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MARTINGALES AND INSURANCE RISK

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ABSTRACT

In [10] M.H.A. Davis introduced a class of non-diffusion models, called piecewise-deterministic Markov processes. As was pointed out by Embrechts [14] in the discussion to Davis's paper, these processes should provide a standard theory for studying applications in insurance risk theory. It is our aim to explain this in more detail by unifying the analysis of stochastic insurance models. Some new results will also be provided.

In Section 1 we introduce the mathematics of the basic model together with a formulation of the classical risk processes as piecewise deterministic Markov (PD) processes. We distinguish between two different approaches. The first approach is used in Section 2 to find expressions for the probability of ruin in the classical Andersen model. Some new results are obtained for varying claim arrival rate, e.g. under periodicity assumptions. A general model including service payments is also analysed. Finally in Section 3 we discuss the second approach to establish exact results for Gamma claim sizes and investment policies, including borrowing. It should be stressed early on that the piecewise deterministic structure is essentially a vehicle for obtaining

interesting martingales, which can then be used to calculate relevant functionals of the process.

If the Davis theory is to be used for the analysis of more realistic insurance models, then it is important to understand in detail its scope and versatility. The present paper stresses the request made by E. Arjas [2] : "... I hope to see in the future many worked out examples of how the PD theory can be used as an aid in solving practical OR problems".

1. PD PROCESSES AND INSURANCE MODELS

In general risk theory, one is interested in the so-called surplus process. It represents the surplus (liquid funds, accumulated capital) of a company and consists essentially of three different processes.

(i) The income process (premiums etc.) $\{U_t : t \geq 0\}$. The variable t will always denote time; it can either be discrete or continuous or indeed be operational time. For reasons of uniformity, we assume $t \in \mathbb{R}^+$ in this paper.

(ii) The counting process of claim arrivals $\{N_t : t \geq 0\}$.

(iii) An infinite sequence of random variables $(Y_i)_i$, the claim sizes.

The aggregate claim size process is defined by $Y_t = \sum_{i=1}^{N_t} Y_i$. So that the surplus process equals $Z_t = U_t - Y_t$.

Whenever we shall speak about *the classical model*, we shall assume the following extra assumptions.

A1. U_t is deterministic, i.e. $u_t = u + ct$ for $t \geq 0$, where $c > 0$ is interpreted as the constant premium income rate and u is the initial capital.

A2. $\{N_t : t \geq 0\}$ is a homogeneous Poisson process with parameter λ, say.

A3. The claim sizes $Y_1, Y_2, \ldots$ are i.i.d. with distribution function G(y).

A4. The processes $\{N_t : t \geq 0\}$ and $(Y_i)_i$ are independent.

A5. Net-profit condition : $\exists\, t_1 > 0, \forall t \geq t_1 : E(Z_t) - Z_0 > 0$.

A6. Assume $G(y)$ is supported by $[0,\infty[$ and its Laplace-Stieltjes transform $\phi(\nu) = \int_0^\infty e^{\nu x} dG(x)$ exists and is twice differentiable on an interval $[0,\alpha[$ where $\alpha > 0$ and $\lim_{\nu \uparrow \alpha} \phi(\nu) = +\infty$.

Remarks :

Remark a : The conditions A5. implies that Z_t is transient. For the models in this paper, it will always be obvious how to insure that transience holds. For more complicated examples we refer to Dassios [11].

Remark b : The surplus process Z_t obviously has right-continuous sample paths. The condition A6 excludes such examples as Pareto and lognormal claim sizes. The gamma family, truncated normal and distributions with finite support are typical classes of distribution functions satisfying A6.

Remark c : Throughout this paper, we shall use standard terminology from insurance mathematics. For the reader not acquainted with this terminology, we suggest the books by Gerber [16] and Heilmann [20] for further background.

One of the goals of current research in insurance risk theory is to analyse the probability of ruin when the conditions A1-6 are relaxed. In this paper we shall concentrate mostly on the notion of ruin, although more general optimization problems and investment strategies can be incorporated in the model.

Denote by $\tau = \inf\{t : t > 0 \text{ and } Z_t \leq 0\}$ the instance of ruin, $\tau = \infty$ if for all $t > 0$, $Z_t > 0$. The probability of ruin in the infinite horizon case is

$$\psi(u) = P\{\tau < \infty \mid Z_0 = u\}$$

whereas in the finite horizon case

$$\psi(u,t) = P\{\tau < t \mid Z_0 = u\}.$$

Problems of this kind have received a lot of interest from probabilists over the recent years, especially in the OR literature. See for instance Brockwell et al. [7], [8], Harrison and Resnick [18], [19] and references therein. See also Delbaen and Haezendonck [12] and Papatriandafylou and Waters [22]. Below we briefly explain the basic definition of a piecewise deterministic Markov process (henceforth abbreviated to PD process). A detailed, mathematical discussion is rather complicated and can be found, together with some examples, in Davis [10]. We shall only concentrate on those aspects of the definition which are important for practical applications. A PD process is a Markov process $\mathbf{X}_t$ with two components (η_t, ξ_t) where η_t takes values in a countable set K, labelling the evolution of the process through different stages (for instance $K = \{0,1\}$ where 0 denotes ruin and 1 denotes non-ruin) and given $\eta_t = n \in K$, ξ_t takes values in an open set $M_n \subset \mathbb{R}^{d(n)}$ for some function $d : \mathbb{N} \to \mathbb{N}$. For the present paper, K will be a subset of $\mathbb{N}$, though applications exist with $K = \mathbb{Q}$. The state space of X_t equals $E = \{(n,z) : n \in K, z \in M_n\}$. We further assume that for every point $x = (n,z) \in E$, there is a unique, deterministic integral curve $\phi_n(t,z) \subset M_n$, determined by a differential operator χ_n on $\mathbb{R}^{d(n)}$, such that $z \in \phi_n(t,z)$. If for some $t_o \in \mathbb{R}^+$, $X_{t_o} = (n_o, z_o) \in E$, then ξ_t, $t \geq t_o$, follows $\phi_{n_o}(t,z_o)$ until either $t = T_0$, some random time with hazard rate function λ, or until $\xi_t \in \partial M_{n_o}$, the boundary of M_{n_o}. In both cases, the process X_t jumps, according to a Markov transition measure Q on E, to a point $(n_1, z_1) \in E$. ξ_t again follows the deterministic path ϕ_{n_1} till a random time T_1 (independent of T_0) or till $\xi_t \in \partial M_{n_1}$, etc... The jump times T_i are assumed to satisfy the following condition (Davis [10]) :

$$\forall t > 0, \quad E(\sum_i I\ (T_i \leq t)) < \infty. \tag{1.1}$$

The stochastic calculus that will enable us to analyse various general insurance models rests on the notion of (extended) generator A of $\mathbf{X}_t$, see for instance Dynkin [13], Rosenkrantz [23] and Davis [10], Definition 5.2. Let Γ denote the set of boundary points of E, $\Gamma = \{(n,z) : n \in K, z \in \partial M_n\}$; and let A be an operator acting on measurable functions $f : E \cup \Gamma \to \mathbb{R}$ satisfying, see Davis [10], Theorem 5.5 and p.367 :

D1. The function $t \to f(n, \phi_n(t,z))$ is absolutely continuous for $t \in [0,t(n,z)]$, for all $(n,z) \in E$. (1.2)

D2. For all $x \in \Gamma$, $f(x) = \int_E f(y)\, Q(x;dy)$ (Boundary condition). (1.3)

D3. For all $t \geq 0$, $E(\sum_{T_i \leq t} | f(X_{T_i}) - f(X_{T_i^-})| < \infty$ $(X_{T_i^-} = \lim_{t \uparrow T_i} X_t)$. (1.4)

It should be noted that condition (1.4) is stronger than the one used in Davis [10]. Hence the set of measurable functions satisfying D1, D2 and D3 form a subset of the domain of the extended generator A, denoted by D(A). Also, in view of (1.1), (1.4) is trivially satisfied if f is bounded, though in general, f need not be bounded in order to belong to D(A). From the applications point of view, it is now important that for PD processes one can explicitly calculate A as was done by Davis [10], Theorem 5.5 :

$$\forall f \in D(A) \ : \ Af(x) = \chi f(x) + \lambda(x) \int_E [f(y) - f(x)]\, Q(x;dy). \qquad (1.5)$$

In some cases, it is important to have time t as an explicit component of the PD process. In those cases A can be decomposed as $\frac{\partial}{\partial t} + A_t$ where A_t is given by (1.5) with possibly time-dependent coefficients, see Davis [10], p.369.

An application of Dynkin's formula now gives the following important results from Davis [10]. (Martingales will always be with respect to the natural filtration $\sigma\{X_s : s \leq t\}$).

<u>*Proposition 1*</u>

(a) *If for all t, f(.,t) belongs to the domain of* A_t *and* $\frac{\partial}{\partial t} f(x,t) + A_t f(x,t) = 0$, *then the process* $f(X_t,t)$ *is a martingale.*

(b) *If f belongs to the domain of A and* $Af(x) = 0$, *then* $f(X_t)$ *is a martingale.*

□

<u>*Proposition 2*</u>

Let $\alpha : [0,t] \times E \to \mathbb{R}_+$, $c : [0,t] \times E \to \mathbb{R}$ *and* $f_{ter} : E \to \mathbb{R}$ *be measurable functions. Suppose* $f : E \times [0,t] \to \mathbb{R}$ *satisfies*

(i) $\forall s$, $0 \leq s < t$, *f(.,s) belongs to the domain of* A_s;

(ii) $\forall x \in E,\ f(x,t) = f_{ter}(x)$ *(terminal cost condition);*

(iii) $\forall (s,x) \in [0,t] \times E$:

$$\frac{\partial f}{\partial s} + A_s f(x,s) - \alpha(s,x)\, f(x,s) + c(s,x) = 0. \qquad (1.6)$$

Then

$$f(x,0) = E\Big(\int_0^t \exp\big(-\int_0^s \alpha(u,X_u)du\big)\, c(s,X_s)ds + \exp\big(-\int_0^t \alpha(u,X_u)du\big)\, f_{ter}(X_t) \mid X_0 = x\Big). \qquad (1.7)$$

□

See Theorem 6.3 of Davis [10]. In all applications, we shall have $\alpha(s,x) \equiv \alpha$ which is to be interpreted as a discount factor and then (1.7) becomes

$$f(x,0) = E\Big(\int_0^t e^{-\alpha s} c(s,X_s)ds + e^{-\alpha t} f_{ter}(X_t) \mid X_0 = x\Big). \qquad (1.8)$$

We now turn to the application of these results in insurance models. As a first example we reformulate the classical surplus process in terms of a PD process. There is no unique way for doing so. The flexibility of the definition of a PD process allows several, essentially different formulations which can be used according to what the objective is. Depending on the labelling component η_t we shall consider two possible options.

In the first model (called MODEL 1 throughout) we assume that there is no absorbtion of the process in zero, hence the state space is just $\mathbb{R}$ and therefore η_t only takes one value, $\eta_t \equiv 1$ say so that $K = \{1\}$. In the second case, MODEL 2, we have absorption in zero, therefore set $\eta_t = 1$ if at time t the company has not yet been ruined (i.e. positive surplus) and $\eta_t = 0$ (with absorption) if the surplus at time t is negative or zero, so that $K = \{0,1\}$.

The reasons for studying two different models are i) to show the versatility of the PD approach, ii) each model has its own special mathematical structure with respect to solving ruin problems and in many cases, the nature of the problem will imply a choice to be made between them.

We first study MODEL 1. As ξ_t we define the surplus process Z_t. Now in between jumps $\xi_t = z + ct$ for some value z so that the integral curves from the definition of a PD process satisfy $\frac{d}{dt}(z + ct) = c\frac{\partial}{\partial z}(z + ct)$ hence $\chi = c\frac{\partial}{\partial z}$ where c is the constant (!) premium income rate. Hence the process evolves as $Z_t = Z_{T_i} + ct$ where T_i is the time of the last jump (claim) before t. The jump (claim) intensity is constant, λ (Poisson process parameter). Also $M_1 = \mathbb{R}$ and $\Gamma = \phi$, the empty set. The measure Q is defined as follows. Let G(y) be the claim size distribution, then $Z_{T_i} = Z_{T_i-} - Y_i$. The generator of the process, acting on absolutely continuous functions f(z,t) so that for all $t,z > 0$, $E\,|f(z-Y,t)-f(z,t)| < \infty$; where Y has distribution function G, is given by

$$\frac{\partial}{\partial t}f(z,t) + c\frac{\partial}{\partial z}f(z,t) + \lambda\left(\int_0^\infty f(z-y,t)\,dG(y) - f(z,t)\right). \quad (1.9)$$

Turning to MODEL 2, here we distinguish between the events of ruin ($\eta_t = 0$) and non-ruin ($\eta_t = 1$). Whenever $\eta_t = 1$, Z_t takes values in $(0,\infty)$ $(= M_1)$ and evolves as $Z_t = Z_{T_i} + ct$ (T_i being the time of the latest jump before t). Again the associated vector field is $\chi_1 = c\frac{\partial}{\partial z}$. If $\eta_t = 0$, the process is absorbed in 0, i.e. $M_0 = \{0\}$. (We could of course enlarge M_0 so as to make it an open set). To define the associated measure Q, take the random variable Y_i as before (i-th claim size) and define a new random variable Y_i^* such that

$$Y_i^* = \begin{cases} Y_i & \text{if } Y_i < Z_{T_i-}, \\ Z_{T_i-} & \text{if } Y_i \geq Z_{T_i-}. \end{cases}$$

(No jumps can occur if $\eta_t = 0$). Including time as an explicit component of the process, the generator acting on f(n,z,t) is, if $\eta_t = 1$,

$$\frac{\partial}{\partial t} f(1,z,t) + c\frac{\partial}{\partial z} f(1,z,t) + \lambda\Big(\int_0^z f(1,z-y,t)dG(y) +$$
$$\bar{G}(z)f(0,0,t)-f(1,z,t)\Big) \quad (1.10)$$

where $\bar{G}(y) = 1 - G(y)$, and if $\eta_t = 0$, $\frac{\partial}{\partial t} f(0,0,t)$.

Since $0 \notin M_1$ we don't need a continuity requirement (see D1) at 0 for f to belong to the domain of the generator. Therefore we can supress dependence on n and set $\forall t \geq 0$, $f(0,t) = 0$. (1.10) can be rewritten as

$$\frac{\partial}{\partial t} f(z,t) + c \frac{\partial}{\partial z} f(z,t) + \lambda\left(\int_0^z f(z-y,t)\, d\, G(y) - f(z,t)\right), \; z > 0. \tag{1.11}$$

The calculation of ruin probabilities (either finite or infinite horizon case) is now based upon the construction of suitable martingales $f(Z_t)$ or $f(Z_t,t)$ via Proposition 1 and the use of Doob's Optional Stopping Theorem, Karlin and Taylor [21], p. 370.

Proposition 3

Let $\{X_t : t \geq 0\}$ be a martingale with respect to a suitable filtration and τ a stopping time. If $P(\tau < \infty) = 1$ and $E[\sup_{t\geq 0} |X_t|] < \infty$ (uniform integrability) then $E(X_\tau) = E(X_0)$. □

We would like to stress at this point that our methods do not save work in any way. However, the PD approach gives a much more systematic method for generating relevant martingales. In the paragraphs to come we shall indicate this point more clearly. A possible disadvantage of the [10] is its generality, making it difficult for the more applied reader to use. For that matter, we shall illustrate the use of PD processes in MODELS 1 and 2 for the easiest classical surplus process, i.e. where the claim size distribution is exponential with density $G'(y) = \alpha e^{-\alpha y}$. The more confident reader should skip these illustrations and immediately turn to the next paragraph.

Illustration 1

Proposition 4

Consider the classical ruin problem in the infinite horizon case with exponential(α) claim sizes. If τ denotes the time of ruin $\tau = \inf\{t > 0 : Z_t \leq 0\}$, then

$$P(\tau < \infty) = \frac{\lambda}{c\alpha} e^{-(\alpha-\lambda/c)u}.$$

(Note that we omit the conditioning on $Z_0 = u$, this is done for notational convenience).

First proof

Use MODEL 1 as described above. Therefore, the generator (1.9) takes on the form :

$$Af(z) = cf'(z) + \lambda \left(\int_0^\infty f(z-y)\, \alpha e^{-\alpha y}\, dy - f(z)\right).$$

Now for $f(Z_t)$ to be martingale, by Theorem 1b), we need a solution f of the equation $Af = 0$, with A given above. (1.12)

The solution of this well-known integro-differential equation can be found by substituting a trial-solution of the form $e^{-\nu_o z}$ and showing that $\nu_o = \alpha - \frac{\lambda}{c}$, which is strictly positive by the net-profit condition. By the same condition, it is easy to check that $e^{-\nu_o z}$ satifies D3 and indeed $e^{-\nu_o z}$ belongs to the domain of A, consequently $e^{-(\alpha - \frac{\lambda}{c})Z_t}$ is a martingale. The time of ruin τ is a stopping time, however since $P(\tau < \infty) < 1$, we cannot directly apply Proposition 3. A classical truncation argument however will work here. Put $\tau \wedge t = \min(\tau, t)$ for t fixed, then by Proposition 3,

$$e^{-(\alpha - \frac{\lambda}{c})u} = E(e^{-(\alpha - \frac{\lambda}{c})Z_{\tau \wedge t}}) \tag{1.13}$$

where u is the initial capital. So

$$e^{-(\alpha - \lambda/c)u} = E(e^{-(\alpha - \lambda/c)Z_\tau}; \tau \le t) + E(e^{-(\alpha - \lambda/c)Z_t}; \tau > t). \tag{1.14}$$

Then, as $t \to \infty$, by the net-profit condition it follows that $Z_t \to \infty$ a.s.. Therefore, by the dominated convergence theorem,

$$P(\tau < \infty) = \frac{e^{-(\alpha - \lambda/c)u}}{E(e^{-(\alpha - \lambda/c)Z_\tau} \mid \tau < \infty)}. \tag{1.15}$$

At this point, one still needs an argument based on the memoryless property of the exponential distribution to calculate the denominator; see Billingsley [5], p. 274. Finally we find the classical formula :

$$P(\tau < \infty) = \frac{\lambda}{c\alpha} e^{-(\alpha - \lambda/c)u}, \tag{1.16}$$

□

A derivation of this result similar to ours, using martingale theory, is to be found in Chapter 9 of Gerber [16]. Gerber pioneered the use of martingale theory in insurance modelling. For more complicated applications of the PD theory, it is important to keep in mind the important steps outlined above.

Illustration 2.

Second proof.

One can also use MODEL 2 to derive (1.16) for instance. In that case, the generator A acting on f(z) is now of convolution type. For $z > 0$,

$$Af(z) = cf'(z) + \lambda\left(\int_0^z f(z-y)\alpha e^{-\alpha y}dy - f(z)\right), \qquad (1.17)$$

where f is such that $f(0) = 0$ (but not $f(0+) = 0$). So for $f(Z_t)$ to be a martingale, we have to solve

$$cf'(z) + \lambda\left(\int_0^z f(z-y)\alpha e^{-\alpha y}dy - f(z)\right) = 0. \qquad (1.18)$$

Differentiating (1.18) one obtains

$$cf''(z) + (c\alpha - \lambda)\, f'(z) = 0. \qquad (1.19)$$

so that $f(z) = B_1 + B_2 e^{-(\alpha - \lambda/c)z}$ $(z > 0)$ where B_1 and B_2 are some constants to be determined by letting $z \downarrow 0$ in (1.18). We find $cf'(0+) - \lambda f(0+) = 0$, so that $B_1 = -\frac{c\alpha}{\lambda} B_2$. Putting $B_2 = 1$ (we are not interested in multiplicative constants) it follows that

$$f(Z_t) = \begin{cases} -\frac{c\alpha}{\lambda} + e^{-(\alpha - \lambda/c)Z_t} & , Z_t > 0 \\ 0 & , Z_t = 0 \end{cases} \qquad (1.20)$$

is a martingale. Using $Z_\tau = 0$, as before we find that

$$-\frac{c\alpha}{\lambda} + e^{-(\alpha - \lambda/c)u} = E(f(Z_{\tau \wedge t}))$$
$$= E\left(-\frac{c\alpha}{\lambda} + e^{-(\alpha - \lambda/c)Z_t}; t < \tau\right).$$

Letting $t \to \infty$, we find $-\frac{c\alpha}{\lambda} + e^{-(\alpha - \lambda/c)u} = -\frac{c\alpha}{\lambda} P(\tau = \infty)$ and therefore (1.16) follows. □

Remarks :

Remark a : A solution of (1.18) can also be found by taking Laplace transforms so that

$$\hat{f}(s) = -\frac{c\alpha}{\lambda}\frac{1}{s} + \frac{1}{s + (c\alpha - \lambda)/c} . \tag{1.21}$$

Inversion leads to the martingale (1.20).

Remark b : There are some obvious, essential differences between the two proofs (i.e. two model-approaches) of (1.16). These differences carry over to more general examples, so that it pays to learn more about the special properties of both MODEL 1 and 2.

Remark c : From these easy examples we can see that the course of action, when one wants to study certain insurance models by the above methods, is as follows :

I. Formulate the insurance model as a PD process X_t say.

II. Determine the generator A of X_t and solve Af = 0 for f in the domain of A. It should be noted that we only need a suitable, particular solution !

III. Apply Dynkin's theorem to find that $f(X_t)$ is a martingale. This solves the problem of spotting the martingale.

IV. Use martingale theory (limit theorems, optional stopping, inequalities, ...) to make inference about the original model.

2. MODEL 1

2.1 The classical model

Theorem 5

Consider the classical model with general claim-size distribution function G and Laplace-Stieltjes transform $\phi(\nu) = \int_0^\infty e^{\nu y}\, dG(y)$ so that $c > \lambda\phi'(0)$. For all $\theta \geq 0$, there exists a unique root ν_θ of $-\theta - \nu c + \lambda(\phi(\nu)-1) = 0$ and the process

$$e^{-\theta t}\, e^{-\nu_\theta Z_t}$$

is a martingale.

Proof.

We recall (1.9), the generator of the process. For $f(Z_t,t)$ to be a martingale, f should belong to the domain of the generator and, by Proposition 1.a), satisfy

$$\frac{\partial}{\partial t}f(z,t) + c\frac{\partial}{\partial z}f(z,t) + \lambda(\int_0^\infty f(z-y,t)d\,G(y) - f(z,t)) = 0. \qquad (2.1)$$

We try a solution of the form $e^{-\theta t}e^{-\nu z}$. It follows that θ and ν have to satisfy

$$-\theta - c\nu + \lambda(\phi(\nu) - 1) = 0. \qquad (2.2)$$

Recall that $\phi(\nu)$ is twice differentiable. Using convexity of ϕ one can show that for all $\theta \geq 0$ there exists a unique positive ν_θ satisfying (2.2) provided that the net profit condition $c > \lambda\phi'(0)$ holds. Under this condition, it also follows that $e^{-\theta t}e^{-\nu_\theta z}$ belongs to the domain of A. In particular D3 holds.

So

$$e^{-\theta t}e^{-\nu_\theta Z_t} \text{ is a martingale,} \qquad (2.3)$$

where ν_θ is the unique positive root of (2.2). □

When $\theta = 0$, the root ν_0 is called the adjustment coefficient in insurance mathematics; see Beard et al. [4], p.312. Finally,

$$e^{-\nu_0 Z_t} \text{ is a martingale.} \qquad (2.4)$$

If again u is the initial capital, we can use Proposition 3 for the martingale (2.4) as in the previous paragraph.

We get

$$P(\tau < \infty) = \frac{e^{-\nu_0 u}}{E(e^{-\nu_0 Z_\tau} \mid \tau < \infty)}, \qquad (2.5)$$

a formula also to be found in Gerber [16], Chapter 9. From it, it immediately follows that

$$P(\tau < \infty) \leq e^{-\nu_0 u}, \qquad (2.6)$$

using the rather crude inequality $E(e^{-\nu_0 Z_\tau} \mid \tau < \infty) \geq 1$. In the general (non-exponential) case, $E(e^{-\nu_0 Z_\tau} \mid \tau < \infty)$ is not easily evaluated and indeed no closed formula as in (1.16) holds here. However, the estimate (2,6) can be refined as follows.

Theorem 6.

Consider the classical model with the net-profit condition $c > \lambda\phi'(0)$ and inital capital u, then

$$\frac{e^{-\nu_0 u}}{m_2(\nu_0)} \le P(\tau < \infty) \le \frac{e^{-\nu_0 u}}{m_1(\nu_0)}$$

where m_1 *and* m_2 *are defined below.*

Proof.

Let $Z_{\tau-} = b$ (where $Z_{\tau-} = \lim_{t\uparrow\tau} Z_t$). Then, conditional on $Z_{\tau-} = b$, the distribution of the last claim $- Z_\tau + b$ equals

$$\begin{cases} \dfrac{G(x) - G(b)}{1 - G(b)} & , x > b, \\ 0 & , x \le b. \end{cases}$$

Therefore

$$E(e^{-\nu_0 Z_\tau} \mid \tau < \infty,\ Z_{\tau-} = b) = \frac{e^{-\nu_0 b}\int_b^\infty e^{\nu_0 x}\, dG(x)}{1 - G(b)} . \tag{2.7}$$

Now define $m_1(\nu) = \inf_{b\ge 0} \left(\dfrac{e^{-\nu b}\int_b^\infty e^{\nu x} dG(x)}{(1 - G(b))} \right)$, and $m_2(\nu)$ the same expression replacing inf by sup. Then $m_1(\nu_0) \le E(e^{-\nu_0 Z_\tau} \mid \tau < \infty) \le m_2(\nu_0)$, so

$$\frac{e^{-\nu_0 u}}{m_2(\nu_0)} \le P(\tau < \infty) \le \frac{e^{-\nu_0 u}}{m_1(\nu_0)} . \tag{2.8}$$

□

As an example, take G Gamma(2,α) distributed with density $\alpha^2 y e^{-\alpha y}$, $y > 0$, then (2.8) becomes

$$\left(\frac{\alpha - \nu_0}{\alpha}\right)^2 e^{-\nu_0 u} \le P(\tau < \infty) \le \left(\frac{\alpha - \nu_0}{\alpha}\right) e^{-\nu_0 u} . \tag{2.9}$$

(Compare the estimate (2.9) with (3.6) later on).

If we want more information on the distributional properties of τ, we have to consider the martingale in (2.3). Again conditioning on $(\tau \le t)$ and $(\tau > t)$ and using Doob's stopping theorem (Proposition 3), we find

$$E(e^{-\theta\tau} e^{-\nu_\theta Z_\tau} \mid \tau < \infty) = \frac{e^{-\nu_\theta u}}{P(\tau < \infty)} .$$

This implies that

$$E(e^{-\theta\tau} e^{-\nu_\theta Z_\tau} \mid \tau < \infty) = E(e^{-\nu_0 Z_\tau} \mid \tau < \infty) \cdot e^{-(\nu_\theta - \nu_0)u} . \quad (2.10)$$

This formula may be of limited use because of the possible correlation between $e^{-\theta\tau}$ and $e^{-\nu_\theta Z_\tau}$. Again in the case of exponential claim sizes τ and Z_τ are independent; see Gerber [16], from which we can deduce the Laplace transform of τ, conditional on $(\tau < \infty)$. Indeed if G is Exponential(α) then (2.10) becomes

$$E(e^{-\theta\tau} \mid \tau < \infty) \frac{\alpha}{\alpha - \nu_\theta} = \frac{c\alpha}{\lambda} e^{-(\nu_\theta - \nu_0)u}$$

where $\nu_0 = \alpha - \frac{\lambda}{c}$. Therefore

$$E(e^{-\theta\tau} \mid \tau < \infty) = \frac{c(\alpha - \nu_\theta)}{\lambda} e^{-(\nu_\theta - \alpha + \lambda/c)u} . \quad (2.11)$$

This can help us in deriving the moments of τ. By differentiating this expression an appropriate number of times with respect to θ, we can evaluate the conditional moments of τ. For instance, see also Gerber [16], p. 138, $E(\tau \mid \tau < \infty) = \frac{1 + (\lambda/c)u}{c\alpha - \lambda}$. In more general situations, (2.10) will only yield suitable approximations. As an example, below we derive a "no-overshoot" approximation to $E(\tau \mid \tau < \infty)$, valid for large u.

Theorem 7.

Under the conditions of Theorem 6, it follows that

$$\lim_{u\to\infty} \frac{E(\tau \mid \tau < \infty)}{u} = \lambda \phi'(\nu_0) - c.$$

Proof.

Rewrite (2.10) more conveniently as :

$$\frac{E(e^{[c\nu_\theta - \lambda(\phi(\nu_\theta) - 1)]\tau} e^{-\nu_\theta Z_\tau} \mid \tau < \infty)}{E(e^{-\nu_0 Z_\tau} \mid \tau < \infty)} = e^{-(\nu_\theta - \nu_0)u} . \quad (2.12)$$

Differentiating (2.12) with respect to θ, and setting $\theta = 0$ we find

$$\frac{E((\lambda\phi'(\nu_0) - c)\tau e^{-\nu_0 Z_\tau} \mid \tau < \infty) + E(Z_\tau e^{-\nu_0 Z_\tau} \mid \tau < \infty)}{E(e^{-\nu_0 Z_\tau} \mid \tau < \infty)} = 0$$

so that as $u \to \infty$,

$$E(\tau \mid \tau < \infty) \sim u\{\lambda\phi'(\nu_0) - c\}, \tag{2.13}$$

hence $E(\tau \mid \tau < \infty)$ tends to be proportional to u. □

We conclude this section with the derivation of a bound for $\psi(u,t) = P(\tau \le t \mid Z_0 = u)$.

Theorem 8.

Under the conditions of Theorem 6, it follows that

$$\psi(u,t) \le \inf_{\theta \ge 0} \left\{ \frac{e^{-[c\nu_\theta - \lambda(\phi(\nu_\theta) - 1)]t - \nu_\theta u}}{m_1(\nu_\theta)} \right\}, \tag{2.14}$$

where m_1 is defined as in Theorem 6.

Proof.

Recall from Theorem 5 that $e^{-\theta t}e^{-\nu_\theta Z_t}$ is a martingale and therefore

$$e^{-\nu_\theta u} = E(e^{-\theta\tau}e^{-\nu_\theta Z_\tau} \mid \tau \le t)\, P(\tau \le t) + E(e^{-\theta t}e^{-\nu_\theta Z_t} \mid t < \tau)\, P(t < \tau)$$

so that

$$e^{-\nu_\theta u} \ge E(e^{-\theta\tau}e^{-\nu_\theta Z_\tau} \mid \tau \le t)\, \psi(u,t).$$

This can be written as

$$e^{-\nu_\theta u} \ge E(e^{[c\nu_\theta - \lambda(\phi(\nu_\theta) - 1)]\tau}\, e^{-\nu_\theta Z_\tau} \mid \tau \le t)\, \psi(u,t). \tag{2.15}$$

Therefore for all $\theta \ge 0$, since $e^{-\nu_\theta Z_\tau} \ge 1$,

$$e^{-\nu_\theta u} \ge e^{[c\nu_\theta - \lambda(\phi(\nu_\theta) - 1)]t}\, \psi(u,t).$$

So that finally $\psi(u,t) \le \inf_{\theta \ge 0} \{ e^{-[c\nu_\theta - \lambda(\phi(\nu_\theta) - 1)]t - \nu_\theta u} \}$.

This inequality can be further improved by the use of (2.15), indeed (2.15) yields $e^{-\nu_\theta u} \ge e^{[c\nu_\theta - \lambda(\phi(\nu_\theta) - 1)]t}\, E(e^{-\nu_\theta Z_\tau} \mid \tau \le t)$. $\psi(u,t)$. Now by the same argument used to obtain (2.8) we find $E(\exp(-\nu_\theta Z_\tau) \mid \tau \le t) \ge m_1(\nu_\theta)$ so that (2.14) follows.

□

2.2 A generalization : periodicity

A first generalization of the above model will allow for periodicity in the claim arrival intensity λ. A general time dependence of the basic quantities c and G(y) can also be incorporated. However, it is easy to see that by a suitable time change, there is no loss in generality if we set one of c and λ to be constant in time. So we replace λ by $\lambda(t)$ where $\lambda(t)$ is defined and is positive for all $t \geq 0$. We should not have $\lambda(t) \to \infty$ as $t \to t_1 < \infty$ as this would violate assumption (1.1). Also G(y) is replaced by G(y,t) where for every t, G(.,t) is a distribution function with support $[0,\infty[$. For this paragraph, we shall adopt the following basic assumption :

(BA) $\forall t \geq 0$, $\phi(\nu,t) = \int_0^\infty e^{\nu y}\, dG(y,t)$ exists for ν in a suitable domain D containing 0 in its interior D°. On D°, we also assume that $\phi(\nu,t)$ is twice differentiable with respect to ν. Furthermore, $\forall \nu \in D$, $\phi(\nu,.)$ should be Riemann integrable.

The formulation of the above process as a PD process via MODEL 1 is as before and hence the generator of the process acting on a function f(z,t) is

$$Af(z,t) = \frac{\partial}{\partial t} f(z,t) + c\frac{\partial}{\partial z} f(z,t)+\lambda(t) \left(\int_0^\infty f(z-y,t)\, dG(y,t)-f(z,t)\right).$$

Hence the equation to be solved to obtain a martingale is $Af = 0$. (2.16)

If we assume a solution of the form $f_1(t)e^{-\nu z}$, then (2.16) gives

$$f_1'(t) - c\nu f_1(t) + \lambda(t)\, (f_1(t)\phi(\nu,t) - f_1(t)) = 0.$$

Solving this equation yields the following solution

$$f_1(t) = \exp\{c\nu t - \int_0^t \lambda(s)[\phi(\nu,s) - 1]ds\}.$$

Therefore

$$\exp\{c\nu t - \int_0^t \lambda(s)[\phi(\nu,s) - 1]\, ds\}\, e^{-\nu Z_t} \tag{2.17}$$

is a martingale.

It is clear that a time dependent claim arrival rate $\lambda(t)$ should allow both for clustering and for seasonality. Take for example car- and fire-insurance where these components are obviously present. A further generalization needed should take randomness in λ, through a doubly stochastic arrival process say, into account. The PD-theory developed so far forms the ideal background for the analysis of such processes. As an example of the use of (2.17), we shall concentrate on seasonality, see also Beard et al. [4], p.110-114. More complicated models are discussed in Dassios [11], we shall return to these in future publications.

Theorem 9.

Assume that the claim size distribution G is independent of t, and that the claim arrival process is a non-homogeneous Poisson process with intensity parameter $\lambda(t) = \lambda_o + \lambda_1 \sin [\frac{2\pi}{T}(t+t_o)]$ *where* λ_o *is to be interpreted as the average claim arrival rate,* λ_1 *is the amplitude of the periodic component, T the period and* t_o *the phase. If we denote* $\psi_T(u) = P(\tau < \infty \mid Z_o = u)$ *in this model, it then follows that*

$$\psi_T(u) = \frac{e^{-\nu_0 u} \exp\left[\frac{c\nu_0}{\lambda_0} \frac{\lambda_1 T}{2\pi} \cos \frac{2\pi t_0}{T}\right]}{E[\exp(\frac{c\nu_0}{\lambda_0} \frac{\lambda_1 T}{2\pi} \cos \frac{2\pi}{T}(\tau + t_0))\, e^{-\nu_0 Z_\tau} \mid \tau < \infty]}, \qquad (2.18)$$

where ν_0 *satisfies* $c\nu_0 = \lambda_0 [\phi(\nu_0) - 1]$.

Proof.

With the above notation, (2.17) takes on the form

$$\exp\{c\nu t - [\phi(\nu) - 1] \int_0^t [\lambda_0 + \lambda_1 \sin\frac{2\pi}{T}(s + t_0)]ds\} e^{-\nu Z_t}$$

$$= \exp\{c\nu t - [\phi(\nu) - 1](\lambda_0 t - \frac{\lambda_1 T}{2\pi}\cos\frac{2\pi}{T}(t+t_0) + \frac{\lambda_1}{2\pi} \cos \frac{2\pi t_0}{T})\} e^{-\nu Z_t}.$$

We choose ν_0 so that $c\nu_0 = \lambda_0[\phi(\nu_0) - 1]$, i.e. ν_0 is the adjustment coefficient for the average rate λ_0. Hence

$$\exp\{\frac{c\nu_0}{\lambda_0} \frac{\lambda_1 T}{2\pi} (\cos \frac{2\pi}{T}(t + t_0) - \cos \frac{2\pi}{T} t_0)\}\, e^{-\nu_0 Z_t} \qquad (2.19)$$

is a martingale. Using the same argument as before, i.e. conditioning on $(\tau \le t)$, $(\tau > t)$ and letting $t \to \infty$, we obtain (2.18).

□

In general, it is difficult to calculate the denominator in (2.18), even in the exponential case where τ and Z_τ are independent. However, inequalities can be obtained readily. For instance with probability one $|\cos(2\pi(\tau + t_0)/T)| < 1$ so that

$$E(e^{-\nu_0 Z_\tau} | \tau < \infty)\, \psi_T(u) < e^{-\nu_0 u} \exp\{\frac{c\nu_0\lambda_1 T}{\lambda_0 2\pi}\left(\cos\frac{2\pi t_0}{T} + 1\right)\}.$$

A lower bound is obtained upon replacing + 1 by - 1. As special cases we obtain

$$\text{for } t_0 = 0,\ \psi_T(u) > e^{-\nu_0 u} / \{E(e^{-\nu_0 Z_\tau} | \tau < \infty)\},$$

$$\text{and if } t_0 = \frac{T}{2},\ \psi_T(u) < e^{-\nu_0 u} / \{E(e^{-\nu_0 Z_\tau} | \tau < \infty)\}.$$

In comparison to (1.15), we interpret these results as follows. If a company starts operating before a dense period of claims it is more probable to get ruined than if no periodic component were involved, while if it starts operating before a sparse period of claims, it is less likely to be ruined. An interesting simulation illustration of this is to be found in Beard et al. [4], p. 246.

2.3 A general renewal model

In the existing literature, various models beyond the classical one have been investigated. In this section we shall incorporate some of these in the PD set-up.

(i) The Poisson-arrival condition will be relaxed to a general renewal arrival process; see for instance Andersen [1], Thorin [26] and references therein and Takács [25]. The claim interarrival distribution H is assumed to have a density h.

(ii) The second assumption to be relaxed is independence between the claim arrival process and the claim sizes. We introduce the following structure : claim sizes are independent but not identically distributed. Given that the time elapsed since the (n-1)st claim is v, Y_n(the nth claim) has a distribution function G(y,v) (n = 1,2,3,...). Its Laplace transform $\phi(\nu;v)$ again satisfies the usual differentiability condition.

We start by formulating the model as a PD process. The problem is that, because of (ii), the process is non-Markovian. However, it can be made Markovian by introducing a supplementary variable; this idea can for instance be found in Cox [9]. The countable component η_t in MODEL 1 is constant and the uncountable one consists of Z_t (surplus) and V_t (time elapsed since the last claim). Time t is included too. The process evolves deterministically as $z_t = z_0 + ct$, $v_t = v_0 + t$ till the time of the first claim T_1. Then $V_{T_1} = 0$ and $Z_{T_1} = Z_{T_1-} - Y_1$ where Y_1 is the size of the first claim, and so on. Hence $M_1 = \mathbb{R} \times \mathbb{R}_+^2$. The generator of the process, acting on absolutely continuous functions $f(z,v,t)$ so that for all t, z, v > 0, $E\,|f(z-Y_v,0,t) - f(z,v,t)| < \infty$, where Y_v has distribution function $G(y;v)$, equals :

$$\frac{\partial}{\partial t} f(z,v,t) + c\frac{\partial}{\partial z} f(z,v,t) + \frac{\partial}{\partial v} f(z,v,t)$$
$$+ \lambda(v)\left(\int_0^\infty f(z-y,0,t)\, dG(y;v) - f(z,v,t)\right) \quad (2.20)$$

where $\lambda(v)$ is the hazard rate of $H(v)$, i.e. $\lambda(v) = h(v)/(1 - H(v))$.

Theorem 10.

With the above notations and under the net-profit condition

$$c\int_0^\infty xh(x)\, dx > \int_0^\infty \phi'(0;x)\, h(x)\, dx$$

then

$$e^{-\theta t} e^{-\nu_\theta Z_t} \frac{e^{(\theta+c\nu_\theta)V_t}}{1 - H(V_t)} \int_{V_t}^\infty e^{-(\theta+c\nu_\theta)x} \phi(\nu_\theta;x)\, h(x)\, dx$$

is a martingale, where ν_θ is the solution of

$$\int_0^\infty e^{-(\theta+c\nu)x} \phi(\nu;x)\, h(x)\, dx = 1.$$

Proof.

Again to find a martingale we must equate (2.20) to 0 and solve for f. Assume now a solution of the form $e^{-\theta t}e^{-\nu z}q(v)$. Substituting this in (2.24) we get

$$-(\theta + c\nu)q(v) + q'(v) + \lambda(v)[\phi(\nu;v)q(0) - q(v)] = 0.$$

Set $k(v) = (1 - H(v))\, q(v)$, then

$$-(\theta + c\nu)k(v) + k'(v) + h(v)\phi(\nu;v)k(0) = 0$$

with as general solution

$$k(v) = C\, e^{(\theta+c\nu)v} + k(0)e^{(\theta+c\nu)v}\int_v^\infty e^{-(\theta+c\nu)x}\phi(\nu;x)h(x)dx.$$

Expressing this in terms of q and putting $v = 0$, we find

$$C = q(0)[1 - \int_0^\infty e^{-(c\nu+\theta)x}\, \phi(\nu;x)h(x)dx]$$

where we assume the integrals to converge. In order that (1.4) holds for f in the domain of the generator (2.20), we need that $C = 0$, therefore

$$\int_0^\infty e^{-(\theta+c\nu)x}\phi(\nu;x)h(x)dx = 1. \tag{2.21}$$

Since $c\int_0^\infty xh(x)dx > \int_0^\infty \phi'(0;x)h(x)dx$ (net-profit condition) one shows by using a convexity argument that for all $\theta \geq 0$ there exists a unique solution $\nu_\theta > 0$ for (2.21). Hence we conclude that

$$e^{-\theta t}\, e^{-\nu_\theta Z_t}\, \frac{e^{(\theta + c\nu_\theta)V_t}}{1 - H(V_t)} \int_{V_t}^\infty e^{-(\theta + c\nu_\theta)x}\, \phi(\nu_\theta;x)h(x)dx \tag{2.22}$$

is a martingale. In particular

$$e^{-\nu_0(Z_t - cV_t)} \int_{V_t}^\infty e^{-c\nu_0 x}\, \phi(\nu_0;x)h(x)dx/(1 - H(V_t)) \tag{2.23}$$

is a martingale.

□

Using the above result we can derive the relevant ruin estimates as before. For instance

$$P(\tau < \infty) = \frac{e^{-\nu_0 u}}{E(e^{-\nu_0 Z_\tau} | \tau < \infty)} \cdot \frac{e^{c\nu_0 v_0} \int_{v_0}^\infty e^{-c\nu_0 x}\, \phi(\nu_0;x)h(x)dx}{1 - H(v_0)}$$

and

$$E(e^{-\theta\tau} e^{-\nu_\theta Z_\tau} | \tau < \infty) = \frac{e^{-\nu_\theta u}}{P(\tau < \infty)} \cdot \frac{e^{(\theta+c\nu_\theta)v_0} \int_{v_0}^{\infty} e^{-(\theta+c\nu_\theta)x} \phi(\nu_\theta;x)h(x)dx}{1 - H(v_0)} .$$

The methods of Section 2.1 are applicable for obtaining inequalities for ruin probabilities or approximations for moments of τ. For completeness we give the analogue of (2.14), which is

$$\psi(u,t) \le \inf_{\theta\ge 0} \{\frac{\exp(\theta t-\nu_\theta u+(\theta+c\nu_\theta)v_0 \int_{v_0}^{\infty} e^{-(\theta+c\nu_\theta)x} \phi(\nu_\theta;x)h(x)dx)}{m_1(\nu_\theta)\ (1 - H(v_0))}\}.$$

2.4 Service payments

In this section we assume the model of Section 2.3 with the alteration that if the company sees that a time T passes since the last claim and no new claim has occured during that period, it decides to make service payments (or bonus payments). These payments count as a claim, so that V_t (the time elapsed since the last claim) is set to be 0. This model should be compared with the replacement model in Karlin and Taylor [21], p.203. Our model is as follows. Claims (proper) arrive as a renewal process with claim interarrival time intervals having distribution function H(v), with density h(v) ($v \ge 0$). The proper claim sizes are as before with distribution function $G_1(y;v)$ with the usual condition on the Laplace transform $\phi_1(\nu;v)$. The income rate is a constant c and if no claim has been made during a time interval of length T, service payments independent of proper claims are paid with $G_2(y)$ being their distribution function. Again $\phi_2(\nu)$ should exist on some interval $[0, \alpha_2)$ and be twice differentiable. After a service payment is made, the claim arrival process starts afresh. The model is not a special case of the one considered in the previous section as the distribution function of the actual interarrival times has a point mass at T. A typical example where the above model occurs is the case of a company (in the wide sense) possessing a certain source of income with a constant income rate. However, this source gets 'damaged' at random times and the company is forced to pay for repairs (proper claims). If no damages occur during a specified time interval, then the company wants or is required by law to carry out service payments to the source.

The formulation as a PD process goes as in Section 2.3. In the present case, (Z_t, V_t, t) takes values in $\mathbb{R} \times [0,T) \times \mathbb{R}_+$. Furthermore Γ, the set through which deterministic curves exit is not empty anymore ! We have that $\Gamma = \mathbb{R} \times \{T\} \times \mathbb{R}_+$. So all functions f in the domain of the generator of the process have to satisfy requirement (1.3), that is

$$f(z,T,t) = \int_0^\infty f(z-y,0,t)d\, G_2(y). \tag{2.24}$$

The generator acting on suitable functions f is, for $v < T$:

$$Af(z,v,t) = \frac{\partial}{\partial t} f(z,v,t) + c\frac{\partial}{\partial z} f(z,v,t) + \frac{\partial}{\partial v} f(z,v,t) + \frac{h(v)}{1-H(v)} \left(\int_0^\infty f(z-y,0,t)dG_1(y;v) - f(z,v,t)\right). \tag{2.25}$$

To obtain a martingale, we have to solve $Af = 0$ subject to (2.24). As in the previous section, we try a solution of the form $e^{-\theta t}$ $e^{-\nu z}q(v)$ and find as before, for $v < T$:

$$q(v) = \{[1 - \int_0^\infty e^{-(\theta+c\nu)x}\phi_1(\nu;x)h(x)dx]\, e^{(\theta+c\nu)v} + e^{(\theta+c\nu)v}\int_v^\infty \phi_1(\nu;x)\, e^{-(\theta+c\nu)x}h(x)dx\} \,/\, (1 - H(v)). \tag{2.26}$$

We may assume that $q(0) = 1$. Since $q(v)$ is continuous and defined on $[0,T]$, there is no need to require boundedness. Hence we should concentrate on (1.3) (that is (2.24)), (1.4) being trivial. From (2.24) it follows that

$$e^{-\theta t}\, e^{-\nu z}q(T) = e^{-\theta t}e^{-\nu z}\int_0^\infty e^{\nu y}dG_2(y)q(0),$$

hence $q(T) = \phi_2(\nu)$. Therefore (see (2.26)),

$$q(v) = \{1 - \int_0^v e^{-(\theta+c\nu)x}\phi_1(\nu;x)h(x)dx\}\, e^{(\theta+c\nu)v}/(1 - H(v)),$$

from which it follows that

$$(1 - \int_0^T e^{-(\theta+c\nu)x}\phi_1(\nu,x)h(x)dx)e^{(\theta+c\nu)T} = \phi_2(\nu)\ (1 - H(T)). \qquad (2.27)$$

By convexity, we can again prove that, given the net profit condition

$$c\int_0^T xh(x)dx + c(1 - H(T)) > \int_0^T \phi_1'(0;x)h(x)dx + \phi_2'(0)(1 - H(T)),$$

that for all $\theta \geq 0$ there exists a unique positive ν_θ satisfying (2.27). As a conclusion, we find that

$$\frac{e^{-\theta t}e^{-\nu_\theta Z_t}}{1 - H(V_t)}\ \{1 - \int_0^{V_t} e^{-(\theta+c\nu_\theta)x}\phi_1(\nu_\theta;x)h(x)dx\}e^{(\theta+c\nu_\theta)V_t}$$

is a martingale and in particular that

$$\frac{e^{-\nu_0 Z_t}}{1 - H(V_t)}\ \{1 - \int_0^{V_t} e^{-c\nu_0 x}\phi_1(\nu_0;x)h(x)dx\}e^{c\nu_0 V_t}$$

is a martingale. Many relevant formulae can be derived from these results. For instance :

$$P\{\tau < \infty\} = \frac{e^{-\nu_0 u}}{E(e^{-\nu_0 Z_\tau}\mid \tau<\infty)}\ \{\frac{1 - \int_0^{v_0} e^{-c\nu_0 x}\phi_1(\nu_0;x)h(x)dx}{1 - H(v_0)}\}\ e^{c\nu_0 v_0},$$

$$E(e^{-\theta\tau}e^{-\nu_\theta Z_\tau}\mid \tau<\infty) = \frac{e^{-\nu_\theta u}}{P(\tau<\infty)}\ \{\frac{1 - \int_0^{v_0} e^{-(\theta+c\nu_0)x}\phi_1(\nu_\theta;x)h(x)dx}{1 - H(v_0)}\}\ .$$

$$e^{(\theta+c\nu_\theta)v_0}$$

and

$$\psi(u,t) \leq \inf_{\theta\geq 0}\ \{\frac{e^{\theta t-\nu_\theta u}}{m_1(\nu_\theta)}\ .\ (\frac{1 - \int_0^{v_0} e^{-(\theta+c\nu_\theta)x}\phi_1(\nu_\theta;x)h(x)dx}{1 - H(v_0)})\}\ .$$

$$e^{(\theta+c\nu_\theta)v_0}.$$

Although these formulae look rather formidable, they are surprisingly explicit and straightforward to approximate in special cases. The main aim in writing them down explicitly is to indicate the usefulness of PD theory in obtaining explicit (though perhaps complicated) analytic expressions for ruin probabilities in general insurance models. The approach of this section can be generalized to cover claim size distributions with point masses. The latter typically occurs whenever certain types of reinsurance policies are applied. For instance, assume that the company only covers claims that are smaller than a fixed amount M say (the 'retention limit'). If $Y_i \geq M$, then $Y_i - M$ is reinsured with another firm.

3. MODEL 2

3.1 The infinite horizon ruin problem in the classical model revisited

In this section we show how MODEL 2 leads to exact results in insurance models. As a first example we revisit the classical model with claim size distribution G(y) on [0,∞) having density g(y). Hence the generator of the associated PD process becomes

$$\frac{\partial}{\partial t}f(z,t) + c\frac{\partial}{\partial z}f(z,t) + \lambda\left(\int_0^z f(z-y,t)g(y)dy - f(z,t)\right) \quad (3.1)$$

with $z > 0$. If we restrict attention to the infinite horizon case, we can supress dependence on t. So there is no $\frac{\partial}{\partial t}$ - term and therefore for $z > 0$,

$$Af(z) = cf'(z) + \lambda\left(\int_0^z f(z-y)g(y)dy-f(y)\right). \quad (3.2)$$

Equating (3.2) to 0 and solving for f we obtain a solution f_0. Therefore $f_o(Z_t)\, I(Z_t > 0)$, where I is the indicator function, is a martingale. Recall that f in the domain of the generator (3.2) has to satisfy $f(0+) \neq f(0) = 0$ (see Illustration 2 in Section 1). Hence if u is the initial capital,

$$f_0(u) = E[f_0(Z_\tau) \mid \tau\leq t]P(\tau\leq t) + E[f_0(Z_t) \mid t<\tau]P(t<\tau). \quad (3.3)$$

Since $Z_\tau = 0$, (3.3) implies that $f_0(u) = E[f_0(Z_t);\, t < \tau]$. Letting $t \to \infty$ and assuming $\lim_{z\to\infty} f_0(z) = \ell$ it follows that $f_0(u) = \ell P(\tau = \infty)$. Therefore

$$P(\tau < \infty) = 1 - \frac{f_0(u)}{\ell} . \quad (3.4)$$

The solution f_0 can be found easily by taking Laplace transforms in (3.2), so that

$$\hat{f}(s) = \frac{f(0+)}{s + \frac{\lambda}{c}(\hat{g}(s)-1)} . \qquad (3.5)$$

(In this paragraph, we use Laplace tranforms rather than generating functions). This is the wellknown Khintchine-Pollaczek formula from queueing theory; see Feller [15], p. 445. Inversion of (3.5) in general yields an infinite series expansion. Letting $u \downarrow 0$ in (3.4) and using $\lim_{s\to 0} s\hat{f}(s) = \ell$ in (3.5) we obtain $P(\tau < \infty) = \lambda\kappa/c$ where κ is the mean of the claim size distribution G (since $\ell = (1-\lambda\kappa/c)$). Of course $\lambda\kappa < c$ by the net profit condition. And so a general formula for the probability of ruin becomes

$$P(\tau < \infty) = 1 - \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{us}}{s + \frac{\lambda}{c}(\hat{g}(s)-1)} ds \Big/ (1 - \frac{\lambda\kappa}{c}) .$$

For example, if $g(y) = \alpha^2 y e^{-\alpha y}$ (Gamma(2,α)) then

$$P(\tau < \infty) = \Big(\frac{(\alpha+\nu_1)^2}{-\nu_1(\nu_1-\nu_2)} e^{\nu_1 u} - \frac{(\alpha+\nu_2)^2}{-\nu_2(\nu_1-\nu_2)} e^{\nu_2 u}\Big)/(1 - \frac{c\alpha}{2\lambda}) \qquad (3.6)$$

where ν_1, ν_2 are the (negative) roots of the equation $s^2 + (2\alpha - \lambda/c)s + \alpha^2 - 2\alpha\lambda/c = 0$. One should compare (3.6) with (2.9).

3.2 Income rate as a function of surplus lending and absolute ruin

To consider a more realistic model, we allow for a stochastic premium income. The company is allowed or forced to choose a different premium rate for different values of the surplus Z_t. Assume that $c = c(z)$ is continuously differentiable. (This condition can be relaxed) For the sake of simplicity we restrict attention to an exponential claim size distribution. Using MODEL 2 (with $0 = f(0) \neq f(0+)$) we find as the generator of the process

$$c(z)f'(z) + \lambda \Big(\int_0^z f(z-y)\alpha e^{-\alpha y} dy - f(z)\Big), \qquad z > 0.$$

Our aim is to evaluate the probability of ruin in the infinite horizon case, $\psi(u)$. For that reason we want to find, as before, a martingale of the form

$$\begin{cases} f(Z_t) & , Z_t > 0, \\ 0 & , Z_t = 0. \end{cases}$$

Similar results to ours can be found in Harrison and Resnick [19], Theorem 4 and equation (20) and Asmussen and Petersen [3] and references therein.

Theorem 11

Consider the classical model with exponential claim sizes, and with c =c(z) continuously differentiable on]0,∞[and

$$\forall x > 0, \quad C(x) = \int_0^x \frac{1}{c(y)}\, dy < \infty.$$

If $P(\tau = \infty) > 0$, then it follows that

$$\psi(u) = \frac{\displaystyle\int_u^\infty \frac{e^{-\alpha x + \lambda C(x)}}{c(x)}\, dx}{\displaystyle\int_0^\infty \frac{e^{-\alpha x + \lambda C(x)}}{c(x)}\, dx + \frac{1}{\lambda}} \tag{3.7}$$

Proof

In this case, the differential field in the definition of the PD process equals $c(z)\frac{d}{dz}$, therefore we have to find f as a suitable solution of the equation

$$c(z)f'(z) + \lambda \left(\int_0^z f(z-y)\alpha e^{-\alpha y} dy - f(z)\right) = 0, \quad z > 0. \tag{3.8}$$

Differentiating the latter equation yields for $z > 0$:

$$c(z)f''(z) + (c'(z) - \alpha c(z) - \lambda) f'(z) = 0, \tag{3.9}$$

which has as a general solution, for $z > 0$,

$$f'(z) = K\left(\exp\left(-\alpha z + \int_0^z \lambda c^{-1}(x)dx\right)\right) / c(z).$$

If we assume that $K = 1$, then

$$f(z) = \int_0^z \left(\exp\left(-\alpha x + \int_0^x \lambda c^{-1}(v)dv\right) / c(x)\right)dx + b$$

where b is a constant to be determined. Letting $z \downarrow 0$ in (3.8) we find $c(0+)f'(0+) - \lambda f(0+) = 0$, i.e. $b = \lambda^{-1}$ since $f'(0+) = \frac{1}{c(0+)}$ and $f(0+) = b$. Therefore

$$\begin{cases} \int_0^{Z_t} (\exp(-\alpha x + \lambda C(x) / c(x))dx + \frac{1}{\lambda} & , \quad Z_t > 0 \\ 0 & , \quad Z_t = 0 \end{cases}$$

is a martingale. Hence

$$\int_0^u (\exp(-\alpha x + \int_0^x \lambda/c(v)dv) / c(x))dx + \frac{1}{\lambda} =$$

$$E(\int_0^{Z_t} (\exp(-\alpha x + \int_0^x \lambda/c(v)dv) / c(x))dx + \frac{1}{\lambda} \; ; \; t < \tau).$$

Letting $t \to \infty$, the result folows.

□

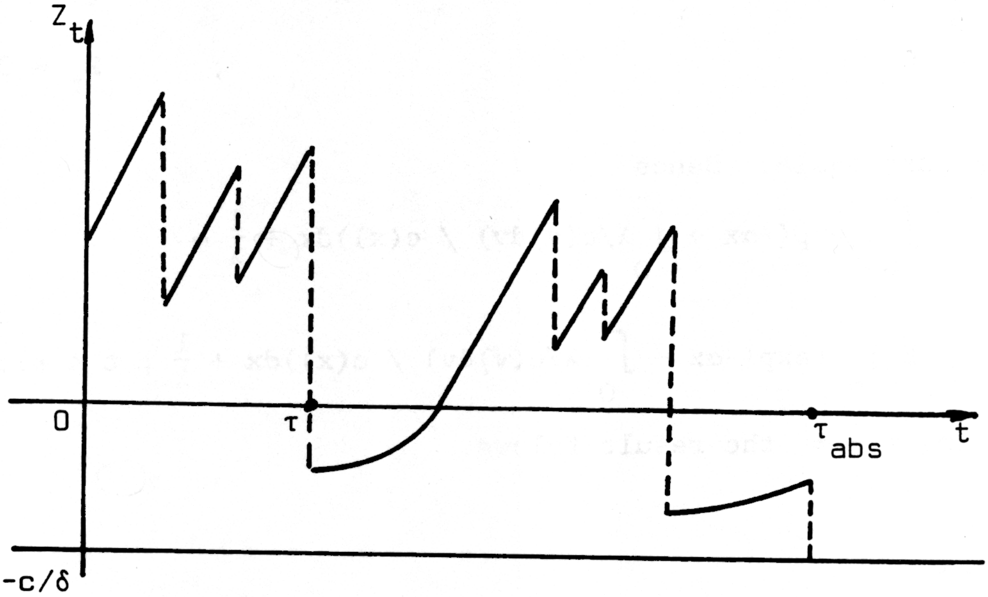
Remark :

As an example take $c(z) = c + \delta z$, that is interest is being paid to the company for its surplus, then (3.7) becomes

$$\psi(u) = [\int_u^\infty e^{-\alpha x}(1+\frac{\delta x}{c})^{(\lambda/\delta)-1}dx]/[\int_0^\infty e^{-\alpha x}(1+\frac{\delta x}{c})^{(\lambda/\delta)-1}dx + \frac{c}{\lambda}], \quad (3.10)$$

a result also to be found in Segerdahl [24].

We shall now consider a more complicated example. Suppose that the company does not cease to operate when its surplus becomes negative, but borrows money instead, paying interest with a constant force δ say; this means that the interest i equals $100(e^\delta - 1)\%$. When the company's surplus is positive, no interest is paid though one could incorporate interest earnings on the company's net surplus as in [3]. Suppose that c is the (constant) premium rate. Then the income is $c(z) = c + \delta z I(z < 0)$ where I stands for the indicator function. The related differential field becomes $(c + \delta z I(z < 0)) \frac{d}{dz}$. We note that if $Z_t \le -c/\delta$, the company will not be able to pay the interest on its debts. Hence we call the event $\{\exists t > 0 : Z_t \le -c/\delta\}$ absolute ruin and denote by $\tau_{abs} = \inf\{t > 0 : Z_t \le -c/\delta\}$ the time of absolute ruin. The probability of absolute ruin in the infinite horizon case is $\psi_{abs}(u) = P(\tau_{abs} < \infty)$. A typical realisation of the process Z(t) is given in the figure below. The equation of the curves can be found by solving $\frac{dz}{dt} = c + \delta z I(z < 0)$.



It follows that, if T_i is the i-th claim time, for $T_i \le t < T_{i+1}$:

$$\begin{cases} Z_t = Z_{T_i} + c\,(t - T_i) & \text{if } Z_{T_i} \ge 0; \\ Z_t = \begin{cases} (\frac{c}{\delta} + Z_{T_i})\, e^{\delta(t-T_i)} - \frac{c}{\delta}, & t < t^* \\ c\,(t - t^*) & , t \ge t^* \end{cases} & \text{if } -\frac{c}{\delta} < Z_{T_i} < 0, \end{cases}$$

where $t^* = T_i + \frac{1}{\delta} \log \left[\frac{c/\delta}{c/\delta + Z_{T_i}}\right]$.

Theorem 12

With the above notation, in the exponential, classical model :

$$\psi_{abs}(u) = e^{-(\alpha-\lambda/c)u} \left(1 + \frac{c\alpha - \lambda}{c} \int_{-c/\delta}^{0} e^{-\alpha x} \left(1 + \frac{\delta x}{c}\right)^{\frac{\lambda}{\delta} - 1} dx\right)^{-1}. \tag{3.11}$$

Proof

The natural approach to calculate ψ_{abs} is to set $Z'_t = Z_t + c/\delta$ and consider the process Z'_t which has the properties described at the beginning of this section with

$$c(z') = \begin{cases} c & , z' \geq c/\delta, \\ \delta z' & , z' < c/\delta. \end{cases}$$

Hence absolute ruin for Z_t is 'classical' ruin for Z'_t. However we cannot apply (3.7) since $\int_0^x \lambda/c(v)dv$ is divergent (this means that the wrong martingale was chosen). We therefore define $Z'_{\epsilon t} = Z_t + c/\delta - \epsilon$ for some $\epsilon > 0$ and set $\psi_\epsilon(u) = P\ (\exists t > 0 : Z'_{\epsilon t} < 0)$ where now

$$c(z') = \begin{cases} c & , z' \geq c/\delta - \epsilon, \\ \delta(z' + \epsilon) & , z' < c/\delta - \epsilon. \end{cases}$$

Defining the events $K_\epsilon = \{\exists t > 0 : Z_t < -c/\delta + \epsilon\}$ and $K = \{\exists t > 0 : Z_t < -c/\delta\}$, it is obvious that $K \subset \bigcap_{\epsilon > 0} K_\epsilon$. Because the claim size distribution is exponential (i.e. unbounded support) and because of the net profit condition ($\lambda < \alpha c$) for $Z_t > 0$, it follows that $P(\bigcap_{\epsilon > 0} (K_\epsilon \backslash K)) = 0$. Consequently $\psi_{abs}(u) = \lim_{\epsilon \downarrow 0} \psi_\epsilon(u)$. Now for the process $Z'_{\epsilon t}$ (given $\epsilon > 0$) we can apply (3.9), because

$$\int_0^x \lambda/c(v)dv = \begin{cases} \frac{\lambda}{\delta} \log(1 + \frac{x}{\epsilon}) & , x < \frac{c}{\delta} - \epsilon, \\ \frac{\lambda}{c} (x - \frac{c}{\delta} + \epsilon) + \frac{\lambda}{\delta} \log \frac{c}{\epsilon\delta} & , x \geq \frac{c}{\delta} - \epsilon. \end{cases}$$

Therefore for $u > 0$, $\epsilon > 0$

$$\psi_\epsilon(u) = e^{(\frac{\lambda}{c} - \alpha)u} . \{1 + \frac{c\alpha - \lambda}{c} e^{-\alpha\epsilon} \int_{-\frac{c}{\delta}}^{\epsilon} e^{-\alpha x} (\epsilon\delta + \frac{x\delta}{c} + 1)^{\frac{\lambda}{\delta} - 1} dx$$

$$+ \frac{1}{\lambda} \frac{c\alpha - \lambda}{c} (\frac{\epsilon\delta}{c})^{\lambda/\delta} e^{-\alpha c/\delta} e^{-\alpha\epsilon}\}^{-1}.$$

Letting $\epsilon \downarrow 0$, (3.11) follows.

□

For $u \leq 0$, we can similarly establish (of course $u > -\frac{c}{\delta}$)

$$\psi_{abs}(u) = \frac{1 + \frac{c\alpha - \lambda}{c} \int_u^0 e^{-\alpha x}(1 + \frac{\delta x}{c})^{\frac{\lambda}{\delta} - 1} dx}{1 + \frac{c\alpha - \lambda}{c} \int_{-c/\delta}^0 e^{-\alpha x}(1 + \frac{\delta x}{c})^{\frac{\lambda}{\delta} - 1} dx} . \qquad (3.12)$$

Remarks :

Remark a : The integral involved in (3.11) and (3.12) is the same as in (3.10) and can be expressed in terms of incomplete gamma functions, though for $\frac{\lambda}{\delta}$ small its calculation poses some problems. Yet for $\frac{\lambda}{\delta}$ large (as is usually the case in practice) it can be approximated as follows. Observe that

$$\int_{-c/\delta}^{z} e^{-\alpha x}(1 + \frac{\delta x}{c})^{\frac{\lambda}{\delta} - 1} dx =$$

$$\frac{c}{\delta} e^{\alpha c/\delta}(\alpha c/\delta)^{-\lambda/\delta}\Gamma(\lambda/\delta) \int_0^{1+\delta z/c} \frac{(\alpha c/\delta)^{\lambda/\delta}}{\Gamma(\lambda/\delta)} y^{\lambda/\delta - 1} e^{-(\alpha c/\delta)y} dy$$

where $\Gamma(.)$ is the gamma function. The integrand is now the gamma $(\lambda/\delta, \alpha c/\delta)$ density which for λ/δ large can be approximated by a normal distribution via the Central Limit Theorem. Therefore for λ/δ large,

$$\int_{-c/\delta}^{z} e^{-\alpha x}(1 + \delta x/c)^{\lambda/\delta - 1} dx$$

$$\simeq \frac{c}{\delta} e^{\alpha c/\delta}(\alpha c/\delta)^{-\lambda/\delta}\Gamma(\lambda/\delta)\ \Phi((c\alpha - \lambda + \alpha\delta z) / \sqrt{\lambda\delta})$$

where Φ is the standard normal distribution function.

Remark b : Note that formula (3.11) has the following structure : $\psi_{abs}(u) = k\psi(u)$ where $0 < k < 1$, a constant, and $\psi(u)$ is the probability of ruin in the classical case.

3.3 An optimization example

Suppose a company is willing to pay out dividends to its shareholders from time to time. Its objective is to pay out "as much as possible" while keeping ruin prospects as "unlikely" as possible. The following model is proposed. The company pays out dividends continuously with a varying rate which is its own choice at any time. The amount paid out is continuously discounted with discount factor β (which can be interpreted as an inflation rate for example). The company pays out dividends till (if ever) it is ruined, either in the classical sense or by absolute ruin. If we assume a dividend rate $r(Z_t, t)$ then the expected amount paid out during the company's lifespan $(0,\tau]$ is :

$$E(\int_0^{\tau} r(Z_s, s)e^{-\beta s}ds). \tag{3.13}$$

Among the various possible strategies governing the payment of dividends, the so-called barrier strategies are important, see Bühlmann [6] and Gerber [16]. They are as follows. The company fixes a level W, then if $Z_t < W$ the company does not pay out at all and if $Z_t = W$, the company pays out its whole income till a claim occurs. As far as the classical model is concerned, barrier strategies are optimal and there exists $W_0 > 0$ associated with a particular barrier strategy with barrier W_0 which maximizes (3.13). This provided $u \le W_0$, where as before u denotes the initial capital. Furthermore, if the claim sizes are exponential, even if $u > W_0$, then the optimal strategy is to pay out $u - W_0$ immediately and then follow the barrier strategy associated with W_0. For further details on this, see Bühlmann [6], p. 168-171 and the references given therein.

Theorem 13

Consider the absolute ruin model of Theorem 12 in which the above dividend structure (with discount factor β) is incorporated. The optimal barrier strategy then correponds to

$$W_0 = (\nu_1 - \nu_2)^{-1} \log \left(\frac{(L-\nu_1)\,\nu_2^2}{(L-\nu_2)\,\nu_1^2}\right)$$

where ν_1, ν_2 are the roots of $s^2 + (\alpha - (\lambda+\beta)/c)s - \alpha\beta/c = 0$ and L is defind below ((3.19)).

Proof

The rule with which the company pays out dividends is $cI(Z_t = W)$, and the income rate is $(c + \delta Z_t\, I(Z_t < 0))I(Z_t < W)$. The formulation of the model as a PD process yields the generator

$$Af(z) = (c + \delta z I(z<0))I(z<W)f'(z) + \lambda\left(\int_0^{z+c/\delta} f(z-y)\alpha e^{-\alpha y}dy - f(z)\right) \tag{3.14}$$

where $-c/\delta < z \le W$ and $f(-c/\delta) = 0$. The right-closedness of $]-c/\delta, W]$ should not pose a problem with respect to the condition that the M_n's in the definition of a PD-process are open. We can always 'overdefine' the range of Z_t to get around this point. In this case, ruin (absolute ruin) is certain, i.e. $P(\tau_{abs} < \infty) = 1$. Indeed, if the claim size distribution has unbounded support, then with probability one a claim larger than $W + c/\delta$ will occur eventually. If on the other hand the claim size distribution has bounded support, ruin will occur at a barrage of claims, using the Poisson process assumption. We are now interested in maximizing $E(\int_0^{\tau_{abs}} cI(Z_s=W)e^{-\beta s}ds)$ with respect to W. Integration over the interval $]0,\tau_{abs}]$ can be replaced by integration over $]0,\infty[$. We can apply Proposition 2. If f belongs to the domain of the generator (3.14) and satisfies

$$Af(z) = \beta f(z) - cI(z=W) \tag{3.15}$$

where $-c/\delta < z \le W$, $0 < u < W$ then

$$f(u) = E\left(\int_0^t cI(Z_s=W)e^{-\beta s}ds + e^{-\beta t}f_{ter}(Z_t)\right).$$

Letting $t \to \infty$, $f_{ter}(Z_t) \to f_{ter}(-c/\delta)$ a.s. since $Z_t \to -c/\delta$ eventually and so $e^{-\beta t}f_{ter}(Z_t) \to 0$ a.s. Therefore $f(u) = E(\int_0^\infty cI(Z_s=W)e^{-\beta s}ds)$. Hence we need to solve (3.15) for f continuous in $]-c/\delta,W]$ so that $f(W-) = f(W)$ and $f(0+) = f(0) = f(0-)$. We write f as

$$f(z) = f_0(z)I(-c/\delta < z \le 0) + f_1(z)I(0 \le z \le W)$$

so that (3.15) becomes

$$(c+\delta z)f_0'(z) + \lambda\left(\int_0^{z+c/\delta} f_0(z-y)\alpha e^{-\alpha y}dy - f_0(z)\right) = \beta f_0(z), \quad -c/\delta < z \le 0;$$

$$cf_1'(z) + \lambda\left(\int_0^z f_1(z-y)\alpha e^{-\alpha y}dy + \int_z^{z+c/\delta} f_0(z-y)\alpha e^{-\alpha y}dy - f_1(z)\right) = \tag{3.16}$$

$$= \beta f_1(z), \quad 0 \le z \le W; \tag{3.17}$$

$$\lambda\left(\int_0^W f_1(W-y)\alpha e^{-\alpha y}dy + \int_W^{W+c/\delta} f_0(W-y)\alpha e^{-\alpha y}dy - f_1(W)\right) =$$

$$= \beta f_1(W) - c\ . \tag{3.18}$$

The equations (3.17), (3.18) imply $f_1'(W-) = 1$. Note that $\int_z^{z+c/\delta} f_0(z-y)\alpha e^{-\alpha y}dy = He^{-\alpha z}$, H a constant so that (3.17) becomes

$$cf_1'(z) + \lambda\left(\int_0^z f_1(z-y)\alpha e^{-\alpha y}dy + He^{-\alpha z} - f_1(z)\right) = \beta f_1(z).$$

Taking Laplace transforms and solving for $\hat{f}_1$ we find that

$$\hat{f}_1(s) = \frac{f(0+)(\alpha+\nu_1) - H\lambda/c}{\nu_1 - \nu_2}(s-\nu_1)^{-1} - \frac{f(0+)(\alpha+\nu_2) - H\lambda/c}{\nu_1 - \nu_2}(s-\nu_2)^{-1}$$

where $\nu_1 > 0 > \nu_2$ are the roots of $s^2 + (\alpha-(\lambda+\beta)/c)s - \alpha\beta/c = 0$. Thus

$$f_1(z) = \frac{f_1(0)(\alpha+\nu_1) - H\lambda/c}{\nu_1 - \nu_2}e^{\nu_1 z} - \frac{f_1(0)(\alpha+\nu_2) - H\lambda/c}{\nu_1 - \nu_2}e^{\nu_2 z}.$$

The constant H can be calculated by setting $z = 0$ so that $\lambda H/c = ((\lambda+\beta)/c)f(0) - f_1'(0)$, hence

$$f_1(z) = \frac{-f_1(0)\nu_2 + f_1'(0)}{\nu_1 - \nu_2}e^{\nu_1 z} - \frac{-f_1(0)\nu_1 + f_1'(0)}{\nu_1 - \nu_2}e^{\nu_2 z}.$$

A relationship between $f_1'(0)$ and $f_1(0)$ can be obtained by letting $z = 0$ in (3.16). We again get $\lambda H/c = ((\lambda+\beta)/c)f_0(0) - f_0'(0)$. By continuity $f_0(0) = f_1(0)$ and $f_0'(0) = f_1'(0)$ so it suffices to find a relationship between $f_0(0)$ and $f_1'(0)$. So finally we have to solve (3.16). Again taking Laplace transforms as above one shows that

$$f_0(z) = K\int_0^{z+c/\delta} (z+c/\delta-y)^{\beta/\delta}\frac{\alpha^{\lambda/\delta}y^{\lambda/\delta-1}}{\Gamma(\lambda/\delta)}e^{-\alpha y}dy.$$

Hence

$$f_0'(0)/f_0(0) = \frac{(\beta/\delta)\int_0^{c/\delta}(c/\delta-y)^{\beta/\delta-1}y^{\lambda/\delta-1}e^{\alpha y}dy}{\int_0^{c/\delta}(c/\delta-y)^{\beta/\delta}y^{\lambda/\delta-1}e^{-\alpha y}dy} = L, \tag{3.19}$$

say.

Thus

$$f_1(z) = \frac{f_1(0)}{\nu_1 - \nu_2} [(L-\nu_2)e^{\nu_1 z} - (L-\nu_1)e^{\nu_2 z}].$$

Using $f_1'(W-) = 1$ in this equation we find

$$\frac{f_1(0)}{\nu_1 - \nu_2} [(L-\nu_2)\nu_1 e^{\nu_1 W} - (L-\nu_1)\nu_2 e^{\nu_2 W}] = 1.$$

Therefore

$$f_1(0) = (\nu_1 - \nu_2) \{(L-\nu_2)\nu_1 e^{\nu_1 W} - (L-\nu_1)\nu_2 e^{\nu_2 W}\}^{-1}.$$

Notice that $f_1(z)$ depends upon W only via $f_1(0)$, so it suffices to maximize $f_1(0)$ with respect to W. This gives us the optimal value

$$W_0 = (\nu_1 - \nu_2)^{-1} \log \frac{(L-\nu_1)\nu_2^2}{(L-\nu_2)\nu_1^2}$$

and L is given above.

□

Remarks :

Remark a : The equation (3.19) holds as soon as it yields a positive value for W_0. Otherwise the discounting factor is so large that the optimal strategy is to invest as much as possible immediately.

Remark b : The value W_0 above yields the optimal barrier strategy, though there is no evidence of its optimality within a wider class of strategies. However, barrier strategies are feasible and sensible. Furthermore, since they are optimal for the classical model, we conjecture wider optimality properties in the absolute ruin case.

Remark c : Finally, the result for the classical model can be obtained if we let $\delta \to \infty$. Hence

$$W_0 = (\nu_1 - \nu_2)^{-1} \log \left(\frac{(\alpha+\nu_2)\nu_2^2}{(\alpha+\nu_1)\nu_1^2} \right) .$$

See for instance Bühlmann [6], p. 171-174.

Remark d : In principle, the same method can be applied if we modify the process Z_t differently wherever Z_t becomes negative.

The insolvency boundary 0 is just chosen for convenience and we could have taken any positive level.

4. CONCLUSION

In this paper we have tried to indicate how the theory of PD processes together with some standard martingale techniques form an ideal theoretical foundation for the stochastic analysis of insurance models. Our main concern was to develop the classical theory from this common ground as well as to study some new models to show the versatility of the PD approach. We could have discussed many more models and optimization problems; we shall return to these in future publications.

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