# PRICING OF ASIAN OPTIONS ON INTEREST RATES IN THE CIR MODEL

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**ABSTRACT**
In this paper, we study the integral over time of the instantaneous rate, i.e the interest rate accrual, in the Cox Ingersoll Ross model. We derive distributional results for this process, including series representations for the density and probability distribution function. Applications to the valuation of derivatives, including Asian options prices in closed form, are presented here. Numerical examples are included to demonstrate the speed of convergence of the series. We also find that the series provide a more robust tool than numerical Laplace transform inversion for regions of high maturity and volatility. Given the versatility of the square-root process, the results derived in this paper are also of value for various others areas of finance, among which stochastic volatility and credit derivatives.

**KEY WORDS**
Derivatives, valuation, Square-root process, average-rate claims.

## 1 Introduction

The popularity of the square-root process in all main branches of financial modeling stems from its desirable property of positivity, its richness of behaviour and its mathematical properties. As a result, it has been used to model equities ( Cox-Ross [1] alternative process), interest rates ( CIR [2] interest rate model and its time-inhomogeneous [3], multivariate [4] and other derivatives), stochastic volatility ( Heston [5] model and its various extensions [6], [7], [8]) and other financial quantities. Cox, Ingersoll and Ross [2] developed a general equilibrium model based on a mean-reverting square-root process. Belonging to the class of time-homogeneous endogenous processes initially employed to represent the short rate, the CIR model has been a benchmark for many years because of its allying both strictly positive interest rates - unlike the Vasicek [9] model - and a relative analytical tractability, unlike many other positive short rate models.

In this paper, we study the integral over time of the short rate, i.e the interest rate accrual. After recalling some known properties of the square-root process and its integral, we will derive explicitly the density of the latter. The same methodology is then used to obtain other distributional results as well as prices of options on the rate accrual, including Asian options on interest rates, also called average-rate claims.

## 2 The square-root process: some reminders

For any positive initial value $X_0$, there is a unique strong solution to the stochastic differential equation

$$dX_t = (a - bX_t)dt + \sigma\sqrt{X_t}dW_t \tag{1}$$

The joint distribution of $(X_T, Y_T)$ can be characterised by its moment generating function $E\left(e^{-\lambda X_t}e^{-\mu\int_0^t X_u du}\middle| X_0 = x_0\right)$ (see Lamberton and Lapeyre [10] for instance):

$$\mathcal{L}^{X,Y}(\lambda,\mu) = E\left(e^{-\lambda X_t}e^{-\mu\int_0^t X_s ds}\middle| X_0 = x_0\right) = e^{-a\phi(t)-x_0\psi(t)} \tag{2}$$

with

$$\psi(t) = \frac{\lambda((\gamma-b)+e^{-\gamma t}(\gamma+b))+2\mu(1-e^{-\gamma t})}{\sigma^2\lambda(1-e^{-\gamma t})+(\gamma+b)+e^{-\gamma t}(\gamma-b)} \tag{3}$$

and

$$\phi(t) = \frac{-2}{\sigma^2}\ln\left(\frac{2\gamma e^{\frac{(b-\gamma)t}{2}}}{\sigma^2\lambda(1-e^{-\gamma t})+(\gamma+b)+e^{-\gamma t}(\gamma-b)}\right) \tag{4}$$

where

$$\gamma = \sqrt{b^2+2\mu\sigma^2} \tag{5}$$

## 3 The marginal density

We start by studying the marginal density of $Y_t$. This enables us to present in detail the methodology used throughout this paper and applied to option prices and other functionals of $Y_t$ as well.

**Definitions and notations.** For $\varpi \in \mathbb{R}^+\backslash\{0\}$, we construct a sequence $I_{p,q}(\varpi)$ in the following recursive way for positive integers $p$ and $q$

• For $q = 0$

$$I_{p+1,0}(y,\varpi) = I_{p,0}(y,\varpi) - \sqrt{\frac{2}{\pi}}\frac{(b\varpi+p+1)He_p\left(\frac{\varpi-2yb\beta}{\sqrt{2y\beta}}\right)e^{-\frac{\varpi^2}{4y\beta}}}{\sqrt{(2y\beta)^{p+3}}} \tag{6}$$

• For $q = 1$

$$I_{p,1}(y,\varpi) = \sqrt{\frac{2}{\pi}}\frac{e^{-\frac{\varpi^2}{4y\beta}}}{\sqrt{(2y\beta)^{p+1}}}He_p\left(\frac{\varpi-2yb\beta}{\sqrt{2y\beta}}\right) \tag{7}$$

• For $q = 2$

$$I_{p+1,2}(y,\varpi) = \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{\varpi^2}{4y\beta}}}{(\sqrt{2y\beta})^{p+1}} He_p\left(\frac{\varpi - 2yb\beta}{\sqrt{2y\beta}}\right) - bI_{p,2}(y,\varpi) \quad (8)$$

• For $q = 3$

$$I_{p,3}(y,\varpi) = p1_{\{p>0\}}I_{p-1,2}(y,\varpi) - \varpi I_{p,2}(y,\varpi)$$

$$+ \frac{e^{-\frac{\varpi^2}{4y\beta}}\sqrt{2}}{\sqrt{\pi(2y\beta)^{p-1}}} He_p\left(\frac{\varpi - 2yb\beta}{\sqrt{2y\beta}}\right) \quad (9)$$

• For $q > 3$

$$I_{p,q}(y,\varpi) = \frac{p1_{\{p>0\}}I_{p-1,q-1}(y,\varpi) + 2y\beta I_{p,q-2}(y,\varpi) - \varpi I_{p,q-1}(y,\varpi)}{q-2} \quad (10)$$

from the only two initial conditions needed:

$$\begin{cases} I_{0,0}(y,\varpi) = \dfrac{\varpi}{2\sqrt{\pi(y\beta)^3}} e^{-\frac{\varpi^2}{4y\beta}} \\ I_{0,2}(y,\varpi) = \text{erfc}\left(\dfrac{\varpi}{2\sqrt{y\beta}}\right) \end{cases} \quad (11)$$

the Hermite polynomials $He_k(x) = \sum_{s=0}^{\left[\frac{k}{2}\right]} (-1)^s \frac{x^{k-2s}}{2^s} \frac{k!}{(k-2s)!s!}$ being also computed through

$$He_{k+1}(x) = xHe_k(x) - kHe_{k-1}(x), He_0(x) = 1 \text{ and } He_1(x) = x, \forall x \in \mathbb{R} \quad (12)$$

**Theorem 3.1** *The marginal density of the integral $Y_t$ can be rewritten as*

$$f^Y(y) = \beta e^{b\frac{at+x_0}{\sigma^2} - b^2 y\beta} \sum_{k=0}^{\infty} \frac{f_k^Y(y)}{2^k} \quad (13)$$

*where*

$$f_k^Y(y) = \sum_{n=0}^{k}\sum_{m=n}^{k} \binom{k+\frac{2a}{\sigma^2}-1}{k-n}\binom{k-n}{m-n} \frac{(-2x_0)^n}{n!\sigma^{2n}} (-1)^m I_{k,k-n}(y,\varpi_m) \quad (14)$$

*with*

$$\varpi_m = \frac{at+x_0}{\sigma^2} + mt \quad \text{and} \quad \beta = \frac{1}{2\sigma^2}$$

**Proof 3.1** *See Appendix 7.1.*

**Remark.**
**1.** For programming purposes, it might be simpler to use

$$\forall K \geq 0, \sum_{k=0}^{K} \frac{f_k^Y(y)}{2^k} = \sum_{m=0}^{K}\sum_{k=m}^{K}\sum_{n=k-m}^{k} u_{k,n,m} I_{k,n}(y,\varpi_m) \quad (15)$$

where $u_{k,n,m} = (-1)^m \frac{\left(-\frac{x_0}{\sigma^2}\right)^{k-n}}{2^n(k-n)!} \binom{k+\frac{2a}{\sigma^2}-1}{n}\binom{n}{m+n-k}$ can also be efficiently computed though recursions.

**2.** The complementary error function erfc, appearing in (11) has been widely studied in the literature. As a result, there exists several algorithms to compute it numerically with accuracy in a minimal number of operations.

# 4 Applications

In the same way as for the density, it is possible to derive key probability functions and option prices related to $Y_T$.

## i. Two fundamental functionals

For this purpose, we begin by deriving two preliminary results, on which lie the other applications we will be looking at in this paper.

**Theorem 4.1** *For $\lambda \geq 0$, the inverse Laplace transform of $\frac{E(e^{-(\lambda+\mu)Y_t})}{\mu}$ with respect to $\mu \geq 0$ is*

$$\mathcal{G}_{a,b,\sigma}(y,\lambda) = e^{b\frac{at+x_0}{\sigma^2}} \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=n}^{k} u_{k,k-n,m} \tilde{I}_{k,k-n}(y,\lambda,\varpi_m) \quad (16)$$

*while the inverse Laplace transform of $\frac{E(Y_t e^{-(\lambda+\mu)Y_t})}{\mu}$ is*

$$\mathcal{G\!\!/}_{a,b,\sigma}(y,\lambda) = e^{b\frac{at+x_0}{\sigma^2}} \sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=n}^{k} u_{k,k-n,m} \hat{I}_{k,k-n}(y,\lambda,\varpi_m) \quad (17)$$

*where $\tilde{I}_{k,k-n}$ and $\hat{I}_{k,k-n}(y,\lambda,\varpi_m)$ are simple functions defined by recursion in Appendix 7.2.*

**Proof 4.1** *The derivation is based on the same decomposition as for Theorem 3.1 and on the same type of arguments as the derivation of the density. See [11] for the details of intermediate calculations.*

## ii. Probability distribution function of the cumulative interest rate

**Theorem 4.2** *For $y > 0$, the probability distribution function of $Y_t$ is given by*

$$P(Y_t \leq y) = \mathcal{G}_{a,b,\sigma}(y,0) \quad (18)$$

**Proof 4.2** *For $\mu > 0$, the Laplace transform of the probability distribution function is*

$$\int_0^{\infty} e^{-\mu y} E\left(1_{\{Y_t \leq y\}}\right) dy = E\left(\int_{Y_t}^{\infty} e^{-\mu y} dy\right) = E\left(\frac{e^{-\mu Y_t}}{\mu}\right)$$

*which, combined with Theorem 4.1, implies the result.*

## iii. Truncated expectation of the cumulative interest rate

**Theorem 4.3** *For $y > 0$, the truncated expectation of $Y_t$ is given by*

$$E\left(Y_t 1_{\{Y_t \leq y\}}\right) = \mathcal{G\!\!/}_{a,b,\sigma}(y,0) \quad (19)$$

**Proof 4.3** *For $\mu > 0$, the Laplace transform of the probability distribution function is*

$$\int_0^{\infty} e^{-\mu y} E\left(Y_t 1_{\{Y_t \leq y\}}\right) dy = E\left(Y_t \int_{Y_t}^{\infty} e^{-\mu y} dy\right) = E\left(Y_t \frac{e^{-\mu Y_t}}{\mu}\right)$$

## iv. Guaranteed endowment option

This option pays out the shortfall between 1 and the amount accumulated in a standard savings account $(1-Ke^{Y_T})^+$. It is simply used to guarantee a minimal accrual on an initial amount of cash, a common insurance feature.

**Theorem 4.4** *For $1 \geq K > 0$, the guaranteed endowment put option is given by the expectation of the discounted expectation of its payoff*

$$GEO^p(K,T) = E(e^{-Y_T} - K)^+ = \mathcal{G}_{a,b,\sigma}(-\ln K, 1) - K\mathcal{G}_{a,b,\sigma}(-\ln K, 0) \quad (20)$$

**Proof 4.4** *The option price can be split in two parts*

$$GEO^p(K,T) = E(e^{-Y_T} 1_{\{Y_T \leq -\ln K\}}) - KP(Y_T \leq -\ln K)$$

*We denote $k = -\ln K$. Since $1 \geq K > 0$, $k$ is strictly positive and we therefore have, for $\mu > 0$,*

$$\int_0^\infty e^{-\mu k} E\left(e^{-Y_T} 1_{\{Y_T \leq k\}}\right) dk = E\left(e^{-Y_T} \int_{Y_T}^\infty e^{-\mu k} dk\right) = E\left(\frac{e^{-(\mu+1)Y_T}}{\mu}\right)$$

**Remark.** The assumption $K < 1$ is due to the fact that $e^{-Y_T} < 1$. For $K > 1$, the option is simply worthless.

**Theorem 4.5** *The guaranteed endowment call option can be deduced from the parity relation*

$$GEO^c(K,T) - GEO^p(K,T) = P(t,T) - K \quad (21)$$

**Proof 4.5** *This results follow from*

$$(e^{-Y_T} - K)^+ - (K - e^{-Y_T})^+ = e^{-Y_T} - K$$

## v. Binary Asian options

In general, binary - or digital - options are classified in two groups: the ones paying in cash units and the one paying in asset units, i.e. paying an interest rate here. In the context of Asian interest rate derivatives, their respective payoff is $E(e^{-Y_T} 1_{\{Y_T \geq K\}})$ and $E(Y_T e^{-Y_T} 1_{\{Y_T \geq K\}})$ for cap - or call - options. For floor - put - options, those payoffs are $E(e^{-Y_T} 1_{\{Y_T \leq K\}})$ and $E(Y_T e^{-Y_T} 1_{\{Y_T \leq K\}})$.

**Theorem 4.6** *For $K > 0$, the cash binary Asian floor is worth*

$$CBA^f(K,T) = E(e^{-Y_T} 1_{\{Y_T \leq K\}}) = \mathcal{G}_{a,b,\sigma}(K,1) \quad (22)$$

**Proof 4.6** *We have , for $\mu > 0$,*

$$\int_0^\infty e^{-\mu K} E\left(e^{-Y_T} 1_{\{Y_T \leq K\}}\right) dK = E\left(e^{-Y_T} \int_{Y_T}^\infty e^{-\mu K} dK\right) = E\left(\frac{e^{-(\mu+1)Y_T}}{\mu}\right)$$

*Hence the result.*

**Theorem 4.7** *For $K > 0$, the rate binary Asian floor is worth*

$$RBA^f(K,T) = E(Y_T e^{-Y_T} 1_{\{Y_T \leq K\}}) = \hat{\mathcal{G}}_{a,b,\sigma}(K,1) \quad (23)$$

**Proof 4.7** *In the same way,*

$$\int_0^\infty e^{-\mu K} E\left(Y_T e^{-Y_T} 1_{\{Y_T \leq K\}}\right) dK = E\left(\frac{Y_T e^{-(\mu+1)Y_T}}{\mu}\right)$$

We end this section with the call-put parity relation which allows us to deduce cap option prices from the floor option prices given above.

**Theorem 4.8** *The options are linked in the following way*

$$\begin{aligned} CBA^c(K,T) + CBA^f(K,T) &= P(0,T) \\ RBA^c(K,T) + RBA^f(K,T) &= P(0,T)E^T(Y_T) \end{aligned} \quad (24)$$

**Proof 4.8** *Indeed,*

$$\begin{aligned} E(e^{-Y_T} 1_{\{Y_T \leq K\}}) + E(e^{-Y_T} 1_{\{Y_T > K\}}) &= E(e^{-Y_T}) \\ E(Y_T e^{-Y_T} 1_{\{Y_T \leq K\}}) + E(Y_T e^{-Y_T} 1_{\{Y_T \leq K\}}) &= E(Y_T e^{-Y_T}) \end{aligned}$$

## vi. Regular Asian options

Interest rates Asian options have been developed to cover needs similar to those having created equity Asians. Those instruments have been noticeably studied by Leblanc and Scaillet [12] and Chacko and Das [13].

**Theorem 4.9** *The regular Asian option can be computed as the inverse Laplace transform with respect to $\mu$ of*

$$\frac{E(Y_T e^{-Y_T})}{\mu} + \frac{E(e^{-(\mu+1)Y_T} - e^{-\mu Y_T})}{\mu^2} \quad (25)$$

**Proof 4.9** *Straightforward application of the previous results.*

**Remarks.**

**1.** The first term of the expression (25) can easily be obtained explicitly as the derivative of the MGF $\frac{\partial}{\partial \xi}\mathcal{L}^{X,Y}(0,\xi)$ at $\xi = 1$. We do not present the actual formula here as the expression turns out to be quite heavy.

**2.** To compare this Laplace transform result to the existing solutions to this problem, the Laplace transform approach proposed in Theorem 4.9 is simpler. Indeed, Leblanc and Scaillet [12] proposed to first compute the density of $Y_T$ through numerical Laplace transform inversion of its MGF and then integrate it against the discounted payoff of the option, which is a heavy numerical procedure. Chacko and Das [13] expressed the option as a sum of Asian binary calls at ascending strikes and used Fourier inversion. Our expression is more immediate to calculate.

We can yet also propose a completely explicit solution for this option.

**Theorem 4.10** *For $K > 0$, the Asian floor is worth*

$$AO^f(K,T) = E((K - Y_T)^+ e^{-Y_T}) = K\mathcal{G}_{a,b,\sigma}(K,1) - \hat{\mathcal{G}}_{a,b,\sigma}(K,1) \quad (26)$$

**Proof 4.10** *The regular Asian option is the difference between two binary options, one paying in cash unit and the other in rate unit*

$$AO^f(K,T) = K.CBA(K,T) - RBA(K,T)$$

As previously, we present a call-put parity result.

**Theorem 4.11** *For $K > 0$, we have*

$$AO^c(K,T) - AO^f(K,T) = E(Y_T e^{-Y_T}) - KP(0,T) \quad (27)$$

## 5 Numerical results

The works of Chacko and Das [13] on one hand and Leblanc and Scaillet [12] provide us with material for comparison. Of the two methods, the one-step numerical inversion proposed by Chacko and Das is the most efficient, since it involves a single integration as opposed to Leblanc and Scaillet double integration. But, both sets of numerical results are useful for reference. We consider the prices of regular Asian caps, $\text{AO}^c(K,T) = E\left(\left(\frac{Y_T}{T} - K\right)^+ e^{-Y_T}\right)$, cash binary caps, $\text{CBA}^c(K,T) = E\left(e^{-Y_T} 1_{\{Y_T > KT\}}\right)$, and additionally the valuation of rate binary caps, $\text{RBA}^c(K,T) = E\left(Y_T e^{-Y_T} 1_{\{Y_T > KT\}}\right)$, truncated moments defined as $\text{TM}^c(K,T) = E\left(Y_T 1_{\{Y_T > KT\}}\right)$ and probabilities, $P = P(Y_T > KT)$.

Following Chacko and Das [13], we first analyse in Table 1 the evolution with respect to the maturity $T$ and strike $K$. The diffusion parameters of the instantaneous rate are a=0.15, b=1.5, $\sigma = 0.2$ and $r_0 = 0.1$. The results we obtained do not actually exactly tally with the ones presented by Chacko and Das [13]. Yet, numerical Laplace inversion using the Abate and Whitt algorithm confirm our results.

To double-check the validity of our method, we also consider Asian options on yields in the setting of Leblanc and Scaillet [12]. The yield $\mathcal{Y}(T, T+\tau)$ of maturity $\tau$ at time $T$ is defined as $-\frac{\ln(B(T,T+\tau))}{\tau} = \frac{a\phi(\tau)+r_T\psi(\tau)}{\tau}$. Asian call options on yields are then given by

$$C^{\mathcal{Y}} = E\left[\left(\frac{1}{T}\int_0^T \mathcal{Y}(u, u+\tau)du - K\right)^+ e^{-Y_T}\right] \tag{28}$$

They are related to the Asian options on the instantaneous rate through

$$C^{\mathcal{Y}} = \frac{\psi(\tau)}{\tau}\text{AO}^c\left(\frac{\tau K - a\phi(\tau)}{\psi(\tau)}, T\right)$$

Table 2 confirms that the values computed with our series are correct. Indeed, the column LS and Fusai collect the prices respectively produced by Leblanc and Scaillet [12] and Fusai [14].

This cross-checking done, we come back to the parameters proposed by Chacko and Das [13]. We first observe how higher maturities, common in fixed-income, affect the results in Table 3.

The volatility of $\frac{Y_T}{T}$ should start from a low level, increase with $\tau$ and then decrease again for high maturities, the mean-reversion pulling the rate to its long-term level $\frac{a}{b}$. The call prices and the probabilities corroborate this intuition. The monotonous evolutions with respect to $K$ are as expected.

The figures of Table 4 represent the minimal number of terms needed to ensure the relative error is inferior to $10^{-4}$. $\mathcal{G}_0$ stand for $\mathcal{G}_{a,b,\sigma}(KT,0)$, $\mathcal{G}_1$ for $\mathcal{G}_{a,b,\sigma}(KT,1)$, $\hat{\mathcal{G}}_0$ for $\hat{\mathcal{G}}_{a,b,\sigma}(KT,0)$ and $\hat{\mathcal{G}}_1$ for $\hat{\mathcal{G}}_{a,b,\sigma}(KT,1)$. The four series converge more and more quickly as $T$ increase. $N$ remains roughly of the same order throughout the strike curve.

All the results produced in this section have been cross-tested against one-step numerical transform inversion, the most efficient numerical method available so far.

For maturities superior than five years, the Abate-Whitt algorithm starts becoming unstable. Figure 1 shows no hint can obtained as to the location of the real inverse for $T = 10$ with the Abate and Whitt algorithm (See Appendix 7.3 for a description of this algorithm). Long-dated instruments are not rare in fixed-income markets. Our series brings a quick and effective solution for those problematic high maturities regions.

The same is true for high volatilities as expected; our series converge faster as the volatility increases whereas the Abate-Whitt routine has difficulties.

## 6 Conclusion

In this paper, we derived explicit expressions for the distribution of the interest rate accrual as well as options on this underlying. Computing numerically these expressions proves simple and also more robust than numerical inversion in regions of high volatilities and maturities.

If this methodology cannot be easily extended to the general CIR model with time-dependent coefficients - for which even the zero-coupon prices are only formally given as solution to a differential equation but not explicitly known in general -, computing derivatives in the CIR++ model, analytically tractable extension of the CIR model proposed by Brigo and Mercurio [15], is however a straightforward application.

## 7 Appendix

### 7.1 Proof of Theorem 3.1 and completed formulation of the density of the integrated CIR process

#### 7.1.1 Main Expansion

Rearranging the marginal MGF of $Y_t$ ($\lambda = 0$ in equation (2)) allows us to see the importance of the term $\frac{2\gamma}{(\gamma+b)+(\gamma-b)e^{-\gamma t}}$. Indeed, $\mathcal{L}^Y(\mu) =$

$$e^{\frac{atb-x_0(\gamma-b)}{\sigma^2}}\left(\frac{2\gamma e^{-\frac{\gamma t}{2}}}{(\gamma+b)+(\gamma-b)e^{-\gamma t}}\right)^{\frac{2a}{\sigma^2}} e^{\frac{x_0(\gamma-b)}{\sigma^2}\frac{2\gamma e^{-\gamma t}}{(\gamma+b)+(\gamma-b)e^{-\gamma t}}} \tag{29}$$

But, if that term does not by itself give more insight, its reciprocal is $\frac{(\gamma+b)+(\gamma-b)e^{-\gamma t}}{2\gamma} = 1 - \frac{(\gamma-b)(1-e^{-\gamma t})}{2\gamma}$, where the last ratio is an easily invertible Laplace transform. Exploiting this, we represent (29) as

$$e^{-(\gamma-b)\frac{at+x_0}{\sigma^2}}\sum_{k=0}^{\infty}\sum_{n=0}^{k}\binom{k+\frac{2a}{\sigma^2}-1}{k-n}\frac{x_0^n(\gamma-b)^k(1-e^{-\gamma t})^{k-n}e^{-\gamma n t}}{n!\sigma^{2n}(2\gamma)^{k-n}} \tag{30}$$

where the generalised binomial coefficient defined as

$$\binom{n+k+\frac{2a}{\sigma^2}}{k} = \frac{(n+\frac{2a}{\sigma^2})\dots(n+\frac{2a}{\sigma^2}+k-1)}{k!},\ k>0$$

is conventionally equal to $1$ when $k=0$.

### 7.1.2 The recursions

The density can be constructed from the inverse of $\frac{(\gamma-b)^p}{\gamma^q}e^{-\varpi\gamma},\ \varpi > 0,\ p,q \geq 0$. We build this inverse denoted $I_{p,q}(y,\varpi)$ in a recursive manner by exploiting the properties of the inverse gaussian distribution, which has for density

$$f^{\mathrm{IG}}(y) = \sqrt{\frac{\kappa}{2\pi y^3}}\, e^{-\frac{\kappa(y-\omega)^2}{2\omega^2 y}} \tag{31}$$

and for moment generating function

$$\mathcal{L}^{\mathrm{IG}}(y) = e^{\frac{\kappa}{\omega}\left(1-\sqrt{1+\frac{2\omega^2\mu}{\kappa}}\right)} \tag{32}$$

with the parameters $\kappa > 0$ and $\mu$ real.

CASE $q > 0$

From (32), the inverse in this case can be expressed as $I_{p,q}(y,\varpi) =$

$$(-1)^p \frac{\partial^p}{\partial\zeta^p}\bigg|_{\zeta=0}\left[\int_0^\infty \frac{e^{-\frac{(\varpi+\zeta+u)^2}{4y\beta}-b^2y\beta+b\zeta}(\varpi+\zeta+u)u^{q-1}\beta}{2\sqrt{\pi(y\beta)^3}(q-1)!}\,du\right] \tag{33}$$

CASE $q = 0$

For this case, the integro-differential formula (33) is to be replaced by the differential formula

$$I_{p,0}(y) = (-1)^p \frac{\partial^p}{\partial\zeta^p}\left[e^{-\frac{(\varpi+\zeta)^2}{4y\beta}-b^2y\beta+b\zeta}\,\frac{\beta(\varpi+\zeta)}{2\sqrt{\pi(y\beta)^3}}\right] \tag{34}$$

Calculating these expressions enables us to formulate the density as in Theorem 3.1.

## 7.2 Definition of expressions in Theorem 4.1

**First inverse:** $\mathcal{g}_{a,b,\sigma}$

For $p \geq 0$, $q \geq 0$ and $\varpi > 0$, the main term in (16), $\tilde{I}_{p,q}(y,\lambda,\varpi)$ is given by

$$\begin{aligned}
& 1_{\{q=0\}}\left\{2y\beta\left(\vartheta^2(\tilde{I}^1_{p,1}(y,\lambda,\varpi)-\tilde{I}^2_{p,1}(y,\lambda,\varpi))+\tilde{I}^3_{p,1}(y,\lambda,\varpi)\right)\right\}\\
+\ & 1_{\{q=1\}}\left\{2y\beta\vartheta\left(\tilde{I}^1_{p,1}(y,\lambda,\varpi)+\tilde{I}^2_{p,1}(y,\lambda,\varpi)\right)\right\}\\
+\ & 1_{\{q>1\}}\left\{2y\beta\left(\tilde{I}^1_{p,q-1}(y,\lambda,\varpi)-\tilde{I}^2_{p,q-1}(y,\lambda,\varpi)\right)\right\}
\end{aligned} \tag{35}$$

where $\vartheta = \sqrt{b^2+2\lambda\sigma^2}$ and the three sequences $\tilde{I}^3_{p,q}(y,\lambda,\varpi)$, $\tilde{I}^1_{p,q}(y,\lambda,\varpi)$ and $\tilde{I}^2_{p,q}(y,\lambda,\varpi)$ are recursively constructed in the following way for $q \geq 1$ and $p \geq 0$.

- For $\tilde{I}^3_{p,q}(y,\lambda,\varpi)$
  - Recursion

$$\tilde{I}^3_{p,q+1}(y,\lambda,\varpi) = -\frac{\varpi}{q}\tilde{I}^3_{p,q}(y,\lambda,\varpi) + 1_{\{p>0\}}\left\{\frac{p}{q}\tilde{I}^3_{p-1,q}(y,\lambda,\varpi)\right\}$$
$$+1_{\{q>1\}}\left\{\frac{2y\beta\tilde{I}^3_{p,q-1}(y,\lambda,\varpi)}{q}\right\}+1_{\{q=1\}}\left\{\frac{e^{-\frac{\varpi^2}{4y\beta}-\vartheta^2y\beta}\beta\sqrt{2}He_p\left(\frac{\varpi-2yb\beta}{\sqrt{2y\beta}}\right)}{\sqrt{\pi(2y\beta)^{p+1}}}\right\} \tag{36}$$

  - Initial value recursion

$$\tilde{I}^3_{p+1,1}(y,\lambda,\varpi) = -b\tilde{I}^3_{p,1}(y,\lambda,\varpi) + \frac{\beta\sqrt{2}e^{-\frac{\varpi^2}{4y\beta}-\vartheta^2y\beta}}{\sqrt{\pi(2y\beta)^{p+3}}}He_p\left(\frac{\varpi-2yb\beta}{\sqrt{2y\beta}}\right) \tag{37}$$

  - Starting point

$$\tilde{I}^3_{0,1}(y,\lambda,\varpi) = \frac{e^{-\vartheta^2y\beta}}{2y}\mathrm{erfc}\left(\frac{\varpi}{2\sqrt{y\beta}}\right) \tag{38}$$

- For $\tilde{I}^1_{p,q}(y,\lambda,\varpi)$
  - Recursion

$$\tilde{I}^1_{p,q+1}(y,\lambda,\varpi) = \frac{1}{\vartheta}\tilde{I}^1_{p,q}(y,\lambda,\varpi) - \frac{1}{2\vartheta^2}\tilde{I}^3_{p,q+1}(y,\lambda,\varpi) \tag{39}$$

  - Initial value recursion

$$\tilde{I}^1_{p+1,1}(y,\lambda,\varpi) = (\vartheta-b)\tilde{I}^1_{p,1}(y,\lambda,\varpi) + \frac{1}{2\vartheta}\tilde{I}^3_{p,1}(y,\lambda,\varpi) \tag{40}$$

  - Starting point

$$\tilde{I}^1_{0,1}(y,\lambda,\varpi) = \frac{e^{-\vartheta\varpi}}{4\vartheta^2y}\mathrm{erfc}\left(\frac{\varpi-2y\beta\vartheta}{2\sqrt{y\beta}}\right) - \frac{e^{-\vartheta^2y\beta}}{4\vartheta^2y}\mathrm{erfc}\left(\frac{\varpi}{2\sqrt{y\beta}}\right) \tag{41}$$

- For $\tilde{I}^2_{p,q}(y,\lambda,\varpi)$
  - Recursion

$$\tilde{I}^2_{p,q+1}(y,\lambda,\varpi) = -\frac{1}{\vartheta}\tilde{I}^1_{p,q}(y,\lambda,\varpi) + \frac{1}{2\vartheta^2}\tilde{I}^3_{p,q+1}(y,\lambda,\varpi) \tag{42}$$

  - Initial value recursion

$$\tilde{I}^2_{p+1,1}(y,\lambda,\varpi) = -(b+\vartheta)\tilde{I}^2_{p,1}(y,\lambda,\varpi) + \frac{1}{2\vartheta}\tilde{I}^3_{p,1}(y,\lambda,\varpi) \tag{43}$$

  - Starting point

$$\tilde{I}^2_{0,1}(y,\lambda,\varpi) = \frac{e^{-\vartheta^2y\beta}}{4\vartheta^2y}\mathrm{erfc}\left(\frac{\varpi}{2\sqrt{y\beta}}\right) - \frac{e^{\vartheta\varpi}}{4\vartheta^2y}\mathrm{erfc}\left(\frac{\varpi+2y\beta\vartheta}{2\sqrt{y\beta}}\right) \tag{44}$$

**Second inverse:** $\phi_{a,b,\sigma}$

For $p \geq 0$, $q \geq 0$ and $\varpi > 0$, $\hat{I}_{p,q}(y,\lambda,\varpi)$ is given by

$$\begin{aligned}
\hat{I}_{p,q}(y,\lambda,\varpi) =\ & 1_{\{q=0\}}\Big\{2y\beta\Big(\vartheta^2(\hat{I}^1_{p,1}(y,\lambda,\varpi)-\hat{I}^2_{p,1}(y,\lambda,\varpi))\\
& +\hat{I}^3_{p,1}(y,\lambda,\varpi)\Big) - 2y(\hat{I}^1_{p,1}(y,\lambda,\varpi)-\hat{I}^2_{p,1}(y,\lambda,\varpi))\Big\}\\
+\ & 1_{\{q=1\}}\Big\{2y\beta\vartheta\left(\hat{I}^1_{p,1}(y,\lambda,\varpi)+\hat{I}^2_{p,1}(y,\lambda,\varpi)\right)\\
& -\frac{y}{\vartheta}\left(\hat{I}^1_{p,1}(y,\lambda,\varpi)+\hat{I}^2_{p,1}(y,\lambda,\varpi)\right)\Big\}\\
+\ & 1_{\{q>1\}}\left\{2y\beta\left(\hat{I}^1_{p,q-1}(y,\lambda,\varpi)-\hat{I}^2_{p,q-1}(y,\lambda,\varpi)\right)\right\}
\end{aligned} \tag{45}$$

where the three new sequences $\hat{I}^3_{p,q}(y,\lambda,\varpi)$, $\hat{I}^1_{p,q}(y,\lambda,\varpi)$ and $\hat{I}^2_{p,q}(y,\lambda,\varpi)$ are constructed recursively for $q \geq 1$ and $p \geq 0$ from $\hat{I}^3_{p,q}(y,\lambda,\varpi)$, $\hat{I}^1_{p,q}(y,\lambda,\varpi)$ and $\hat{I}^2_{p,q}(y,\lambda,\varpi)$.

- For $\hat{I}^3_{p,q}(y,\lambda,\varpi)$
  - Recursion

$$\hat{I}^3_{p,q+1}(y,\lambda,\varpi) = -\frac{\varpi}{q}\hat{I}^3_{p,q}(y,\lambda,\varpi) + 1_{\{p>0\}}\left\{\frac{p}{q}\hat{I}^3_{p-1,q}(y,\lambda,\varpi)\right\}$$

$$+1_{\{q>1\}}\left\{\frac{2y\beta\hat{I}^3_{p,q-1}(y,\lambda,\varpi)}{q}\right\}+1_{\{q=1\}}\left\{\frac{e^{-\frac{\varpi^2}{4y\beta}-\vartheta^2 y\beta}He_p\left(\frac{\varpi-2yb\beta}{\sqrt{2y\beta}}\right)}{\sqrt{2\pi(2y\beta)^{p-1}}}\right\} \quad (46)$$

  - Initial value recursion

$$\hat{I}^3_{p+1,1}(y,\lambda,\varpi) = -b\hat{I}^3_{p,1}(y,\lambda,\varpi) + \frac{e^{-\frac{\varpi^2}{4y\beta}-\vartheta^2 y\beta}}{\sqrt{2\pi(2y\beta)^{p+1}}}He_p\left(\frac{\varpi-2yb\beta}{\sqrt{2y\beta}}\right) \quad (47)$$

  - Starting point

$$\hat{I}^3_{0,1}(y,\lambda,\varpi) = \frac{e^{-\vartheta^2 y\beta}}{2}\operatorname{erfc}\left(\frac{\varpi}{2\sqrt{y\beta}}\right) \quad (48)$$

- For $\hat{I}^1_{p,q}(y,\lambda,\varpi)$
  - Recursion

$$\hat{I}^1_{p,q+1}(y,\lambda,\varpi) = \frac{\hat{I}^1_{p,q}(y,\lambda,\varpi)}{\vartheta}+\frac{\sigma^2\hat{I}^1_{p,q}(y,\varpi)}{\vartheta^3}-\frac{\frac{\hat{I}^3_{p,q+1}(y,\lambda,\varpi)}{2}+\sigma^2\hat{I}^3_{p,q+1}(y,\lambda,\varpi)}{\vartheta^2} \quad (49)$$

  - Initial value recursion

$$\hat{I}^1_{p+1,1}(y,\lambda,\varpi) = (\vartheta-b)\hat{I}^1_{p,1}(y,\lambda,\varpi)-\frac{\sigma^2}{\vartheta}\hat{I}^1_{p,1}(y,\lambda,\varpi)+\frac{\hat{I}^3_{p,1}(y,\lambda,\varpi)+\frac{\sigma^2\hat{I}^3_{p,1}(y,\lambda,\varpi)}{\vartheta}}{2\vartheta} \quad (50)$$

  - Starting point

$$\hat{I}^1_{0,1}(y,\lambda,\varpi) = \frac{\sigma^2 e^{-\vartheta\varpi}}{2\vartheta^3 y}\operatorname{erfc}\left(\frac{\varpi-2y\beta\vartheta}{2\sqrt{y\beta}}\right)\left[\frac{\varpi}{2}+\frac{1}{\vartheta}\right]$$

$$-\sqrt{\frac{\beta}{\pi y}}\frac{\sigma^2 e^{-\frac{\varpi^2}{4y\beta}-\vartheta^2 y\beta}}{2\vartheta^3}-\frac{e^{-\vartheta^2 y\beta}}{2\vartheta^2}\operatorname{erfc}\left(\frac{\varpi}{2\sqrt{y\beta}}\right)\left[\frac{1}{2}+\frac{\sigma^2}{\vartheta^2 y}\right] \quad (51)$$

- For $\hat{I}^2_{p,q}(y,\lambda,\varpi)$
  - Recursion

$$\hat{I}^2_{p,q+1}(y,\lambda,\varpi) = -\frac{\hat{I}^1_{p,q}(y,\lambda,\varpi)}{\vartheta}-\frac{\sigma^2\hat{I}^1_{p,q}(y,\varpi)}{\vartheta^3}+\frac{\hat{I}^3_{p,q+1}(y,\lambda,\varpi)}{2\vartheta^2}+\frac{\sigma^2\hat{I}^3_{p,q+1}(y,\lambda,\varpi)}{\vartheta^4} \quad (52)$$

  - Initial value recursion

$$\hat{I}^2_{p+1,1}(y,\lambda,\varpi) = \frac{\sigma^2}{\vartheta}\hat{I}^2_{p,1}(y,\lambda,\varpi)-(b+\vartheta)\hat{I}^2_{p,1}(y,\lambda,\varpi)+\frac{\hat{I}^3_{p,1}(y,\lambda,\varpi)+\frac{\sigma^2\hat{I}^3_{p,1}(y,\lambda,\varpi)}{\vartheta}}{2\vartheta} \quad (53)$$

  - Starting point

$$\hat{I}^2_{0,1}(y,\lambda,\varpi) = \frac{e^{-\vartheta^2 y\beta}}{2\vartheta^2}\operatorname{erfc}\left(\frac{\varpi}{2\sqrt{y\beta}}\right)\left[\frac{1}{2}+\frac{\sigma^2}{\vartheta^2 y}\right]$$

$$+\frac{\sigma^2 e^{\vartheta\varpi}}{2\vartheta^3 y}\operatorname{erfc}\left(\frac{\varpi+2y\beta\vartheta}{2\sqrt{y\beta}}\right)\left[\frac{\varpi}{2}-\frac{1}{\vartheta}\right]+\sqrt{\frac{\beta}{y\pi}}\frac{\sigma^2 e^{-\frac{\varpi^2}{4y\beta}-\vartheta^2 y\beta}}{2\vartheta^3} \quad (54)$$

### 7.3 The Abate-Whitt algorithm

Denoting $f(\gamma) = \int_0^\infty e^{-\gamma t}F(t)dt$, the Laplace-inverse can be obtained by

$$F(t) = \frac{2e^{at}}{\pi}\int_0^\infty \operatorname{Re}(f(a+iu))\cos(ut)du \quad (55)$$

for any a to the right of all the singularities of $f(\cdot)$.

The most popular inversion method, the Abate and Whitt algorithm, rely on a trapezoidal rule to invert this oscillatory integral. Defining $A = at$, the Abate and Whitt is the sum of the series

$$F^{\mathrm{AB}}(t) = \frac{e^{\frac{A}{2}}}{2t}\operatorname{Re}\left(f\left(\frac{A}{2t}\right)\right)+\frac{e^{\frac{A}{2}}}{t}\sum_{k=1}^{\infty}\operatorname{Re}\left(f\left(\frac{A+2ki\pi}{2t}\right)\right) \quad (56)$$

The series can critically depend on the choice of A: the series may not converge to the correct values for too small A. On the other hand, it might become too difficult to evaluate numerically for large A. The correct value for the inverse can be located on a interval of A for which the sum (56) remains constant. This length of this interval of stability can vary greatly from one application to another.

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| Maturity | Type | Strike 0.08 | 0.09 | 0.10 | 0.11 | 0.12 |
|---|---|---|---|---|---|---|
| 0.1 | $P$ | 0.1061 | 0.2936 | 0.5337 | 0.7427 | 0.8790 |
| | CBA$^C$ | 0.9631 | 0.8104 | 0.4812 | 0.1763 | 0.0387 |
| | TM$^C$ | 0.0098 | 0.0085 | 0.0053 | 0.0021 | 0.0006 |
| | RBA$^C$ | 0.0097 | 0.0084 | 0.0052 | 0.0021 | 0.0005 |
| | AO$^c$ | 0.0199 | 0.0109 | 0.0043 | 0.0011 | 0.0002 |
| 0.5 | $P$ | 0.1549 | 0.3261 | 0.5280 | 0.7107 | 0.8441 |
| | CBA$^C$ | 0.8018 | 0.6377 | 0.4452 | 0.2718 | 0.1459 |
| | TM$^C$ | 0.0444 | 0.0285 | 0.0275 | 0.0180 | 0.0103 |
| | RBA$^C$ | 0.0421 | 0.0351 | 0.0260 | 0.0169 | 0.0097 |
| | AO$^c$ | 0.0201 | 0.0128 | 0.0074 | 0.0039 | 0.0018 |
| 1 | $P$ | 0.1878 | 0.3535 | 0.5354 | 0.6979 | 0.8209 |
| | CBA$^C$ | 0.7301 | 0.5779 | 0.4125 | 0.2662 | 0.1565 |
| | TM$^C$ | 0.0867 | 0.0726 | 0.0553 | 0.0383 | 0.0242 |
| | RBA$^C$ | 0.0777 | 0.0647 | 0.0490 | 0.0337 | 0.0211 |
| | AO$^c$ | 0.0193 | 0.0127 | 0.0078 | 0.0044 | 0.0023 |
| 2 | $P$ | 0.1789 | 0.3509 | 0.5395 | 0.7050 | 0.8276 |
| | CBA$^C$ | 0.6644 | 0.5193 | 0.3633 | 0.2291 | 0.1317 |
| | TM$^C$ | 0.1744 | 0.1451 | 0.1093 | 0.0746 | 0.0465 |
| | RBA$^C$ | 0.1402 | 0.1155 | 0.0859 | 0.0578 | 0.0354 |
| | AO$^c$ | 0.0170 | 0.0110 | 0.0066 | 0.0037 | 0.0019 |

Table 1. Evolution with $T$, Chacko and Das parameters.

| $T$ | $\tau$ | LS | Fusai | Series |
|---|---|---|---|---|
| 1 | 10 | 0.000949 | 0.00094927 | 0.000949272 |
| 0.25 | 10 | 0.00012 | 0.00012019 | 0.00012019 |
| 1 | 0.25 | 0.008131 | 0.00813132 | 0.00813132 |
| 0.25 | 0.25 | 0.008131 | 0.00477464 | 0.00477464 |

Table 2. Asian options on yield: $a = 0.02$, $b = 0.2$, $\sigma^2 = 0.02$ and $K = 0.1$

| Maturity | Type | Strike 0.08 | 0.09 | 0.10 | 0.11 | 0.12 |
|---|---|---|---|---|---|---|
| 5 | $P$ | 0.1061 | 0.2936 | 0.5337 | 0.7427 | 0.8790 |
| | CBA$^C$ | 0.5354 | 0.4130 | 0.2637 | 0.1399 | 0.0631 |
| | TM$^C$ | 0.4607 | 0.3806 | 0.2665 | 0.1571 | 0.0791 |
| | RBA$^C$ | 0.2730 | 0.2208 | 0.1499 | 0.0851 | 0.0410 |
| | AO$^c$ | 0.0118 | 0.0070 | 0.0036 | 0.0016 | 0.0006 |
| 10 | $P$ | 0.0436 | 0.2201 | 0.5255 | 0.7936 | 0.9343 |
| | CBA$^C$ | 0.3504 | 0.2756 | 0.1576 | 0.0634 | 0.0185 |
| | TM$^C$ | 0.9668 | 0.8154 | 0.5248 | 0.2442 | 0.0834 |
| | RBA$^C$ | 0.3495 | 0.2853 | 0.1732 | 0.0747 | 0.0234 |
| | AO$^c$ | 0.0069 | 0.0037 | 0.0016 | 0.0005 | 0.0001 |

Table 3. Higher maturities, Chacko and Das parameters.

| Type | T | .08 | .09 | .10 | .11 | .12 | T | .08 | .09 | .10 | .11 | .12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathcal{G}_0$ | .1 | 50 | 51 | 54 | 57 | 59 | .5 | 16 | 19 | 19 | 21 | 21 |
| $\mathcal{G}_1$ | | 49 | 51 | 53 | 56 | 53 | | 16 | 19 | 19 | 21 | 21 |
| $\not{\mathcal{G}}_0$ | | 49 | 51 | 53 | 57 | 53 | | 16 | 19 | 19 | 21 | 21 |
| $\not{\mathcal{G}}_1$ | | 50 | 51 | 54 | 57 | 59 | | 16 | 20 | 19 | 21 | 21 |
| $\mathcal{G}_0$ | 1 | 15 | 14 | 12 | 14 | 14 | 2 | 11 | 10 | 11 | 10 | 9 |
| $\mathcal{G}_1$ | | 15 | 14 | 12 | 14 | 14 | | 11 | 10 | 11 | 10 | 9 |
| $\not{\mathcal{G}}_0$ | | 15 | 14 | 12 | 15 | 14 | | 11 | 10 | 11 | 10 | 9 |
| $\not{\mathcal{G}}_1$ | | 15 | 14 | 12 | 15 | 14 | | 11 | 10 | 11 | 10 | 9 |
| $\mathcal{G}_0$ | 5 | 9 | 7 | 8 | 7 | 8 | 10 | 8 | 6 | 6 | 6 | 6 |
| $\mathcal{G}_1$ | | 9 | 3 | 8 | 7 | 6 | | 8 | 6 | 6 | 6 | 5 |
| $\not{\mathcal{G}}_0$ | | 9 | 7 | 8 | 7 | 8 | | 8 | 6 | 6 | 6 | 6 |
| $\not{\mathcal{G}}_1$ | | 9 | 7 | 8 | 7 | 6 | | 8 | 6 | 6 | 6 | 6 |

Table 4. Evolution of the speed of convergence with T.

| K | N $\mathcal{G}_0$ | $\mathcal{G}_1$ | $\not{\mathcal{G}}_0$ | $\not{\mathcal{G}}_1$ | Value $P$ | CBA$^C$ | TM$^C$ | RBA$^C$ | AO$^c$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.08 | 8 | 8 | 11 | 11 | 0.3040 | 0.6204 | 0.0803 | 0.0711 | 0.0215 |
| 0.09 | 11 | 10 | 12 | 11 | 0.4308 | 0.5039 | 0.0695 | 0.0612 | 0.0158 |
| 0.1 | 11 | 11 | 11 | 11 | 0.5534 | 0.3924 | 0.0579 | 0.0506 | 0.0114 |
| 0.11 | 10 | 10 | 10 | 10 | 0.6625 | 0.2942 | 0.0464 | 0.0403 | 0.0079 |
| 0.12 | 10 | 10 | 10 | 10 | 0.7533 | 0.2133 | 0.0360 | 0.0310 | 0.0054 |

Table 5. $\sigma = 0.3$, $T = 1$, Chacko and Das parameters.

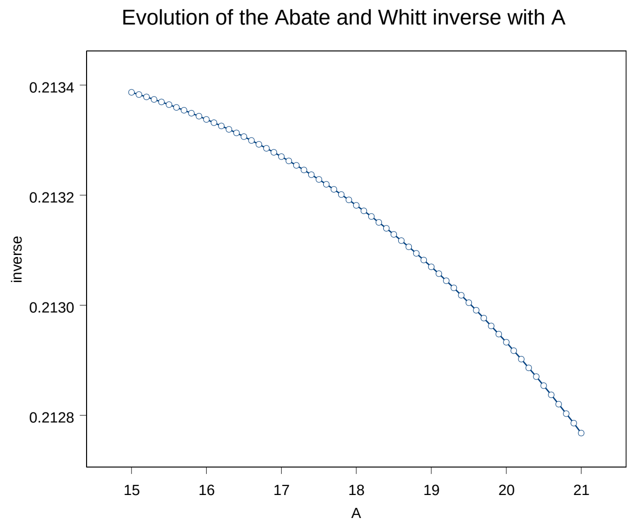


Figure 1. *Parameters:* $r_0 = 0.1$, $a = 0.15$, $b = 1.5$, $\sigma = 0.2$ *and* $T = 10$