On the quantiles of the Brownian motion and their hitting times.

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February 2004

Abstract

The distribution of the $\alpha$-quantile of a Brownian motion on an interval $[0, t]$ has been obtained motivated by a problem in financial mathematics. In this paper we generalise these results by calculating an explicit expression for the joint density of the $\alpha$-quantile of a standard Brownian motion, its first and last hitting times and the value of the process at time $t$. Our results can be easily generalised for a Brownian motion with drift. It is shown that the first and last hitting times follow a transformed arcsine law.

Keywords: Quantiles of Brownian motion, arcsine law, hitting times.

1 Introduction

Let $(X(s), s \geq 0)$ be a real valued stochastic process on a probability space $(\Omega, \mathcal{F}, \text{Pr})$. For $0 < \alpha < 1$, define the $\alpha-$quantile of the path of $(X(s), s \geq 0)$ up to a fixed time $t$ by

$$M_X(\alpha, t) = \inf \left\{ x : \int_0^t 1(X(s) \leq x) \, ds > \alpha t \right\}.$$  \hspace{1cm} (1)

The study of the quantiles of various stochastic processes has been undertaken as a response to a problem arising in the field of mathematical finance, the pricing of a particular path-dependent financial option; see Miura (1992), Akahori (1995) and Dassios (1995). This involves calculating quantities such as $E(h(M_X(\alpha, t)))$, where $h(x) = (e^x - b)^+$ or some other appropriate function. This requires obtaining the distribution of $X(t)$. In the case where $(X(s), s \geq 0)$ is a process with exchangeable increments the following result was obtained:

**Proposition 1** Let $X'(s) = X(\alpha t + s) - X(\alpha t)$. Then,

$$(M_X(\alpha, t), X(t)) \overset{(law)}{=} \left( N_X(\alpha, t), X(\alpha t) + X'((1-\alpha)t) \right),$$ \hspace{1cm} (2)

where

$$N_X(\alpha, t) = \sup_{0 \leq s \leq \alpha t} X(s) + \inf_{0 \leq s \leq (1-\alpha)t} X'(s).$$ \hspace{1cm} (3)
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Note that if \((X (s), s \geq 0)\) is a Lévy process (having stationary and independent increments), then \(X' (s)\) is an independent copy of \(X (s)\).

When \((X (s), s \geq 0)\) is a Brownian motion, we can use this result and obtain an explicit formula for the joint density of \(M_X (\alpha, t)\) and \(X (t)\). This result was first proved for a Brownian motion with drift; see Dassios (1995) and Embrechts, Rogers and Yor (1995) and for Lévy processes by Dassios (1996). There is also a similar result for discrete time random walks first proved by Wendel (1960).

We will demonstrate that Chaumont’s results point to a representation involving \(\alpha\) and \(t\), extending Proposition 1 that are extensions of Wendel’s results in discrete time. In the case where the random walk steps can only take the values +1 or -1, a representation for the analogues of \(L_X (\alpha, t)\) and \(K_X (\alpha, t)\) is obtained. Finally he derives a continuous time representation for the triple law of \(M_X (\alpha, t), L_X (\alpha, t)\) and \(X (t)\), extending Proposition 1 when \(X (t)\) has continuous paths; in particular when it is a Brownian motion. We will demonstrate that Chaumont’s results point to a representation involving \(K_X (\alpha, t)\) as well. We will use this to obtain an explicit form in the last section. We will also derive alternative representations and prove a remarkable arc-sine law.

For the rest of the paper we assume that \((X (s), s \geq 0)\) is a standard Brownian motion, unless otherwise specified. Without loss of generality, we will restrict our attention to the case \(t = 1\) taking advantage of the Brownian scaling. For simplicity we set \(M_X (\alpha, t) = M_X (\alpha), L_X (\alpha, t) = L_X (\alpha)\) and \(K_X (\alpha, t) = K_X (\alpha)\). We will derive the joint density of \((M_X (\alpha), L_X (\alpha), K_X (\alpha), X (1))\). If we denote this density by \(f (y, x, u, v)\), our results can be generalised for a Brownian motion with drift \(m\), using a Cameron-Martin-Girsanov transformation. The corresponding density will be

\[
f (y, x, u, v) \exp (mx - m^2/2).
\]

Before we obtain the density of \((M_X (\alpha), L_X (\alpha), K_X (\alpha), X (1))\), we will first show that the law of \(L_X (\alpha)\) (and \(K_X (\alpha)\)) is a transformed arc-sine law.

### 2 An arcsine law for \(L_X (\alpha, t)\).

Let \(S_X (t) = \sup_{0 \leq s \leq t} \{X (s)\}\) and \(\theta_X (t) = \sup \{s \in [0, t] : X (s) = S_X (t)\}\). Define also the stopping time \(\tau_c = \inf \{s > 0 : X (s) = c\}\). We will first obtain the joint distribution of \((M_X (\alpha), L_X (\alpha))\) (also of \((M_X (\alpha) - X (1), 1 - K_X (\alpha))\)).

**Theorem 1** For \(b > 0\),

\[
\Pr (M_X (\alpha) \in db, L_X (\alpha) \in du) = \\
\Pr (S_X (1) \in db, \theta_X (1) \in du) \mathbf{1} (0 < u < \alpha),
\]

\[\tag{4}\]
and for $b < 0$,
\[
\Pr(M_X(\alpha) \in db, L_X(\alpha) \in du) = \\
\Pr(S_X(1) \in d|b|, \theta_X(1) \in du) \mathbf{1}(0 < u < (1 - \alpha)).
\] (5)

Furthermore, $(M_X(\alpha), L_X(\alpha))$ and $(M_X(\alpha) - X(1), 1 - K_X(\alpha))$
have the same distribution.

**Proof** Let $b > 0$ and $u < \alpha$. We then have that
\[
\Pr(M_X(\alpha) > b, L_X(\alpha) > u) = \Pr(S_X(u) < M_X(\alpha), M_X(\alpha) > b) = \\
\Pr(b < S_X(u) < M_X(\alpha)) + \Pr(S_X(u) < b < M_X(\alpha)).
\] (6)

Let $\tau_b = \inf \{s > 0 : X(s) = b\}$ and $X^*(s) = X(\tau_b + s) - b$. $(X^*(s), s \geq 0)$ is a standard Brownian motion which is independent of $(X(s), 0 \leq s \leq \tau_b)$. We then have,
\[
\Pr(b < S_X(u) < M_X(\alpha)) = \\
\Pr(S_X(u) > b, \int_0^1 \mathbf{1}(X(s) \leq S_X(u)) \, ds < \alpha) = \\
\Pr(S_X(u) > b, \int_0^1 \mathbf{1}(X(s) - X(u) \leq S_X(u) - X(u)) \, ds < \alpha - u).
\] (7)

We now condition on $\sigma \{X(s), 0 \leq s \leq u\}$. Let $X^*(s) = X(u + s) - X(u)$. $(X^*(s), s \geq 0)$ is a standard Brownian motion which is independent of $(X(s), 0 \leq s \leq u)$. We condition on $S_X(u) - X(u) = c$, and set $\tau_c = \inf \{s > 0 : X^*(s) = c\}$ and $X^{**}(s) = X^*(\tau_c + s) - c$. $(X^{**}(s), s \geq 0)$ is a standard Brownian motion which is independent of both $(X(s), 0 \leq s \leq u)$ and $(X^*(s), 0 \leq s \leq \tau_c)$. We have that
\[
\Pr\left(\int_0^{1-u} \mathbf{1}(X^*(s) \leq c) \, ds < \alpha - u\right) = \\
\int_0^{\alpha-u} \Pr(\tau_c \in dr) \Pr\left(\int_0^{1-u-r} \mathbf{1}(X^{**}(s) \leq 0) \, ds < \alpha 1 - u - r\right)
\]
and since $\int_0^{1-u-r} \mathbf{1}(X^{**}(s) \leq 0) \, ds$ has the same (arcsine) law as
$\theta_X^*(1 - u - r)$, this is equal to
\[
\int_0^{\alpha-u} \Pr(\tau_c \in dr) \Pr(\theta_X^*(1 - u - r) < \alpha - u - r) = \\
\int_0^{\alpha-u} \Pr(\tau_c \in dr) \Pr\left(\sup_{0 \leq s \leq \alpha - u - r} X^{**}(s) > \sup_{\alpha - u - r \leq s \leq 1-u-r} X^{**}(s)\right) = \\
\Pr\left(\sup_{0 \leq s \leq \alpha - u} X^*(s) > \sup_{\alpha - u - r \leq s \leq 1-u-r} X^*(s), \sup_{0 \leq s \leq \alpha - u} X^*(s) > c\right)
\]
and so (7) is equal to
\[
\Pr\left(\sup_{0 \leq s \leq \alpha} X(s) - X(u) > \sup_{\alpha \leq s \leq 1} X(s) - X(u), \sup_{0 \leq s \leq \alpha} X(s) - X(u) > \sup_{\alpha \leq s \leq u} X(s) - X(u), \sup_{0 \leq s \leq u} X(s) > b\right) =
\]

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\[ \Pr (S_X (u) > b, u < \theta_X (1) < \alpha) . \]  

(8)

Furthermore,

\[ \Pr (S_X (u) < b < M_X (\alpha)) = \Pr \left( S_X (u) < b, \int_0^1 1 (X (s) \leq b) \, ds < \alpha \right) = \]

\[ \int_u^\alpha \Pr (\tau_b \in dr) \Pr \left( \int_0^{1-r} 1 (X^* (s) \leq 0) < \alpha - r \right) \]

\[ = \int_u^\alpha \Pr (\tau_b \in dr) \Pr (\theta_X (1 - r) < \alpha - r) = \]

\[ \Pr \left( u < \theta_X (1) < \alpha, S_X (u) < b, \sup_{u \leq s \leq \alpha} X (s) > b \right) . \]  

(9)

Adding (8) and (9) together, we see that (6) is equal to

\[ \Pr \left( u < \theta_X (1) < \alpha, \sup_{u \leq s \leq \alpha} X (s) > b \right) = \Pr (u < \theta_X (1) < \alpha, S_X (1) > b) \]

which leads to (4).

Since \((-X (s), s \geq 0)\) is a standard Brownian motion and \(M_{-X} (\alpha) = -M_X (1 - \alpha)\) almost surely, we use \(-X (s)\) instead of \(X (s)\) and we get that for \(b < 0\),

\[ \Pr (M_X (\alpha) < b, L_X (\alpha) > u) = \Pr (u < \theta_X (1) \leq (1 - \alpha), S_X (1) > |b|) , \]

which leads to (5).

To see that \((t - K_X (\alpha), M_X (\alpha) - X (1))\) has the same distribution as \((L_X (\alpha), M_X (\alpha))\), set again \(\tilde{X} (s) = X (1 - s) - X (1)\). Clearly \((\tilde{X} (s), 0 \leq s \leq t)\) is a standard Brownian motion and we can easily see that \(M_{\tilde{X}} (\alpha) = M_X (\alpha) - X (1), M_{\tilde{X}} (\alpha) - \tilde{X} (1) = M_X (\alpha)\) and \(K_{\tilde{X}} (\alpha) = 1 - L_X (\alpha) . \) □

Remarks

1. The distribution of \((\theta_X (1), S_X (1))\) is well known (see for example Karatzas and Shreve (1988, page 102). From this and Theorem 2, we can deduce the density of \((L_X (\alpha), M_X (\alpha))\). This is given by

\[ \Pr (M_X (\alpha) \in db, L_X (\alpha) \in du) = \frac{|b|}{\pi \sqrt{u^3 (1 - u)}} \exp \left( -\frac{b^2}{2u} \right) . \]

\[ [1 (0 < u < \alpha, b > 0) + 1 (0 < u < 1 - \alpha, b < 0)] \, dbdu. \]  

(10)

2. Theorem 1 also leads to an alternative expression for the distribution of \(M_X (\alpha)\); that is

\[ \Pr (M_X (\alpha) \in db) = \Pr (S_X (1) \in db, 0 < \theta_X (1) < \alpha) , \]

for \(b > 0\) and

\[ \Pr (M_X (\alpha) \in db) = \Pr (S_X (1) \in d |b|, 0 < \theta_X (1) < 1 - \alpha) , \]

for \(b < 0\).
3. From Theorem 1, we can immediately obtain the following corollary:

**Corollary 1** For $u > 0$, 

$$
\Pr(L_X(\alpha) > u) = \Pr(u < \theta_X(1) \leq \alpha) + \Pr(u < \theta_X(1) \leq 1 - \alpha) \quad (11)
$$

and

$$
\Pr(L_X(\alpha) \in du) = \frac{1(u \leq \alpha) + 1(u \leq 1 - \alpha)}{\pi \sqrt{u(1 - u)}} du. \quad (12)
$$

Furthermore, $K_X(\alpha)$ has the same distribution as $1 - L_X(\alpha)$.

3 **The joint law of** $(L_X(\alpha), K_X(\alpha), M_X(\alpha), X(1))$.

From now on we will denote the density of $\tau_b$ by $k(\cdot, \cdot)$; that is for $v > 0$,

$$
\Pr(\tau_b \in dv) = k(v, b) dv = \frac{|b|}{\sqrt{2\pi v^3}} \exp\left(-\frac{b^2}{2v}\right) dv. \quad (13)
$$

We will also denote the joint density of $(M_X(\frac{v}{t}, t), X(t))$ by $g(\cdot, \cdot, \cdot, \cdot)$; that is for $0 < v < t$,

$$
\Pr\left(M_X\left(\frac{v}{t}, t\right) \in db, X(t) \in da\right) = g(b, a, v, t) db da.
$$

From proposition 1 this is also the density of

$$
\left(N_X(\alpha, t), X(\alpha t) + X^\prime((1 - \alpha)t)\right),
$$

where $N_X(\alpha, t)$ is defined by (3).

We can calculate $g(\cdot, \cdot, \cdot, \cdot)$ by using proposition 1. Note that

$$
\inf_{0 \leq s \leq (1-\alpha)t} X^\prime(s) = -\sup_{0 \leq s \leq (1-\alpha)t} (-X^\prime(s))
$$

and that the density of $(S_X(t), X(t))$ is given by

$$
\Pr(S_X(t) \in db, X(t) \in da) = \frac{2(2b - a)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2b - a)^2}{2t}\right) \mathbf{1}(b \geq 0, b \geq a) dadb \quad (14)
$$

(see Karatzas and Shreve, 1988, p.95). We observe that since (14) is bounded, $g(\cdot, \cdot, \cdot, \cdot)$ is a bounded density. We first need to calculate $g(0, 0, v, t)$. This is the same as the value of the density of $(M_X(\frac{v}{t}, t), M_X(\frac{v}{t}, t) - X(t))$ at $(0, 0)$. From (14) we see that

$$
\Pr(S_X(t) \in dy, S_X(t) - X(t) \in dx) = \frac{2(y + x)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(y + x)^2}{2t}\right) \mathbf{1}(y \geq 0, x \geq 0) dy dx \quad (15)
$$

and it is a simple exercise to verify that
Let Proposition 2 of Chaumont (1999) (see Theorem 7 and the remark after Theorem 4 in his paper): which leads to (17).

\[ g(0,0,v,t) = \int_0^\infty \int_0^\infty \frac{2(y+x)}{\sqrt{2\pi v^3}} \exp \left( -\frac{(y+x)^2}{2v} \right) \frac{2(y+x)}{\sqrt{2\pi (t-v)^3}} \exp \left( -\frac{(y+x)^2}{2(t-v)} \right) \, dx \, dy = \frac{4\sqrt{v(t-v)}}{\pi t^2}. \]  

(16)

We will also use the following lemma

Lemma 1 Let \( (X(s), s \geq 0) \) be a standard Brownian motion, \( \tau_x = \inf \{ s > 0 : X(s) = x \} \) and \( \tau_y = \sup \{ s \leq t : X(s) = y \} \). Then, for \( 0 < x < z \) and \( w < y < z \),

\[ \Pr (\tau_x \in du, \tau_y \in dv, S_X(t) \in dz, X(t) \in dw) = k(u,x)k(t-v,y-w)\Pr (S_X(v-u) \in d(z-y), X(v-u) \in d(x-y)) \]  

(17)

Proof Using the strong Markov property as in the previous section, we see that the right hand side of (17) is equal to

\[ \Pr (\tau_x \in du) \Pr (\tau_{y-x} \in d(v-u), S_X(t-u) \in d(z-x), X(t-u) \in d(w-x)) \]

and replacing \( X(s) \) by the standard Brownian motion \( X(t-u-s) - X(t-u) \) this is equal to

\[ \Pr (\tau_x \in du) \Pr (\tau_{y-w} \in d(1-v), S_X(t-u) \in d(z-w), X(t-u) \in d(x-w)) \]

which leads to (17). \( \square \)

The following extension to Proposition 1 can be derived as a direct consequence of the results of Chaumont (1999) (see Theorem 7 and the remark after Theorem 4 in his paper):

Proposition 2 Let \( (X(s), s \geq 0) \) be a continuous process with exchangeable increments and \( X'(s) = X(\alpha + s) - X(\alpha) \). Then,

\[ (L_X(\alpha), K_X(\alpha), M_X(\alpha), X(1)) \overset{(law)}{=} (T_X(\alpha), U_X(\alpha), N_X(\alpha), X(\alpha) + X'(1-\alpha)), \]  

(18)

where

\[ T_X(\alpha) = \inf\{ s \geq 0 : X(s) = N_X(\alpha) \} \mathbf{1}(N_X(\alpha) \geq 0) + \inf\{ s \geq 0 : X'(s) = N_X(\alpha) \} \mathbf{1}(N_X(\alpha) \leq 0) \]

and

\[ U_X(\alpha) = \left( 1 - \alpha + \sup\{ s \leq \alpha : X(s) = N_X(\alpha) - X'(1-\alpha) \} \right) \mathbf{1}(N_X(\alpha) \geq X(\alpha) + X'(1-\alpha)) + \left( \alpha + \sup\{ s \leq 1 - \alpha : X'(s) = N_X(\alpha) - X(\alpha) \} \right) \mathbf{1}(N_X(\alpha) \leq X(\alpha) + X'(1-\alpha)) \]

Note that the expression for \( U_X(\alpha) \) is a slight modification of the one in Chaumont’s paper that better serves our purpose. We now deduce the law of \( (L_X(\alpha), K_X(\alpha), M_X(\alpha), X(1)) \).
Theorem 2 For the standard Brownian motion \((X(s), s \geq 0)\),

\[
\Pr(L_X(\alpha) \in du, K_X(\alpha) \in dv, M_X(\alpha) \in db, X(1) \in da) = \frac{2|b|b - a|dudvdbda}{\pi^2(v - u)^2 \sqrt{u^3(1 - v)^3}} \exp\left(-\frac{b^2}{2u} - \frac{(b - a)^2}{2(1 - v)}\right) \times \begin{cases} 
\sqrt{(u - v - (1 - \alpha))(1 - \alpha)}I(u > 0, u + (1 - \alpha) < v < 1) & b > 0, b > a \\
\sqrt{\alpha - u}(v - \alpha)I(0 < u < \alpha < v < 1) & b > 0, b < a \\
\sqrt{v - u - \alpha}I(u > 0, u + \alpha < v < 1) & b < 0, b > a \\
\sqrt{(1 - \alpha - u)(v - (1 - \alpha))}I(0 < u < 1 - \alpha < v < 1) & b < 0, b < a.
\end{cases}
\]

(19)

Proof We start with the case \(b > 0, b > a\); we use Proposition 2 and Lemma 1 with \(z = b - \inf_{0 \leq s \leq (1 - \alpha)\tau} X'(s), w = a - X'(1 - \alpha), x = b\) and \(y = b - X'(1 - \alpha)\). This leads to

\[
k(b, u)k(b - a, 1 - v)g(0, 0, v - u - (1 - \alpha), v - u)1(u > 0, u + (1 - \alpha)t < v < t)dudvdbda.
\]

(20)

Substituting (13) and (15) in (20), we get the first leg of the right hand side of (19). For the case \(b > 0, b < a\), note that we can rewrite \(U_X(\alpha)\) in Proposition 2 as

\[
U_X(\alpha) = 1 - \inf\{s \geq 0 : X''(s) = \sup_{0 \leq s \leq \alpha} X(s) + \inf_{0 \leq s \leq 1 - \alpha} X''(s) - X(\alpha)\},
\]

where \(X''(s) = X'(1 - \alpha - s) - X'(1 - \alpha)\). The left hand side of (19) is then the density of

\[
\inf\{s \geq 0 : X(s) = b\}, \inf\{s \geq 0 : X''(s) = b - a\}, \sup_{0 \leq s \leq \alpha} X(s) + \inf_{0 \leq s \leq 1 - \alpha} X''(s) - X''(1 - \alpha), X(\alpha) - X''(1 - \alpha)
\]

at \((u, 1 - v, b, a)\). This in turn is equal to \(k(b, u)k(|b - a|, 1 - v)\) multiplied by the density of

\[
\sup_{0 \leq s \leq (\alpha - u)} X(s) + \inf_{0 \leq s \leq (v - \alpha)} X''(s) - X''(v - \alpha), X(\alpha - u) - X''(v - \alpha)
\]

which leads to the second leg of the right hand side of (19).

Considering the process \((-X(s), 0 \leq s \leq 1)\) and observing that \(M_{-X}(\alpha) = -M_X(1 - \alpha)\), \(L_{-X}(\alpha) = L_X(1 - \alpha)\) and \(K_{-X}(\alpha) = K_X(1 - \alpha)\) yields the rest of (19).

Remark
One could derive Theorem 1 from Theorem 2, by integrating out two variables. However, it is difficult to obtain the result, without knowing it in advance.

Acknowledgement
The author wishes to thank an anonymous referee for many useful comments.
References


