha \Pr\left(\tau_b \in d\,r\right)\Pr\left(\theta_{X^*}(1-r) < \alpha - r\right) =$$

$$\Pr\left(u < \theta_X(1) < \alpha, S_X(u) < b, \sup_{u\le s\le\alpha} X(s) > b\right). \tag{6}$$

Adding (5) and (6) together, we see that (3) is equal to

$$\Pr\ \left(u\ <\ \theta_X(1)\ <\ \alpha,\ \sup_{u\le s\le\alpha}\ X(s)\ >\ b\right)\ =\ \Pr\ \left(u\ <\ \theta_X(1) < \alpha, S_X(1) > b\right)$$

which leads to (1).

Since $(-X(s), s \geq 0)$ is a standard Brownian motion and $M_{-X}(\alpha) = - M_X(1-\alpha)$ almost surely, we use $-X(s)$ instead of $X(s)$ and we get that for $b<0$,

$$\Pr\left(M_X(\alpha) < b, L_X(\alpha) > u\right) = \Pr\left(u < \theta_X(1) \leq (1-\alpha),\right.$$
$$\left. S_X(1) > |b|\right),$$

which leads to (2).
To see that $(t - K_X(\alpha), M_X(\alpha) - X(1))$ has the same distribution as $(L_X(\alpha),\ M_X(\alpha))$, set again $\tilde{X}(s) = X(1-s) - X(1)$. Clearly $\left(\tilde{X}(s), 0 \leq s \leq t\right)$ is a standard Brownian motion and we can easily see that $M_{\tilde{X}}(\alpha) = M_X(\alpha) - X(1)$, $M_{\tilde{X}}(\alpha) - \tilde{X}(1) = M_X(\alpha)$ and $K_{\tilde{X}}(\alpha) = 1 - L_X(\alpha)$. $\square$

Remarks

1. The distribution of $(\theta_X(1),\ S_X(1))$ is well known (see for example Karatzas and Shreve (1988, page 102). From this and Theorem 2, we can deduce the density of $(L_X(\alpha), M_X(\alpha))$. This is given by

   $$\Pr\ (M_X(\alpha)\in d\ b,\ L_X(\alpha)\in d\ u) = \frac{|b|}{\pi\sqrt{u^3(1-u)}}\exp\left(-\frac{b^2}{2u}\right)\cdot$$

   $$[1(0<u<\alpha, b>0)+1(0<u<1-\alpha, b<0)]\,d\,b\,d\,u. \tag{7}$$

2. Theorem 1 also leads to an alternative expression for the distribution of $M_X(\alpha)$; that is

   $$\Pr\,(M_X(\alpha)\in d\,b) = \Pr\,(S_X(1)\in d\,b, 0<\theta_X(1)<\alpha),$$

   for $b>0$ and

   $$\Pr\ (M_X(\alpha)\in d\ b) = \Pr\ (S_X(1)\in d|b|,\ 0<\theta_X(1)<1-\alpha),$$

   for $b<0$.

3. From Theorem 1, we can immediately obtain the following corollary:

Corollary 6. *For* $u > 0$,

$$\Pr(L_X(\alpha) > u) = \Pr(u < \theta_X(1) \leq \alpha) + \Pr(u < \theta_X(1) \leq 1 - \alpha) \tag{8}$$

and

$$\Pr(L_X(\alpha) \in d\,u) = \frac{1(u \leq \alpha) + 1(u \leq 1 - \alpha)}{\pi\sqrt{u(1-u)}} d\,u. \tag{9}$$

Furthermore, $K_X(\alpha)$ *has the same distribution as* $1 - L_X(\alpha)$.

The joint law of $(L_X(\alpha), K_X(\alpha), M_X(\alpha)$

Theorem 7. *For the standard Brownian motion* $(X(s), s \geq 0)$,

$\Pr\,(L_X(\alpha) \in d\,u,\, K_X(\alpha) \in d\,v,\, M_X(\alpha) \in d\,b,\, X(1) \in d\,a) =$

$$\frac{2|b||b-a|\,d\,u\,d\,v\,d\,b\,d\,a}{\pi^2 (v-u)^2 \sqrt{u^3 (1-v)^3}} \exp\left(-\frac{b^2}{2u} - \frac{(b-a)^2}{2(1-v)}\right) \times$$

$$\begin{cases} \sqrt{(v-u-(1-\alpha))(1-\alpha)}\,1(u>0, u+(1-\alpha)<v<1) & b>0, \\ \sqrt{(\alpha-u)(v-\alpha)}\,1(0<u<\alpha<v<1) & b>0, \\ \sqrt{(v-u-\alpha)\alpha}\,1(u>0, u+\alpha<v<1) & b<0, \\ \sqrt{(1-\alpha-u)(v-(1-\alpha))}\,1(0<u<1-\alpha<v<1) & b<0, \end{cases}$$

SOME PROPERTIES OF THE MEDIAN

(General Lévy processes)

1. $E\left(M_X\left(\frac{1}{2}, t\right)\right) = E\left(\frac{X(t)}{2}\right)$.(assuming existence)

2. $M_X\left(\frac{1}{2}, t\right)$ is stochastically more variable than $\frac{X(t)}{2}$.

3. $M_X\left(\frac{1}{2}, t\right)$ is stochastically less variable than $X\left(\frac{t}{2}\right)$. (but not that different at the tail for a Brownian motion)

For the standard Brownian motion $M_X\left(\frac{1}{2}, t\right)$is stochastically more variable than $X(\beta t)$, where $\beta \leq 6 - 4\sqrt{2} = 0.343$.

Note$\frac{\int_0^t X(s)ds}{t}$ has the same distribution as $X\left(\frac{t}{3}\right)$ and $\frac{1}{3} < 6 - 4\sqrt{2}$, so $M_X\left(\frac{1}{2}, t\right)$is stochastically more variable than $\frac{\int_0^t X(s)ds}{t}$.

Are median options more expensive than geometric average options? (also more expensive than Asian when way out of the money ?)

In the tables we calculate $e^{-r}E^*\left((V(1)-k)^+\right)$, for a geometric average option, an Asian option (values taken from Rogers and Shi ,lower bounds given with upper bounds in brackets) $Y(0) = 100$

TABLE 1. $\sigma = .05$

r	k	Geom.	Asian	Median
.05	95	7.147	7.178 (7.183)	7.156
	100	2.689	2.716 (2.722)	2.708
	105	0.324	0.337 (0.343)	0.410
.09	95	8.757	8.809 (8.821)	8.767
	100	4.256	4.308 (4.318)	4.275
	105	0.922	0.958 (0.968)	1.059
.15	95	10.988	11.094 (11.114)	11.001
	100	6.689	6.794 (6.810)	6.704
	105	2.646	2.744 (2.761)	2.765

TABLE 2. $\sigma = .1$

r	k	Geom.	Asian	Median
.05	90	11.862	11.951 (11.973)	11.894
	100	3.573	3.641 (3.663)	3.617
	110	0.306	0.331 (0.353)	0.413
.09	90	13.274	13.385 (13.410)	13.301
	100	4.816	4.915 (4.942)	4.863
	110	0.583	0.630 (0.657)	0.745
.15	90	15.235	15.399 (15.445)	15.265
	100	6.869	7.028 (7.066)	6.919
	110	1.310	1.413 (1.451)	1.553

TABLE 3. $\sigma = .2$

r	k	Geom.	Asian	Median
.05	90	12.318	12.595 (12.687)	12.469
	100	5.547	5.762 (5.854)	5.651
	110	1.845	1.989 (2.080)	2.045
.09	90	13.520	13.831 (13.927)	13.652
	100	6.518	6.777 (6.872)	6.628
	110	2.359	2.545 (2.641)	2.593
.15	90	15.267	15.641 (15.748)	15.383
	100	8.073	8.408 (8.515)	8.193
	110	3.292	3.554 (3.661)	3.571

TABLE 4. $\sigma = .1$

r	k	Geom.	Asian	Median
.05	90	13.404	13.952 (14.161)	13.657
	100	7.496	7.944 (8.153)	7.674
	110	3.722	4.070 (4.279)	3.981
.09	90	14.388	14.983 (15.194)	14.627
	100	8.324	8.827 (9.039)	8.510
	110	4.291	4.695 (4.906)	4.574
.15	90	15.838	16.512 (16.732)	16.062
	100	9.612	10.208 (10.429)	9.812
	110	5.229	5.728 (5.948)	5.548

EXTENSION OF THE RESULTS.

Let $X(t)$ be a Lévy process. Define $\Gamma_x(t) = \int_0^t 1(X(s) \leq x) d\,s$ and let Γ_x^{-1} be its inverse. So far

$$\Gamma_x^{-1}(u) - u \overset{(law)}{=} R_u$$

where

$$R_u = \inf\left(r\colon \sup_{0 \leq s \leqslant r} X^{(1)}(s) + \inf_{0 \leq s \leqslant u} X^{(2)}(s)\right)$$

Extension

$$\left\{ \begin{array}{c} \Gamma_x^{-1}(u) - u \\ u \geq 0 \end{array} \right\} \overset{(law)}{=} \left\{ \begin{array}{c} R_u \\ u \geq 0 \end{array} \right\}$$

Corollary

$$\sup_{0 \leq t \leq T} M_X(\alpha, \quad t) \overset{(law)}{=} \sup_{0 \leq t \leq T} \left\{ \sup_{0 \leq s \leqslant \alpha t} X^{(1)}(s) + \inf_{0 \leq s \leqslant (1-\alpha t)} X^{(2)}(s) \right\}$$