On the quantiles of the Brownian motion and their hitting times.

Angelos Dassios

Department of Statistics, London School of Economics
Houghton Street, London WC2A 2AE, United Kingdom
email: A.Dassios@lse.ac.uk

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Abstract

The distribution of the α-quantile of a Brownian motion on an interval $[0, t]$ has been obtained motivated by a problem in financial mathematics. In this paper we generalise these results by calculating an explicit expression for the joint density of the α-quantile of a standard Brownian motion, its first and last hitting times and the value of the process at time t. Our results can be easily generalised for a Brownian motion with drift. It is shown that the first and last hitting times follow a transformed arcsine law.
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1 Introduction

Let $(X(s), s \geq 0)$ be a real valued stochastic process on a probability space $(\Omega, \mathcal{F}, \Pr)$. For $0 < \alpha < 1$, define the $\alpha-$quantile of the path of $(X(s), s \geq 0)$ up to a fixed time t by

$$M_X(\alpha, t) = \inf\left\{x : \int_0^t \mathbf{1}(X(s) \leq x)\, ds > \alpha t\right\}. \tag{1}$$

The study of the quantiles of various stochastic processes has been undertaken as a response to a problem arising in the field of mathematical finance, the pricing of a particular path-dependent financial option; see Miura (1992), Akahori (1995) and Dassios (1995). This involves calculating quantities such as $E(h(M_X(\alpha, t)))$, where $h(x) = (e^x - b)^+$ or some other appropriate function. This requires obtaining the distribution of $X(t)$. In the case where $(X(s), s \geq 0)$ is a process with exchangeable increments the following result was obtained:

Proposition 1 *Let $X'(s) = X(\alpha t + s) - X(\alpha t)$. Then,*

$$(M_X(\alpha, t), X(t)) \overset{(law)}{=} \left(N_X(\alpha, t), X(\alpha t) + X'((1-\alpha)t)\right), \tag{2}$$

where

$$N_X(\alpha, t) = \sup_{0 \leq s \leq \alpha t} X(s) + \inf_{0 \leq s \leq (1-\alpha)t} X'(s). \tag{3}$$

Note that if $(X(s), s \geq 0)$ is a Lévy process (having stationary and independent increments), then $X'(s)$ is an independent copy of $X(s)$.

When $(X(s), s \geq 0)$ is a Brownian motion, we can use this result and obtain an explicit formula for the joint density of $M_X(\alpha, t)$ and $X(t)$. This result was first proved for a Brownian motion with drift; see Dassios (1995) and Embrechts, Rogers and Yor (1995) and for Lévy processes by Dassios (1996). There is also a similar result for discrete time random walks first proved by Wendel (1960).

We now let

$$L_X(\alpha, t) = \inf\{s \in [0, t] : X(s) = M_X(\alpha, t)\}$$

be the first, and

$$K_X(\alpha, t) = \sup\{s \in [0, t] : X(s) = M_X(\alpha, t)\},$$

the last time the process hits $M_X(\alpha, t)$. One can now introduce a 'barrier' element to the financial application by making the option worthless if the quantile is hit too early or too late. For example, this can involve calculating quantities such as $E(h(M_X(\alpha, t)) \mathbf{1}(L_X(\alpha, t) > v, K_X(\alpha, t) < u))$.

The first study of these quantities can be found in Chaumont (1999). By using combinatorial arguments he derives results of the same type as Proposition 1 that are extensions of Wendel's results in discrete time. In the case where the random walk steps can only take the values +1 or -1, a representation for the analogues of $L_X(\alpha, t)$ and $K_X(\alpha, t)$ is obtained. Finally he derives a continuous time representation for the triple law of $M_X(\alpha, t)$, $L_X(\alpha, t)$ and $X(t)$, extending Proposition 1 when $X(t)$ has continuous paths; in particular when it is a Brownian motion. We will demonstrate that Chaumont's results point to a representation involving $K_X(\alpha, t)$ as well. We will use this to obtain an explicit form in the last section. We will also derive alternative representations and prove a remarkable arc-sine law.

For the rest of the paper we assume that $(X(s), s \geq 0)$ is a standard Brownian motion, unless otherwise specified. Without loss of generality, we will restrict our attention to the case $t = 1$ taking advantage of the Brownian scaling. For simplicity we set $M_X(\alpha, t) = M_X(\alpha)$, $L_X(\alpha, t) = L_X(\alpha)$ and $K_X(\alpha, t) = K_X(\alpha)$.We will derive the joint density of $M_X(\alpha)$, $L_X(\alpha)$, $K_X(\alpha)$ and $X(1)$. If we denote this density by $f(y, x, u, v)$, our results can be generalised for a Brownian motion with drift m, using a Cameron-Martin-Girsanov transformation. The corresponding density will be

$$f(y, x, u, v) \exp\left(mx - m^2/2\right).$$

Before we obtain the density of $(M_X(\alpha), L_X(\alpha), K_X(\alpha), X(1))$, we will first show that the law of $L_X(\alpha)$ (and $K_X(\alpha)$) is a transformed arcsine law.

2 An arcsine law for $L_X(\alpha, t)$.

Let $S_X(t) = \sup_{0 \leq s \leq t}\{X(s)\}$ and $\theta_X(t) = \sup\{s \in [0, t] : X(s) = S_X(t)\}$. Define also the stopping time $\tau_c = \inf\{s > 0 : X(s) = c\}$. We will first obtain the joint distribution of $(M_X(\alpha), L_X(\alpha))$ (also of $(M_X(\alpha) - X(1), 1 - K_X(\alpha))$.

Theorem 1 *For $b > 0$,*

$$\Pr(M_X(\alpha) \in db, L_X(\alpha) \in du) =$$
$$\Pr(S_X(1) \in db, \theta_X(1) \in du) \, \mathbf{1}(0 < u < \alpha), \tag{4}$$

and for $b < 0$,

$$\Pr\left(M_X(\alpha) \in db, L_X(\alpha) \in du\right) =$$
$$\Pr\left(S_X(1) \in d|b|, \theta_X(1) \in du\right) \mathbf{1}\left(0 < u < (1-\alpha)\right). \quad (5)$$

Furthermore, $(M_X(\alpha), L_X(\alpha))$ *and* $(M_X(\alpha) - X(1), 1 - K_X(\alpha))$ *have the same distribution.*

Proof Let $b > 0$ and $u < \alpha$. We then have that

$$\Pr\left(M_X(\alpha) > b, L_X(\alpha) > u\right) = \Pr\left(S_X(u) < M_X(\alpha), M_X(\alpha) > b\right) =$$
$$\Pr\left(b < S_X(u) < M_X(\alpha)\right) + \Pr\left(S_X(u) < b < M_X(\alpha)\right). \quad (6)$$

Let $\tau_b = \inf\{s > 0 : X(s) = b\}$ and $X^*(s) = X(\tau_b + s) - b$. $(X^*(s), s \geq 0)$ is a standard Brownian motion which is independent of $(X(s), 0 \leq s \leq \tau_b)$. We then have,

$$\Pr\left(b < S_X(u) < M_X(\alpha)\right) =$$
$$\Pr\left(S_X(u) > b, \int_0^1 \mathbf{1}\left(X(s) \leq S_X(u)\right) ds < \alpha\right) =$$
$$\Pr\left(S_X(u) > b, \int_u^1 \mathbf{1}\left(X(s) - X(u) \leq S_X(u) - X(u)\right) ds < \alpha - u\right). \quad (7)$$

We now condition on $\sigma\{X(s), 0 \leq s \leq u\}$. Let $X^*(s) = X(u+s) - X(u)$. $(X^*(s), s \geq 0)$ is a standard Brownian motion which is independent of $(X(s), 0 \leq s \leq u)$. We condition on $S_X(u) - X(u) = c$, and set $\tau_c = \inf\{s > 0 : X^*(s) = c\}$ and $X^{**}(s) = X^*(\tau_c + s) - c$. $(X^{**}(s), s \geq 0)$ is a standard Brownian motion which is independent of both $(X(s), 0 \leq s \leq u)$ and $(X^*(s), 0 \leq s \leq \tau_c)$. We have that

$$\Pr\left(\int_0^{1-u} \mathbf{1}\left(X^*(s) \leq c\right) ds < \alpha - u\right) =$$
$$\int_0^{\alpha-u} \Pr\left(\tau_c \in dr\right) \Pr\left(\int_0^{1-u-r} \mathbf{1}\left(X^{**}(s) \leq 0\right) ds < \alpha 1 - u - r\right)$$

and since $\int_0^{1-u-r} \mathbf{1}\left(X^{**}(s) \leq 0\right) ds$ has the same (arcsine) law as $\theta_{X^{**}}(1-u-r)$, this is equal to

$$\int_0^{\alpha-u} \Pr\left(\tau_c \in dr\right) \Pr\left(\theta_{X^{**}}(1-u-r) < \alpha - u - r\right) =$$
$$\int_0^{\alpha-u} \Pr\left(\tau_c \in dr\right) \Pr\left(\sup_{0 \leq s \leq \alpha-u-r} X^{**}(s) > \sup_{\alpha-u-r \leq s \leq 1-u-r} X^{**}(s)\right) =$$
$$\Pr\left(\sup_{0 \leq s \leq \alpha-u} X^*(s) > \sup_{\alpha-u \leq s \leq t-u} X^*(s), \sup_{0 \leq s \leq \alpha-u} X^*(s) > c\right)$$

and so (7) is equal to

$$\Pr\begin{pmatrix} \sup_{u \leq s \leq \alpha} X(s) - X(u) > \sup_{\alpha \leq s \leq 1} X(s) - X(u), \\ \sup_{u \leq s \leq \alpha} X(s) - X(u) > \sup_{0 \leq s \leq u} X(s) - X(u), \\ \sup_{0 \leq s \leq u} X(s) > b \end{pmatrix} =$$

$$\Pr\left(S_X(u) > b, u < \theta_X(1) < \alpha\right). \tag{8}$$

Furthermore,

$$\Pr\left(S_X(u) < b < M_X(\alpha)\right) = \Pr\left(S_X(u) < b, \int_0^1 \mathbf{1}\left(X(s) \le b\right) ds < \alpha\right) =$$

$$\int_u^\alpha \Pr\left(\tau_b \in dr\right) \Pr\left(\int_0^{1-r} \mathbf{1}\left(X^*(s) \le 0\right) < \alpha - r\right)$$

$$= \int_u^\alpha \Pr\left(\tau_b \in dr\right) \Pr\left(\theta_{X^*}(1-r) < \alpha - r\right) =$$

$$\Pr\left(u < \theta_X(1) < \alpha, S_X(u) < b, \sup_{u \le s \le \alpha} X(s) > b\right). \tag{9}$$

Adding (8) and (9) together, we see that (6) is equal to

$$\Pr\left(u < \theta_X(1) < \alpha, \sup_{u \le s \le \alpha} X(s) > b\right) = \Pr\left(u < \theta_X(1) < \alpha, S_X(1) > b\right)$$

which leads to (4).

Since $(-X(s), s \ge 0)$ is a standard Brownian motion and $M_{-X}(\alpha) = -M_X(1-\alpha)$ almost surely, we use $-X(s)$ instead of $X(s)$ and we get that for $b < 0$,

$$\Pr\left(M_X(\alpha) < b, L_X(\alpha) > u\right) = \Pr\left(u < \theta_X(1) \le (1-\alpha), S_X(1) > |b|\right),$$

which leads to (5).

To see that $(t - K_X(\alpha), M_X(\alpha) - X(1))$ has the same distribution as $(L_X(\alpha), M_X(\alpha))$, set again $\tilde{X}(s) = X(1-s) - X(1)$. Clearly $\left(\tilde{X}(s), 0 \le s \le t\right)$ is a standard Brownian motion and we can easily see that $M_{\tilde{X}}(\alpha) = M_X(\alpha) - X(1)$, $M_{\tilde{X}}(\alpha) - \tilde{X}(1) = M_X(\alpha)$ and $K_{\tilde{X}}(\alpha) = 1 - L_X(\alpha)$. □

Remarks

1. The distribution of $(\theta_X(1), S_X(1))$ is well known (see for example Karatzas and Shreve (1988, page 102). From this and Theorem 2, we can deduce the density of $(L_X(\alpha), M_X(\alpha))$. This is given by

$$\Pr\left(M_X(\alpha) \in db, L_X(\alpha) \in du\right) = \frac{|b|}{\pi\sqrt{u^3(1-u)}} \exp\left(-\frac{b^2}{2u}\right) \cdot$$

$$\left[\mathbf{1}\left(0 < u < \alpha, b > 0\right) + \mathbf{1}\left(0 < u < 1-\alpha, b < 0\right)\right] db du. \tag{10}$$

2. Theorem 1 also leads to an alternative expression for the distribution of $M_X(\alpha)$; that is

$$\Pr\left(M_X(\alpha) \in db\right) = \Pr\left(S_X(1) \in db, 0 < \theta_X(1) < \alpha\right),$$

for $b > 0$ and

$$\Pr\left(M_X(\alpha) \in db\right) = \Pr\left(S_X(1) \in d|b|, 0 < \theta_X(1) < 1-\alpha\right),$$

for $b < 0$.

3. From Theorem 1, we can immediately obtain the following corollary:

Corollary 1 *For $u > 0$,*

$$\Pr\left(L_X(\alpha) > u\right) = \Pr\left(u < \theta_X(1) \leq \alpha\right) + \Pr\left(u < \theta_X(1) \leq 1-\alpha\right) \tag{11}$$

and

$$\Pr\left(L_X(\alpha) \in du\right) = \frac{\mathbf{1}\left(u \leq \alpha\right) + \mathbf{1}\left(u \leq 1-\alpha\right)}{\pi\sqrt{u\left(1-u\right)}} du. \tag{12}$$

Furthermore, $K_X(\alpha)$ has the same distribution as $1 - L_X(\alpha)$.

3 The joint law of $\left(L_X(\alpha), K_X(\alpha), M_X(\alpha), X(1)\right)$.

From now on we will denote the density of τ_b by $k(\cdot,\cdot)$; that is for $v > 0$,

$$\Pr\left(\tau_b \in dv\right) = k\left(v,b\right) dv = \frac{|b|}{\sqrt{2\pi v^3}} \exp\left(-\frac{b^2}{2v}\right) dv. \tag{13}$$

We will also denote the joint density of $\left(M_X\left(\frac{v}{t},t\right), X(t)\right)$ by $g(\cdot,\cdot,\cdot,\cdot)$; that is for $0 < v < t$,

$$\Pr\left(M_X\left(\frac{v}{t},t\right) \in db, X(t) \in da\right) = g\left(b,a,v,t\right) dbda.$$

From proposition 1 this is also the density of

$$\left(N_X(\alpha,t), X(\alpha t) + X'\left(\left(1-\alpha t\right)\right)\right),$$

where $N_X(\alpha,t)$ is defined by (3).

We can calculate $g(\cdot,\cdot,\cdot,\cdot)$ by using proposition 1. Note that

$$\inf_{0\leq s\leq(1-\alpha)t} X'(s) = -\sup_{0\leq s\leq(1-\alpha)t}\left(-X'(s)\right)$$

and that the density of $\left(S_X(t), X(t)\right)$ is given by

$$\Pr\left(S_X(t) \in db, X(t) \in da\right) =$$

$$\frac{2\left(2b-a\right)}{\sqrt{2\pi t^3}} \exp\left(-\frac{\left(2b-a\right)^2}{2t}\right) \mathbf{1}\left(b \geq 0, b \geq a\right) dadb \tag{14}$$

(see Karatzas and Shreve, 1988, p.95). We observe that since (14) is bounded, $g(\cdot,\cdot,\cdot,\cdot)$ is a bounded density. We first need to calculate $g(0,0,v,t)$. This is the same as the value of the density of $\left(M_X\left(\frac{v}{t},t\right), M_X\left(\frac{v}{t},t\right) - X(t)\right)$ at $(0,0)$. From (14) we see that

$$\Pr\left(S_X(t) \in dy, S_X(t) - X(t) \in dx\right) =$$

$$\frac{2\left(y+x\right)}{\sqrt{2\pi t^3}} \exp\left(-\frac{\left(y+x\right)^2}{2t}\right) \mathbf{1}\left(y \geq 0, x \geq 0\right) dydx \tag{15}$$

and it is a simple exercise to verify that

$$g(0,0,v,t) = \int_0^\infty \int_0^\infty \frac{2(y+x)}{\sqrt{2\pi v^3}} \exp\left(-\frac{(y+x)^2}{2v}\right) \frac{2(y+x)}{\sqrt{2\pi(t-v)^3}} \exp\left(-\frac{(y+x)^2}{2(t-v)}\right) dxdy = \frac{4\sqrt{v(t-v)}}{\pi t^2}. \tag{16}$$

We will also use the following lemma

Lemma 1 *Let* $(X(s), s \geq 0)$ *be a standard Brownian motion,* $\tau_x = \inf\{s > 0 : X(s) = x\}$ *and* $\underline{\tau}_y = \sup\{s \leq t : X(s) = y\}$. *Then, for* $0 < x < z$ *and* $w < y < z$,

$$\Pr\left(\tau_x \in du, \underline{\tau}_y \in dv, S_X(t) \in dz, X(t) \in dw\right) = k(u,x)\, k(t-v, y-w) \Pr\left(S_X(v-u) \in d(z-y), X(v-u) \in d(x-y)\right) \tag{17}$$

Proof Using the strong Markov property as in the previous section, we see that the right hand side of (17) is equal to

$$\Pr(\tau_x \in du) \Pr\left(\underline{\tau}_{y-x} \in d(v-u), S_X(t-u) \in d(z-x), X(t-u) \in d(w-x)\right)$$

and replacing $X(s)$ by the standard Brownian motion $X(t-u-s) - X(t-u)$ this is equal to

$$\Pr(\tau_x \in du) \Pr\left(\tau_{y-w} \in d(1-v), S_X(t-u) \in d(z-w), X(t-u) \in d(x-w)\right)$$

which leads to (17). $\square$

The following extension to Proposition 1 can be derived as a direct consequence of the results of Chaumont (1999) (see Theorem 7 and the remark after Theorem 4 in his paper):

Proposition 2 *Let* $(X(s), s \geq 0)$ *be a continuous process with exchangeable increments and* $X'(s) = X(\alpha + s) - X(\alpha)$. *Then,*

$$(L_X(\alpha), K_X(\alpha), M_X(\alpha), X(1)) \overset{(law)}{=} \left(T_X(\alpha), U_X(\alpha), N_X(\alpha), X(\alpha) + X'(1-\alpha)\right), \tag{18}$$

where

$$T_X(\alpha) = \inf\{s \geq 0 : X(s) = N_X(\alpha)\}\, \mathbf{1}(N_X(\alpha) \geq 0) + \inf\{s \geq 0 : X'(s) = N_X(\alpha)\}\, \mathbf{1}(N_X(\alpha) \leq 0)$$

and

$$U_X(\alpha) = \left(1 - \alpha + \sup\{s \leq \alpha : X(s) = N_X(\alpha) - X'(1-\alpha)\}\right) \mathbf{1}\left(N_X(\alpha) \geq X(\alpha) + X'(1-\alpha)\right) + \left(\alpha + \sup\{s \leq 1-\alpha : X'(s) = N_X(\alpha) - X(\alpha)\}\right) \mathbf{1}\left(N_X(\alpha) \leq X(\alpha) + X'(1-\alpha)\right)$$

Note that the expression for $U_X(\alpha)$ is a slight modification of the one in Chaumont's paper that better serves our purpose. We now deduce the law of $(L_X(\alpha), K_X(\alpha), M_X(\alpha), X(1))$.

Theorem 2 *For the standard Brownian motion* $(X(s), s \geq 0)$,

$$\Pr\left(L_X(\alpha) \in du, K_X(\alpha) \in dv, M_X(\alpha) \in db, X(1) \in da\right) =$$
$$\frac{2|b||b-a|\,dudvdbda}{\pi^2 (v-u)^2 \sqrt{u^3(1-v)^3}} \exp\left(-\frac{b^2}{2u} - \frac{(b-a)^2}{2(1-v)}\right) \times$$
$$\left\{ \begin{array}{ll} \sqrt{(v-u-(1-\alpha))(1-\alpha)}\,\mathbf{1}(u>0, u+(1-\alpha)<v<1) & b>0, b>a \\ \sqrt{(\alpha-u)(v-\alpha)}\,\mathbf{1}(0<u<\alpha<v<1) & b>0, b<a \\ \sqrt{(v-u-\alpha)\alpha}\,\mathbf{1}(u>0, u+\alpha<v<1) & b<0, b>a \\ \sqrt{(1-\alpha-u)(v-(1-\alpha))}\,\mathbf{1}(0<u<1-\alpha<v<1) & b<0, b<a \end{array} \right. . \tag{19}$$

Proof We start with the case $b > 0, b > a$; we use Proposition 2 and Lemma 1 with $z = b - \inf_{0 \leq s \leq (1-\alpha)t} X'(s)$, $w = a - X'(1-\alpha)$, $x = b$ and $y = b - X'(1-\alpha)$. This leads to

$$k(b,u)\,k(b-a, 1-v)\,g(0, 0, v-u-(1-\alpha), v-u) \cdot$$
$$\mathbf{1}(u>0, u+(1-\alpha)t<v<t)\,dudvdbda. \tag{20}$$

Substituting (13) and (15) in (20), we get the first leg of the right hand side of (19). For the case $b > 0, b < a$, note that we can rewrite $U_X(\alpha)$ in Proposition 2 as

$$U_X(\alpha) = 1 - \inf\{s \geq 0 : X''(s) = \sup_{0 \leq s \leq \alpha} X(s) + \inf_{0 \leq s \leq\ 1-\alpha} X''(s) - X(\alpha)\},$$

where $X''(s) = X'(1-\alpha-s) - X'(1-\alpha)$. The left hand side of (19) is then the density of

$$\left(\begin{array}{c} \inf\{s \geq 0 : X(s) = b\}, \inf\{s \geq 0 : X''(s) = b-a\}, \\ \sup_{0 \leq s \leq \alpha} X(s) + \inf_{0 \leq s \leq 1-\alpha} X''(s) - X''(1-\alpha), X(\alpha) - X''(1-\alpha) \end{array} \right)$$

at $(u, 1-v, b, a)$. This in turn is equal to $k(b,u)\,k(|b-a|, 1-v)$ multiplied by the density of

$$\left(\sup_{0 \leq s \leq (\alpha-u)} X(s) + \inf_{0 \leq s \leq (v-\alpha)} X''(s) - X''(v-\alpha), X(\alpha-u) - X''(v-\alpha) \right),$$

which leads to the second leg of the right hand side of (19).

Considering the process $(-X(s), 0 \leq s \leq 1)$ and observing that $M_{-X}(\alpha) = -M_X(1-\alpha)$, $L_{-X}(\alpha) = L_X(1-\alpha)$ and $K_{-X}(\alpha) = K_X(1-\alpha)$ yields the rest of (19). □

Remark
One could derive Theorem 1 from Theorem 2 , by integrating out two variables. However, it is difficult to obtain the result, without knowing it in advance.

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