Quantiles of Lévy processes and applications in finance.

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Abstract

This paper provides a survey of results on the quantiles of a Brownian motion with drift as well as a general Lévy process. The motivation is to calculate the price of related financial options. At the end of the paper some new results on variability orderings between various quantities associated with path dependent and European options are presented. This survey is not exhaustive, but intends to provide a flavour of research carried out in the area.

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1 Path dependent options

A path dependent (look-back) option is a statistic which is a functional of the path of the stochastic process $(Y(t), t \geq 0)$ that denotes the price of an underlying asset. Pricing such options involves calculating $E^*(h(V(t_2)) \mid \mathcal{F}_{t_1})$, where the expectation is calculated under a changed measure, h is a known function, $0 \leq t_1 < t_2$ are fixed times, $\mathcal{F}_t$ is the filtration generated by $Y(t)$ and $V(t)$ is an $\mathcal{F}_t$-measurable process.

For example the price of a call option of this kind would be

$$e^{-r(t_2-t_1)}E^*\Big((V(t_2)-b)^+ \mid \mathcal{F}_{t_1}\Big) = e^{-r(t_2-t_1)}E^*(\max(V(t_2)-b,0) \mid \mathcal{F}_{t_1}).$$

There is also the possibility of a floating-strike call that has price

$$e^{-r(t_2-t_1)}E^*\Big((X(t_2)-V(t_2))^+ \mid \mathcal{F}_{t_1}\Big) = e^{-r(t_2-t_1)}E^*(\max(X(t_2)-V(t_2),0) \mid \mathcal{F}_{t_1}).$$

One can similarly write down the prices for put and other types of options. From now on we will drop the distinction between the physical measure and the risk-neutral valuation equivalent measure. We will assume that it just changes the values of the parameters but not the structure of the price process. This is true when the price follows a geometric Brownian motion and the market is complete. We will also for simplicity set $t_1=0$ and concentrate on the calculation of

$$E\Big((V(t)-b)^+\Big) \tag{1}$$

and

$$E\Big((Y(t)-V(t))^+\Big). \tag{2}$$

These clearly require obtaining the distribution of $V(t)$ and in the second case the joint distribution of $V(t)$ and $X(t)$.

We will denote the price process as $Y(t) = Y(0)\exp\,(X(t))$; $X(t)$ is therefore playing the role of the logarithm of the price and follows a Brownian motion with drift in the classical case.

Here are some examples.

1. Arithmetic average options, where
$$V(t)=\frac{\int_0^t Y(s)d\,s}{t}$$
are also called Asian options.
2. It is mathematically simpler to look at the geometric average option. In this case,
$$V(t)=Y(0)\exp\left(\frac{\int_0^t X(s)d\,s}{t}\right).$$
3. We will also look at quantile options. Another statistic that can be used is the median or more generally any α-quantile $(0<\alpha<1)$ of the underlying stochastic process. This was first introduced by Miura . The median sounds rather appealing as it should be similar in some respects to the average, but it has a nice property seen below.The α-quantile is going to be the level at which the process spends a proportion of size at least α of its time below that level and a proportion of size at least $1-\alpha$ above. For $0<\alpha<1$, define $M_X(\alpha,t)$ as
$$M_X(\alpha,t)=\inf\left\{x\colon \int_0^t 1(X(s)\leq x)d\,s>\alpha t\right\}.$$
Note $Y(t) = Y(0)\exp\,(X(t))$; so $M_Y(\alpha,\,t) = Y(0)\exp\,(M_X(\alpha,\,t))$. So the study of the quantiles of $Y(t)$ is equivalent to the study of the quantiles. Also the events $\{M_X(\alpha,\,t) > x\}$ and $\left\{\int_0^t 1(X(s) \leq x)d\,s < \alpha t\right\}$ are identical. This option has not appeared on trading floors very much, but hopefully this will change.

2 Quantile options

As already mentioned these were introduced by Miura in 1992. We repeat the definition of a quantile. For $0<\alpha<1$, define $M_X(\alpha,t)$ as

$$M_X(\alpha,t)=\inf\left\{x\colon \int_0^t 1(X(s)\leq x)d\,s>\alpha t\right\}.$$

Note $Y(t)=Y(0)\exp\,(X(t))$; so $M_Y(\alpha,t)=Y(0)\exp\,(M_X(\alpha,t))$. So the study of the quantiles of $Y(t)$ is equivalent to the study of the quantiles. This is a crucial and important property that makes these options attractive from a mathematics point of view. The fact that the events $\{M_X(\alpha,t)>x\}$ and $\left\{\int_0^t 1(X(s)\leq x)d\,s<\alpha t\right\}$ are identical is the starting point. Note that the second event involves the occupation time of the process (the time it spends below the level x). This means that results on quantile options can be used on options involving occupation times. The following result was originally proved by AD in 1995,

Proposition 1. *Let $X(t)=\sigma B(t)+\mu t$, where $\mu\in R$, $\sigma\in R^+$ and $(B(t),t\geq 0)$ is a standard Brownian motion. Furthermore, let $X^{(1)}(t)$ and $X^{(2)}(t)$ be independent copies of $X(t)$. Then,*

$$M_X(\alpha,t)\overset{(law)}{=}\sup_{0\leq s\leqslant\alpha t}X^{(1)}(s)+\inf_{0\leq s\leqslant(1-\alpha t)}X^{(2)}(s).$$

This means that the two quantities have the same distribution, The original proof is in the 1995 paper. The proof is using the Feynman-Kac formula and you should study it as it is a very good application to demonstrate what a powerful tool the Feynman-Kac formula is. Because of the striking nature of the result various researchers tried to think of a simple explanation. Although a simple explanation has not been found there have been various alternative proofs of the result. The most elegant of those is the proof by Embrechts, Rogers and Yor (1996); it is based on the following result,

Proposition 2. *Let $X(t)=\sigma B(t)+\mu t$, where $\mu\in R$, $\sigma\in R^+$ and $(B(t),t\geq 0)$ is a standard Brownian motion. Define $S_X(t)=\sup_{0\leq s\leq t}\{X(s)\}$ and $\theta_X(t)=\sup\{s\in[0,t]\colon X(s)=S_X(t)\}$. Then,*

$$\int_0^t 1(X(s)>0)\mathrm{ds}\overset{(law)}{=}\theta_X(t)$$

$S_X(t)$ is the maximum of the Brownian motion and $\theta_X(t)$ is the last time it is attained.

The result is well known for the standard Brownian motion and it is called the arc-sine law. This is because the density of the two quantities is $\frac{1}{\pi\sqrt{u(t-u)}}$ and so the distribution function is the arcsine function. In the case of the Brownian motion with drift they observed that although the density function is different, proposition 2 is still true. You can find a proof in their paper.

We now present the proof of proposition 1. This is a modified version of the proof in the paper.

Proof. (of Proposition 1). Without loss of generality, we will assume that $t = 1$. Let $x > 0$ and also let $\tau_x = \inf\{s > 0\colon X(s) = x\}$ and $X^{(1)}(s) = X(\tau_x + s) - x$. $\big(X^{(1)}(s),\, s \geq 0\big)$ is an independent copy of $(X(s), 0 \leq s \leq \tau_x)$. Using the fact that the events $\{M_X(\alpha, t) > x\}$ and $\left\{\int_0^t 1(X(s) \leq x) d\, s < \alpha t\right\}$ are identical, we have

$$\Pr(M_X(\alpha) > x) = \Pr\left(\int_0^1 1(X(s) \leq x) d\, s < \alpha\right) =$$

$$\Pr\left(\int_0^1 1(X(s) > x) d\, s > 1 - \alpha\right) = \Pr\left(\int_{\tau_x}^1 1(X(s) > x) d\, s > 1 - \alpha\right) =$$

$$\Pr\left(\int_{\tau_x}^1 1(X(s) > x) d\, s > 1 - \alpha\right) = \Pr\left(\int_0^{1-\tau_x} 1(X(\tau_x + s) - x > 0) d\, s > 1 - \alpha\right) =$$

$$\int_0^\alpha \Pr(\tau_x \in d\, r) \Pr\left(\int_0^{1-r} 1\big(X^{(1)}(s) > 0\big) \mathrm{ds} > 1 - \alpha\right)$$

and because of () this is equal to

$$\int_0^\alpha \Pr(\tau_x \in d\, r) \Pr\left(\int_0^{1-r} \theta_{X^{(1)}}(1 - r) > 1 - \alpha\right)$$

$$\int_0^\alpha \Pr(\tau_x \in d\, r) \Pr\left(\sup_{0 \leq s\mathrm{l} \leq 1-\alpha} X^{(1)}(s) < \sup_{1-\alpha \leq s \leq 1-r} X^{(1)}(s)\right) =$$

$$\int_0^\alpha \Pr(\tau_x \in d\, r) \Pr\left(\sup_{0 \leq s \leq 1-\alpha} \big(X^{(1)}(s) - X^{(1)}(1-\alpha)\big) < \sup_{1-\alpha \leq s \leq 1-r} \big(X^{(1)}(s) - X^{(1)}(1-\alpha)\big)\right) =$$

$$\int_0^\alpha \Pr(\tau_x \in d\, r) \Pr\left(-\inf_{0 \leq s \leq 1-\alpha} \big(X^{(1)}(1-\alpha) - X^{(1)}(1-\alpha-s)\big) < \sup_{0 \leq u \leq \alpha - r} \big(X^{(1)}(1-\alpha+u) - X^{(1)}(1-\alpha)\big)\right). \tag{3}$$

Now set $X^{(2)}(s) = X^{(1)}(1-\alpha) - X^{(1)}(1-\alpha-s)$ and $X^{(3)}(s) = X^{(1)}(1-\alpha+u) - X^{(1)}(1-\alpha)\big)$. $\big(X^{(2)}(s),\, s \geq 0\big)$ and $\big(X^{(3)}(s),\, s \geq 0\big)$ are two more independent copies of the original Brownian motion. Then (3) is equal to

$$\int_0^{\alpha} \Pr(\tau_x \in dr) \Pr\Big(-\inf_{0 \leq s \leq 1-\alpha} X^{(2)}(s) < \sup_{0 \leq u \leq \alpha - r} X^{(3)}(s) \Big) =$$

$$\Pr\Big(-\inf_{0 \leq s \leq 1-\alpha} X^{(2)}(s) < \sup_{0 \leq u \leq \alpha} X^{(4)}(s) - x \Big) =$$

$$\Pr\Big(\sup_{0 \leq u \leq \alpha} X^{(4)}(s) + \inf_{0 \leq s \leq 1-\alpha} X^{(2)}(s) > x \Big),$$

where $\big(X^{(4)}(s),\, s \geq 0\big)$ is yet another independent copy of the original Brownian motion, The case $x < 0$ can be dealt by from the fact that $M_X(\alpha, 1)$ has the same distribution as $M_{-X}(1-\alpha, 1)$.

□

Using the Girsanov transformation, we can extend Proposition 1 to a result involving the joint distribution of $M(\alpha, t)$ and $X(t)$.

Proposition 3. *Let $X(t)$, $M_X(\alpha,\, t)$, $X^{(1)}(t)$ and $X^{(2)}(t)$ as in Proposition 1. Then,*

$$\begin{pmatrix} M_X(\alpha, t) \\ X(t) \end{pmatrix} \stackrel{(law)}{=} \begin{pmatrix} \sup_{0 \leq s \leqslant \alpha t} X^{(1)}(s) + \inf_{0 \leq s \leqslant (1-\alpha t)} X^{(2)}(s) \\ X^{(1)}(\alpha t) + X^{(2)}((1-\alpha)t) \end{pmatrix}.$$

Using this result one could calculate an expression for the joint probability density of $M(\alpha, t)$ and $X(t)$. This can be used to price a floating strike option.

We can then use the density of $\sup_{0 \leq s \leqslant \alpha t} X^{(1)}(s)$,

$$\frac{\sqrt{2}}{\sigma\sqrt{\pi \alpha t}} \exp\Big(\frac{-(x - \mu\alpha t)^2}{\sigma\sqrt{\alpha t}} \Big) - \frac{2\mu}{\sigma^2} e^{\frac{2\mu}{\sigma^2}x} \Phi\Big(\frac{-x - \mu\alpha t}{\sigma\sqrt{\alpha t}} \Big)$$

and use it to get the price of a quantile option.

Note that a formula for the quantile option was independently found by Akahori (1995).

In the case of the standard Brownian motion we can actually find an expression for the density of $M_X(\alpha, t)$. The he density of $M_B(\alpha, 1)$ is given by

$$\begin{array}{ll} \frac{4}{\sqrt{2\pi}}\bar{\Phi}\left(x\sqrt{\frac{1-\alpha}{\alpha}}\right)\exp\left(-\frac{x^2}{2}\right), & x>0 \\ \frac{4}{\sqrt{2\pi}}\bar{\Phi}\left(|x|\sqrt{\frac{\alpha}{1-\alpha}}\right)\exp\left(-\frac{x^2}{2}\right), & x<0 \end{array} \tag{4}$$

where $\bar{\Phi}(x) = 1 - \Phi(x)$.

Proposition 1 and proposition 3 can be generalised to the case where $(X(t), t \geq 0)$ is a Lévy process (it has independent and stationary increments). The result now becomes

Proposition 4. *Let $(X(t), t \geq 0)$ be a Lévy process . Furthermore, let $X^{(1)}(t)$ and $X^{(2)}(t)$ be independent copies of $X(t)$. Then,*

$$M_X(\alpha, t) \overset{(law)}{=} \sup_{0 \leq s \leqslant \alpha t} X^{(1)}(s) + \inf_{0 \leq s \leqslant (1-\alpha t)} X^{(2)}(s).$$

and proposition 3

Proposition 5. *Let $(X(t), t \geq 0)$ be a Lévy process . Furthermore, let $X^{(1)}(t)$ and $X^{(2)}(t)$ be independent copies of $X(t)$. Then,*

$$\begin{pmatrix} M_X(\alpha, t) \\ X(t) \end{pmatrix} \overset{(law)}{=} \begin{pmatrix} \sup_{0 \leq s \leqslant \alpha t} X^{(1)}(s) + \inf_{0 \leq s \leqslant (1-\alpha t)} X^{(2)}(s) \\ X^{(1)}(\alpha t) + X^{(2)}((1-\alpha)t) \end{pmatrix}.$$

These can be proven as limits of the following discrete time results (For details on how to take the limits and prove propositions 4 and 5 see AD (1996a)

Consider the sequence $x = (x_0, x_1, x_2, \ldots)$. For integers $0 \leq j \leq n$, define the$(j, n)^{th}$ quantile of x for $j = 0, 1, 2, \ldots, n$ by

$$M_{j,n}(x) = \inf\left\{z \colon \sum_{i=0}^{n} 1(x_i \leq z) > j\right\}.$$

It should also be remarked that if $x_{(0)}, x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ is an increasing order permutation of $x_0, x_1, x_2, \ldots, x_n,$ then $M_{j,n}(x) = x_{(j)}$. So, in particular, $M_{0,n}(x) = \min_{i=0,1,\ldots,n} \{x_i\}$ and $M_{n,n}(x) = \max_{i=0,1,\ldots,n} \{x_i\}$. Also note that in this setup $M_{0,0}(x) = x_0$.

The following result is due to Wendel (1960)

Proposition 6. *Let $Y_1, Y_2, ..., Y_n$ be i.i.d. random variables. Define $X = (X_0, X_1, ..., X_n)$ by*

$$X_n = \begin{cases} \sum_{i=1}^{n} Y_i & n = 1, 2, ... \\ 0 & n = 0 \end{cases}.$$

and let $X^{(1)}$ and $X^{(2)}$ be two independent copies of X. Then,

$$\begin{pmatrix} M_{j,n}(X) \\ X_n \end{pmatrix} \overset{(law)}{=} \begin{pmatrix} M_{j,j}(X^{(1)}) + M_{0,n-j}(X^{(2)}) \\ X_j^{(1)} + X_{n-j}^{(2)} \end{pmatrix}.$$

An extension of this result, involving the time the quantile is achieved, was obtained by Port (1963) . He defined the ordering $\prec$ by $X_i \prec X_j$ if $X_i < X_j$ or $X_i = X_j$ but $i < j$. Then, one could alternatively define $M_{0,n}(X)$, $M_{1,n}(X)$, ..., $M_{n,n}(X)$ as the rearrangement of X_0, X_1, ..., X_n such that $M_{0,n}(X) \prec M_{1,n}(X) \prec ... \prec M_{n,n}(X)$. He then defined $L_{k,n}(X)$as the index in X_0, X_1, ..., X_n of $M_{k,n}(X)$ and extended the result to

$$\begin{pmatrix} M_{j,n}(X) \\ L_{j,n}(X) \\ X_n \end{pmatrix} \overset{(law)}{=} \begin{pmatrix} M_{j,j}(X^{(1)}) + M_{0,n-j}(X^{(2)}) \\ L_{j,j}(X^{(1)}) + L_{0,n-j}(X^{(2)}) \\ X_j^{(1)} + X_{n-j}^{(2)} \end{pmatrix}.$$

It is rather difficult to find a continuous time equivalent quantity to $L_{j,n}(X)$. In general (think of the Brownian motion) the quantile will be crossed from above and below many times (in fact infinitely but countably many) and so it is not clear which one corresponds to the limit of $L_{j,n}(X)$. For more details on hitting times of quantiles see AD (2005). For a combinatorics treatment see Chaumont (1999) and some of the references therein.

We will provide a proof of the first leg of proposition 6 (from this one can use an argument similar to the Girsanov transformation and get proposition 6). That is

$$M_{j,n}(X) \overset{(law)}{=} M_{j,j}(X^{(1)}) + M_{0,n-j}(X^{(2)}) = \max_{0 \le i \le j} X_i^{(1)} + \min_{0 \le i \le n-j} X_i^{(2)}. \tag{5}$$

The proof is taken form AD (1996b). We start with an important lemma.

Lemma 7. *Let $Y_1, Y_2, ...$ be a sequence of independent and identically distributed random variables with distribution function F. Let $X_0 = 0$ and $X_n = \sum_{i=1}^{n} Y_i$ for $n = 1, 2, ...$. Furthermore let $0 \le \psi < 1$ and define*

$$H_1(x; \psi) = (1 - \psi) \sum_{n=0}^{\infty} \psi^n \Pr\left(\max_{0 \le i \le n} (X_i) \le x \right) \tag{6}$$

and

$$H_2(x; \psi) = (1 - \psi) \sum_{n=0}^{\infty} \psi^n \Pr\left(\min_{0 \le i \le n} (X_i) \le x \right). \tag{7}$$

Then $H_1(x;\psi)$ satisfies the equation

$$H_1(x;\psi) = 1(x \geq 0)\left((1-\psi) + \psi \int_{-\infty}^{\infty} H_1(x-y;\psi)dF(y)\right) \tag{8}$$

and $H_2(x;\psi)$ satisfies the equation

$$H_2(x;\psi) = 1(x \geq 0) + \psi 1(x<0) \int_{-\infty}^{\infty} H_2(x-y;\psi)dF(y). \tag{9}$$

Proof. For $x<0$, (8) is trivially true. For $x \geq 0$, and using the fact that $Y_1, Y_2, \ldots$ are i.i.d. and therefore exchangeable, observe that

$$\begin{gathered}
(1-\psi)\sum_{n=0}^{\infty} \psi^n \Pr\left(\max_{0\leq i\leq n}(X_i) \leq x\right) = \\
1-\psi+\psi(1-\psi)\sum_{n=1}^{\infty} \psi^{n-1}\Pr\left(\max\left(0, Y_1, \ldots, \sum_{i=1}^{n} Y_i\right) \leq x\right) = \\
1-\psi+\psi(1-\psi)\sum_{n=1}^{\infty} \psi^{n-1}\Pr\left(\max\left(Y_1, Y_1+Y_2, \ldots, \sum_{i=1}^{n} Y_i\right) \leq x\right) = \\
1-\psi+\psi(1-\psi)\left(\sum_{n=1}^{\infty} \psi^{n-1}\Pr\left(\max\left(0, Y_2, \ldots, \sum_{i=2}^{n} Y_i\right) \leq x - Y_1\right)\right) = \\
1-\psi+\psi(1-\psi)\left(\sum_{n=0}^{\infty} \psi^{n}\Pr\left(\max\left(0, Y_2, \ldots, \sum_{i=1}^{n} Y_{i+1}\right) \leq x - Y_1\right)\right) = \\
(1-\psi)+\psi\int_{-\infty}^{\infty} H_1(x-y;\psi)dF(y),
\end{gathered}$$

which proves (8). To prove (9) note that it is trivially true for $x \geq 0$ and that for $x<0$,

$$\begin{gathered}
(1-\psi)\sum_{n=0}^{\infty} \psi^n \Pr\left(\min_{0\leq i\leq n}(X_i) \leq x\right) = \\
\psi(1-\psi)\sum_{n=1}^{\infty} \psi^{n-1}\Pr\left(\min\left(0, Y_1, \ldots, \sum_{i=1}^{n} Y_i\right) \leq x\right) = \\
\psi(1-\psi)\sum_{n=1}^{\infty} \psi^{n-1}\Pr\left(\min\left(Y_1, Y_1+Y_2, \ldots, \sum_{i=1}^{n} Y_i\right) \leq x\right) = \\
\psi(1-\psi)\left(\sum_{n=1}^{\infty} \psi^{n-1}\Pr\left(\min\left(0, Y_2, \ldots, \sum_{i=2}^{n} Y_i\right) \leq x - Y_1\right)\right) = \\
1-\psi+\psi(1-\psi)\left(\sum_{n=0}^{\infty} \psi^{n}\Pr\left(\min\left(0, Y_2, \ldots, \sum_{i=1}^{n} Y_{i+1}\right) \leq x - Y_1\right)\right) = \\
\psi\int_{-\infty}^{\infty} H_2(x-y;\psi)dF(y),
\end{gathered}$$

This completes the proof.

□

You should observe the following:

1. Equations (8) and (9) are Wiener-Hopf type equations like the ones described in the book by Feller section XII.3, with a defective probability measure.

2. From their definition we can see that $H_1(x;\,\psi)$ and $H_2(x;\,\psi)$ are distribution functions. $H_1(x;\,\psi)$ is the distribution function of $\max_{0\le i\le N}(X_i)$ and is the distribution function of $\min_{0\le i\le N}(X_i)$, where N is a random variable independent of $Y_1, Y_2, \ldots$ with a geometric distribution; that is $\Pr(N=n) = (1-\psi)\psi^n,\ n=0,1,2,\ldots.$

We will now prove a very important result.

Lemma 8. *Let $\overline{D}(\mathbb{R})$ be the space of bounded real valued functions of $\mathbb{R}$ that are right continuous with left limits existing for all points and let $G_1(x), G_2(x)$ be distribution functions. Then, for all $0\le\psi<1$ and $0\le\phi<1$, the equation*

$$H(x) = (1-\psi)1(x\ge 0) + \psi 1(x\ge 0)\int_{-\infty}^{\infty} H(x-y)dG_1(y) + \phi 1(x<0)\int_{-\infty}^{\infty} H(x-y)dG_2(y) \tag{10}$$

has a unique solution $H(x;\psi,\phi)$in $\overline{D}(R)$. Furthermore

$$H(x;\psi,\phi) = \int_{-\infty}^{\infty} H_1(x-y;\psi)dH_2(y;\phi), \tag{11}$$

where $H_1(x;\psi)$ is the unique solution of

$$H(x) = (1-\psi)1(x\ge 0) + \psi 1(x\ge 0)\int_{-\infty}^{\infty} H(x-y)dG_1(y) \tag{12}$$

in $\overline{D}(R)$ and $H_2(x;\phi)$ is the unique solution of

$$H(x) = 1(x\ge 0) + \phi 1(x<0)\int_{-\infty}^{\infty} H(x-y;\psi)dG_2(y) \tag{13}$$

in $\overline{D}(R)$.

Proof. Define the metric $d(H,K)$ in $\overline{D}(\mathbb{R})$, by

$$d(H,K) = \sup_{x\in R} |H(x)-K(x)|. \tag{14}$$

Also, define $T: \overline{D}(R) \to \overline{D}(R)$, by

$$TH(x) = (1-\psi)1(x \geq 0) + \psi 1(x \geq 0) \int_{-\infty}^{\infty} H(x-y) d G_1(y) +$$
$$\phi 1(x < 0) \int_{-\infty}^{\infty} H(x-y) d G_2(y).$$

Observe that for $0 \leq \psi < 1$ and $0 \leq \phi < 1$, T is a contraction mapping on $\overline{D}(\mathbb{R})$, using the metric defined by 14. Then by the fixed point theorem for contraction mappings, (10) has a unique solution. Moreover, (12) and (13) have unique solutions, since they are special cases of (10) for $\varphi = 0$ and $\psi = 0$ respectively. By lemma 1 and the second remark following its proof, these solutions, $H_1(x;\, \psi)$ and $H_2(x;\, \phi)$, are distribution functions and let U and V be random variables on a suitable probability space with distribution functions $H_1(x;\, \psi)$ and $H_2(x;\, \phi)$ respectively. Let $H(x;\, \psi,\, \phi)$ be the convolution of $H_1(x;\, \psi)$ and $H_2(x;\, \phi)$, as defined by (11). We will prove that $H(x;\, \psi,\, \phi)$ satisfies (10) and therefore is its unique solution in $\overline{D}(\mathbb{R})$.

Observe that $H(x; \psi, \phi) = \Pr(U + V \leq x)$. Now, suppose $x \geq 0$, and condition on $V = v$, then $\Pr(U + V \leq x \mid V = v) = H_1(x - v; \psi)$. Note that $\Pr(V \leq 0) = 1$ and so we only need to consider $v \leq 0$, in which case $x - v \geq 0$ and from (12) we then get that

$$H_1(x-v; \psi) = (1-\psi) + \psi \int_{-\infty}^{\infty} H_1(x-v-y; \psi) d G_1(y). \tag{15}$$

Integrating over all non-positive v, we get that for $x \geq 0$,

$$H(x; \psi, \phi) = (1-\psi) + \psi \int_{-\infty}^{\infty} H(x-y; \psi, \phi) d G_1(y). \tag{16}$$

Similarly for $x < 0$, condition on $U = u$; then $\Pr(U + V \leq x \mid U = u) = H_2(x - u;\, \psi)$. Note that $\Pr(U \geq 0) = 1$ and so we only need to consider $u \geq 0$, in which case $x - u < 0$ and from 13 we then get that

$$H_2(x-u; \phi) = \phi \int_{-\infty}^{\infty} H_2(x-u-y; \psi) d G_2(y). \tag{17}$$

Averaging over all non-negative u, we get that for $x < 0$,

$$H(x; \psi, \phi) = \phi \int_{-\infty}^{\infty} H(x-y; \psi, \phi) d G_2(y). \tag{18}$$

Combining (16) and (18) we see that $H(x;\, \psi,\, \phi)$ satisfies (10) and therefore is its unique solution in $\overline{D}(\mathbb{R})$.

□

We now deduce (5) as a corollary.

Corollary 9. *Let Y_1, Y_2, ... be a sequence of independent and identically distributed random variables with distribution function F. Let $X_0 = 0$ and $X_n = \sum_{i=1}^{n} Y_i$ for $n = 1, 2, \ldots$. Furthermore let*

$$M_{j,n} = \inf\left\{ z \colon \sum_{i=0}^{n} 1(X_i \leq z) > j \right\}. \tag{19}$$

(So $M_{0,n}$ denotes the smallest of X_0, X_1, ..., X_n, $M_{1,n}$ the second smallest and so on, with $M_{n,n}$ denoting the largest). Let also $X_0^{(1)}, X_1^{(1)}, \ldots$ and $X_0^{(2)}, X_1^{(2)}, \ldots$ be two independent copies of the sequence $X_0, X_1, \ldots$. Then

$$M_{j,n} \overset{(law)}{=} \max_{0 \leq i \leq j} \left(X_i^{(1)} \right) + \min_{0 \leq i \leq n-j} \left(X_i^{(2)} \right). \tag{20}$$

Proof. Consider the occupation time $L_n(x) = \sum_{i=0}^{n} 1(X_i \leq x)$ and let

$$h(x) = \sum_{n=0}^{\infty} \phi^n E\left(\eta^{L_n(x)} \right),$$

where $0 \leq \phi < 1$ and $0 \leq \eta < \frac{1}{\phi}$. Using the fact that $Y_1, Y_2, \ldots$ are independent and therefore exchangeable, condition on $Y_1 = y$ and observe that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \phi^n E\left(\eta^{L_n(x)} \mid Y_1 = y \right) \\
&= \eta^{1(x \geq 0)} + \phi \eta^{1(x \geq 0)} \sum_{n=1}^{\infty} \phi^{n-1} E\left(\eta^{\sum_{i=1}^{n} 1\left(\sum_{r=1}^{i} Y_r \leq x \right)} \mid Y_1 = y \right) = \\
&\eta^{1(x \geq 0)} + \phi \eta^{1(x \geq 0)} \sum_{n=1}^{\infty} \phi^{n-1} E\left(\eta^{1(0 \leq x-y) + \sum_{i=2}^{n} 1\left(\sum_{r=2}^{i} Y_r \leq x-y \right)} \right) = \\
&\eta^{1(x \geq 0)} + \phi \eta^{1(x \geq 0)} \sum_{n=1}^{\infty} \phi^{n-1} E\left(\eta^{1(0 \leq x-y) + \sum_{i=1}^{n-1} 1\left(\sum_{r=1}^{i} Y_r \leq x-y \right)} \right) = \\
&\eta^{1(x \geq 0)} + \phi \eta^{1(x \geq 0)} h(x-y).
\end{aligned}$$

Averaging over all values of y we therefore get

$$h(x) = \eta^{1(x \geq 0)} + \phi \eta^{1(x \geq 0)} \int_{-\infty}^{\infty} h(x-y) dF(y). \tag{21}$$

Now, note that

$$E\left(\eta^{L_n(x)} \right) = 1 - (1-\eta) \sum_{j=0}^{n} \eta^j \Pr\left(L_n(x) > j \right),$$

and therefore

$$\begin{aligned}
h(x) &= \frac{1}{1-\phi} - (1-\eta) \sum_{n=0}^{\infty} \phi^n \sum_{j=0}^{n} \eta^j \Pr\left(L_n(x) > j \right) = \\
&\frac{1}{1-\phi} - (1-\eta) \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \phi^k (\eta\phi)^j \Pr\left(L_{j+k}(x) > j \right).
\end{aligned} \tag{22}$$

Setting $\psi = \eta\phi$ and observing that the events $\{L_{j+k}(x) > j\}$ and $\{M_{j,j+k} \le x\}$ are identical, we can rewrite (22) as

$$h(x) = \frac{1}{1-\phi} - \frac{\phi-\psi}{\phi}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\phi^k\psi^j\mathrm{Pr}\left(M_{j,j+k} \le x\right) = \frac{1}{1-\phi} - \frac{\phi-\psi}{\phi(1-\phi)(1-\psi)}H(x), \tag{23}$$

where

$$H(x) = (1-\phi)(1-\psi)\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\phi^k\psi^j\mathrm{Pr}\left(M_{j,j+k} \le x\right). \tag{24}$$

From (23) and (21), we get

$$H(x) = (1-\psi)1(x \ge 0) + \psi 1(x \ge 0)\int_{-\infty}^{\infty} H(x-y)dF(y) + \phi 1(x<0)\int_{-\infty}^{\infty} H(x-y)dF(y). \tag{25}$$

From Lemma 2 we have that

$$H(x) = \int_{-\infty}^{\infty} H_1(x-y)dH_2(y), \tag{26}$$

where $H_1(x)$ is the unique solution in $\overline{D}(R)$ of

$$H_1(x) = (1-\psi)1(x \ge 0) + \psi 1(x \ge 0)\int_{-\infty}^{\infty} H_1(x-y)dF(y),$$

and $H_2(x)$ is the unique solution in $\overline{D}(R)$ of

$$H_2(x) = 1(x \ge 0) + \phi 1(x<0)\int_{-\infty}^{\infty} H_2(x-y)dF(y).$$

From Lemma 1 we see that

$$H_1(x) = (1-\psi)\sum_{j=0}^{\infty}\psi^j\mathrm{Pr}\left(\max_{0\le i\le j}(X_i) \le x\right) \tag{27}$$

and

$$H_2(x) = (1-\phi)\sum_{k=0}^{\infty}\phi^k\mathrm{Pr}\left(\min_{0\le i\le k}(X_i) \le x\right). \tag{28}$$

From (24), (26), (27), (28) and the uniqueness of the relevant expansion we conclude that

$$M_{j,j+k} \overset{(law)}{=} \max_{0\le i\le j}\left(X_i^{(1)}\right) + \min_{0\le i\le k}\left(X_i^{(2)}\right),$$

for all $j \ge 0$ and $k \ge 0$. This concludes the proof of the corollary. □

One might wonder whether the property in proposition 4 characterises Lévy processes (in other words whether they are the only processes with this property. The same paper AD (1996b) concludes that this is not the case as the following example demonstrates. Let $((T_i, Y_i), i = 1, 2, ...)$ be a sequence of independent and identically distributed pairs of random variables on a probability space $(\Omega, \mathcal{F}, \Pr)$ taking values in $R^+ \times R$ and having joint distribution function $G(u, y)$. Let

$$S_0 = 0, \;\; S_n = \sum_{i=1}^{n} T_i, \;\; n = 1, 2, ...$$

and define the renewal process $(N(t), t \geq 0)$ by

$$N(t) = \sup_{n=0,1,2,...} \{n : S_n \leq t\}.$$

We define $(X(t), t \geq 0)$ by

$$X(t) = \begin{cases} \sum_{i=1}^{N(t)} Y_i & N(t) = 1, 2, ... \\ 0 & N(t) = 0 \end{cases}.$$

It should be noted that $X(t)$ is semi-Markov, but not a Markov process. However, the pair $(X(t),\, U(t))$ is a Markov process. Let $X^{(1)}(t)$, $X^{(2)}(t)$ be independent copies of $X(t)$; then

$$M_X(\alpha, t) \overset{(law)}{=} \sup_{0 \leq s \leq \alpha t} X^{(1)}(s) + \inf_{0 \leq s \leq (1-\alpha)t} X^{(2)}(s).$$

The proof can be found in the paper, but it is not important; it is really another corollary to lemma 8. Note that for the proof of corollary 9 $G_1 = G_2 = F$. The example above is the case where $G_1 \neq G_2$.

We finish this section with some calculated prices of call options for (values ,calculated using the Rogers and Shi lower bound with upper bounds in brackets) $Y(0) = 100$.

TABLE 1. $\sigma = .05$

r	k	Geom.	Asian	Median
.05	95	7.147	7.178 (7.183)	7.156
	100	2.689	2.716 (2.722)	2.708
	105	0.324	0.337 (0.343)	0.410
.09	95	8.757	8.809 (8.821)	8.767
	100	4.256	4.308 (4.318)	4.275
	105	0.922	0.958 (0.968)	1.059
.15	95	10.988	11.094 (11.114)	11.001
	100	6.689	6.794 (6.810)	6.704
	105	2.646	2.744 (2.761)	2.765

TABLE 2. $\sigma = .1$

r	k	Geom.	Asian	Median
.05	90	11.862	11.951 (11.973)	11.894
	100	3.573	3.641 (3.663)	3.617
	110	0.306	0.331 (0.353)	0.413
.09	90	13.274	13.385 (13.410)	13.301
	100	4.816	4.915 (4.942)	4.863
	110	0.583	0.630 (0.657)	0.745
.15	90	15.235	15.399 (15.445)	15.265
	100	6.869	7.028 (7.066)	6.919
	110	1.310	1.413 (1.451)	1.553

TABLE 3. $\sigma = .2$

r	k	Geom.	Asian	Median
.05	90	12.318	12.595 (12.687)	12.469
	100	5.547	5.762 (5.854)	5.651
	110	1.845	1.989 (2.080)	2.045
.09	90	13.520	13.831 (13.927)	13.652
	100	6.518	6.777 (6.872)	6.628
	110	2.359	2.545 (2.641)	2.593
.15	90	15.267	15.641 (15.748)	15.383
	100	8.073	8.408 (8.515)	8.193
	110	3.292	3.554 (3.661)	3.571

TABLE 4. $\sigma = .5$

r	k	Geom.	Asian	Median
.05	90	13.404	13.952 (14.161)	13.657
	100	7.496	7.944 (8.153)	7.674
	110	3.722	4.070 (4.279)	3.981
.09	90	14.388	14.983 (15.194)	14.627
	100	8.324	8.827 (9.039)	8.510
	110	4.291	4.695 (4.906)	4.574
.15	90	15.838	16.512 (16.732)	16.062
	100	9.612	10.208 (10.429)	9.812
	110	5.229	5.728 (5.948)	5.548

3 Hitting times of quantiles

We now let

$$L_X(\alpha,t)=\inf\{s\in[0,t]\colon X(s)=M_X(\alpha,t)\}$$

be the first, and

$$K_X(\alpha,t)=\sup\{s\in[0,t]\colon X(s)=M_X(\alpha,t)\},$$

the last time the process hits $M_X(\alpha,t)$. One can now introduce a 'barrier' element to the financial application by making the option worthless if the quantile is hit too early or too late. For example, this can involve calculating quantities such as $E(h(M_X(\alpha,t))1(L_X(\alpha,t)>v,K_X(\alpha,t)<u))$.

The first study of these quantities can be found in Chaumont (1999).

For this section we assume that $(X(s),s\geq 0)$ is a standard Brownian motion, unless otherwisew specified. Without loss of generality, we will restrict our attention to the case $t=1$ taking advantage of the Brownian scaling. For simplicity we set $M_X(\alpha,t)=M_X(\alpha)$, $L_X(\alpha,t)=L_X(\alpha)$ and $K_X(\alpha,t)=K_X(\alpha)$.We will derive the joint density of $M_X(\alpha)$, $L_X(\alpha)$, $K_X(\alpha)$ and $X(1)$. If we denote this density by $f(y,x,u,v)$, our results can be generalised for a Brownian motion with drift m, using a Cameron-Martin-Girsanov transformation. The corresponding density will be

$$f(y,x,u,v)\exp\left(m\,x-m^2/2\right).$$

Before we obtain the density of $(M_X(\alpha),L_X(\alpha),K_X(\alpha),X(1))$, we will first show that the law of $L_X(\alpha)$ (and $K_X(\alpha)$) is a transformed arcsine law.

3.1 An arc-sine law

Let $S_X(t)=\sup_{0\leq s\leq t}\{X(s)\}$ and $\theta_X(t)=\sup\{s\in[0,t]\colon X(s)=S_X(t)\}$. Define also the stopping time $\tau_c=\inf\{s>0\colon X(s)=c\}$. We will first obtain the joint distribution of

$$(M_X(\alpha),L_X(\alpha))$$

(also of $(M_X(\alpha)-X(1),1-K_X(\alpha))$.

Theorem 10. *For $b>0$,*

$$\Pr(M_X(\alpha)\in d\,b,L_X(\alpha)\in d\,u)=$$

$$\Pr(S_X(1)\in d\,b,\theta_X(1)\in d\,u)1(0<u<\alpha),\tag{29}$$

and for $b<0$,

$$\Pr(M_X(\alpha)\in d\,b, L_X(\alpha)\in d\,u)=$$

$$\Pr(S_X(1)\in d|b|, \theta_X(1)\in d\,u)1(0<u<(1-\alpha)). \tag{30}$$

Furthermore, $(M_X(\alpha), L_X(\alpha))$ *and* $(M_X(\alpha)-X(1), 1-K_X(\alpha))$ *have the same distribution.*

Proof. Let $b>0$ and $u<\alpha$. We then have that

$$\Pr(M_X(\alpha)>b, L_X(\alpha)>u)=\Pr(S_X(u)<M_X(\alpha), M_X(\alpha)>b)=$$

$$\Pr(b<S_X(u)<M_X(\alpha))+\Pr(S_X(u)<b<M_X(\alpha)). \tag{31}$$

Let $\tau_b=\inf\{s>0\colon X(s)=b\}$ and $X^*(s)=X(\tau_b+s)-b$. $(X^*(s), s\geq 0)$ is a standard Brownian motion which is independent of $(X(s), 0\leq s\leq\tau_b)$. Using theorem 1, we have

$$\Pr(b<S_X(u)<M_X(\alpha))=$$

$$\Pr\left(S_X(u)>b, \int_0^1 1(X(s)\leq S_X(u))d\,s<\alpha\right)=$$

$$\Pr\left(S_X(u)>b, \int_u^1 1(X(s)-X(u)\leq S_X(u)-X(u))d\,s<\alpha-u\right). \tag{32}$$

We now condition on $\sigma\{X(s), 0\leq s\leq u\}$. Let $X^*(s)=X(u+s)-X(u)$. $(X^*(s), s\geq 0)$ is a standard Brownian motion which is independent of $(X(s), 0\leq s\leq u)$. We condition on $S_X(u)-X(u)=c$, and set $\tau_c=\inf\{s>0\colon X^*(s)=c\}$ and $X^{**}(s)=X^*(\tau_c+s)-c$. $(X^{**}(s), s\geq 0)$ is a standard Brownian motion which is independent of both $(X(s), 0\leq s\leq u)$ and $(X^*(s), 0\leq s\leq\tau_c)$. We have that

$$\Pr\left(\int_0^{1-u} 1(X^*(s)\leq c)d\,s<\alpha-u\right)=$$

$$\int_0^{\alpha-u}\Pr(\tau_c\in d\,r)\Pr\left(\int_0^{1-u-r} 1(X^{**}(s)\leq 0)d\,s<\alpha 1-u-r\right)$$

and since $\int_0^{1-u-r} 1(X^{**}(s)\leq 0)d\,s$ has the same (arcsine) law as $\theta_{X^{**}}(1-u-r)$, this is equal to

$$\int_0^{\alpha-u}\Pr(\tau_c\in d\,r)\Pr(\theta_{X^{**}}(1-u-r)<\alpha-u-r)=$$

$$\int_0^{\alpha-u}\Pr(\tau_c\in d\,r)\Pr\left(\sup_{0\leq s\leq\alpha-u-r} X^{**}(s)>\sup_{\alpha-u-r\leq s\leq 1-u-r} X^{**}(s)\right)=$$

$$\Pr\left(\sup_{0\leq s\leq\alpha-u} X^*(s)>\sup_{\alpha-u\leq s\leq t-u} X^*(s), \sup_{0\leq s\leq\alpha-u} X^*(s)>c\right)$$

and so (32) is equal to

$$\Pr\begin{pmatrix} \sup_{u\le s\le\alpha} X(s)-X(u) > \sup_{\alpha\le s\le 1} X(s)-X(u), \\ \sup_{u\le s\le\alpha} X(s)-X(u) > \sup_{0\le s\le u} X(s)-X(u), \\ \sup_{0\le s\le u} X(s) > b \end{pmatrix} =$$

$$\Pr(S_X(u) > b, u < \theta_X(1) \le \alpha). \tag{33}$$

Furthermore,

$$\Pr(S_X(u) < b < M_X(\alpha)) = \Pr\left(S_X(u) < b, \int_0^1 1(X(s)\le b)ds < \alpha\right) =$$

$$\int_u^\alpha \Pr(\tau_b \in dr)\Pr\left(\int_0^{1-r} 1(X^*(s)\le 0) < \alpha - r\right)$$

$$= \int_u^\alpha \Pr(\tau_b \in dr)\Pr(\theta_{X^*}(1-r) < \alpha - r) =$$

$$\Pr\left(u < \theta_X(1) < \alpha, S_X(u) < b, \sup_{u\le s\le\alpha} X(s) > b\right). \tag{34}$$

Adding (33) and (34) together, we see that (31) is equal to

$$\Pr\left(u < \theta_X(1) < \alpha, \sup_{u\le s\le\alpha} X(s) > b\right) = \Pr(u < \theta_X(1) < \alpha, S_X(1) > b)$$

which leads to (29).

Since $(-X(s), s\ge 0)$ is a standard Brownian motion and $M_{-X}(\alpha) = -M_X(1-\alpha)$ almost surely, we use $-X(s)$ instead of $X(s)$ and we get that for $b<0$,

$$\Pr(M_X(\alpha) < b, L_X(\alpha) > u) = \Pr(u < \theta_X(1) \le (1-\alpha), S_X(1) > |b|),$$

which leads to (30).

To see that $(t-K_X(\alpha), M_X(\alpha)-X(1))$ has the same distribution as $(L_X(\alpha), M_X(\alpha))$, set again $\tilde{X}(s) = X(1-s) - X(1)$. Clearly $\left(\tilde{X}(s),\, 0\le s\le t\right)$ is a standard Brownian motion and we can easily see that $M_{\tilde{X}}(\alpha) = M_X(\alpha) - X(1)$, $M_{\tilde{X}}(\alpha) - \tilde{X}(1) = M_X(\alpha)$ and $K_{\tilde{X}}(\alpha) = 1 - L_X(\alpha)$. □

Remarks

1. The distribution of $(\theta_X(1), S_X(1))$ is well known (see for example Karatzas and Shreve (1988, page 102). From this and Theorem 2, we can deduce the density of $(L_X(\alpha), M_X(\alpha))$. This is given by

$$\Pr(M_X(\alpha)\in db, L_X(\alpha)\in du) = \frac{|b|}{\pi\sqrt{u^3(1-u)}}\exp\left(-\frac{b^2}{2u}\right)\cdot$$

$$[1(0<u<\alpha, b>0) + 1(0<u<1-\alpha, b<0)]\,db\,du. \tag{35}$$

2. Theorem 1 also leads to an alternative expression for the distribution of $M_X(\alpha)$; that is
$$\Pr(M_X(\alpha)\in d\,b)=\Pr(S_X(1)\in d\,b, 0<\theta_X(1)<\alpha),$$
for $b>0$ and
$$\Pr(M_X(\alpha)\in d\,b)=\Pr(S_X(1)\in d|b|, 0<\theta_X(1)<1-\alpha),$$
for $b<0$.

3. From Theorem 1, we can immediately obtain the following corollary:

Corollary 11. *For* $u>0$,
$$\Pr(L_X(\alpha)>u)=\Pr(u<\theta_X(1)\leq\alpha)+\Pr(u<\theta_X(1)\leq 1-\alpha) \tag{36}$$
and
$$\Pr(L_X(\alpha)\in d\,u)=\frac{1(u\leq\alpha)+1(u\leq 1-\alpha)}{\pi\sqrt{u(1-u)}}d\,u. \tag{37}$$

Furthermore, $K_X(\alpha)$ *has the same distribution as* $1-L_X(\alpha)$.

3.2 The joint law of $(L_X(\alpha), K_X(\alpha), M_X(\alpha), X(1))$

Theorem 12. *For the standard Brownian motion* $(X(s), s\geq 0)$,
$$\Pr(L_X(\alpha)\in d\,u, K_X(\alpha)\in d\,v, M_X(\alpha)\in d\,b, X(1)\in d\,a)=$$
$$\frac{2|b||b-a|\,d\,u\,d\,v\,d\,b\,d\,a}{\pi^2(v-u)^2\sqrt{u^3(1-v)^3}}\exp\left(-\frac{b^2}{2u}-\frac{(b-a)^2}{2(1-v)}\right)\times$$
$$\begin{cases} \sqrt{(v-u-(1-\alpha))(1-\alpha)}\,1(u>0, u+(1-\alpha)<v<1) & b>0, b>a \\ \sqrt{(\alpha-u)(v-\alpha)}\,1(0<u<\alpha<v<1) & b>0, b<a \\ \sqrt{(v-u-\alpha)\alpha}\,1(u>0, u+\alpha<v<1) & b<0, b>a \\ \sqrt{(1-\alpha-u)(v-(1-\alpha))}\,1(0<u<1-\alpha<v<1) & b<0, b<a \end{cases}. \tag{38}$$

For the proof of this see AD (2005).

4 Variability orderings

One might have concluded from the last section that the median is just a special case of a quantile option with $\alpha = \frac{1}{2}$ and does not merit any extra interest from a mathematics point of view. This is not true. In this subsection we will investigate whether the median option is cheaper than a European option. We will also compare the median option to European options with different strike dates (we will see that this is more appropriate) and try to compare it to geometric average options.

The tool we use is the concept of stochastic variability introduced in section 1.3. Note that if the random variable X is stochastically more variable than Y, then $\exp(X)$ is stochastically more variable than $\exp(Y)$, and that $E\Big((V(t)-b)^+\Big)$ is a non-decreasing convex function. If we therefore compare $M_X\Big(\frac{1}{2}, t\Big)$ with $X(t)$, we can decide whether the median option on is cheaper than the European option (remember the underlying stock price is $Y(t) = Y(0)\exp\,(X(t))$). Also comparing $M_X\Big(\frac{1}{2},\ t\Big)$ with $\frac{\int_0^t X(s)ds}{t}$ we can decide whether median or geometric average options are cheaper. This is because $(\exp(\alpha x) - b)^+$ is an increasing convex function of x.

We will mostly work in discrete time; the continuous time results can follow as limiting cases. Let . Let $X_0 = 0$ and $X_n = \sum_{i=1}^n Y_i$ for $n = 1, 2, ...,$ where $Y_1, Y_2, ...$ is a sequence of independent and identically distributed random variables with finite mean $E(Y_1) = m$. In the sequel $Y_1^{(k)}, Y_2^{(k)}$ $k = 1, 2$ are independent copies of the sequence and $X_n^{(k)} = \sum_{i=1}^n Y_i^{(k)}$ Let us consider $M_{n,2n} - \frac{1}{2}X_{2n}$. This has the same distribution as

$$\max_{0\le i\le n}\left(X_i^{(1)}\right) + \min_{0\le i\le n}\left(X_i^{(2)}\right) - \frac{X_n^{(1)}}{2} - \frac{X_n^{(2)}}{2} =$$

$$\max_{0\le i\le n}\left(X_i^{(1)} - \frac{X_n^{(1)}}{2}\right) - \max_{0\le i\le n}\left(\frac{X_n^{(2)}}{2} - X_i^{(2)}\right) =$$

$$\max_{0\le i\le n}\left(X_i^{(1)} - \frac{X_n^{(1)}}{2}\right) - \max_{0\le i\le n}\left(\frac{X_n^{(2)}}{2} - X_{n-i}^{(2)}\right).$$

Note now that

$$X_i^{(1)} - \frac{X_n^{(1)}}{2} = \frac{-Y_1^{(1)} - Y_2^{(1)} - \cdots - Y_i^{(1)} + Y_{i+1}^{(1)} + \cdots + Y_n^{(1)}}{2}$$

and

$$\frac{X_n^{(2)}}{2} - X_{n-i}^{(2)} = \frac{-Y_n^{(2)} - Y_{n-1}^{(2)} - \cdots - Y_{n-i}^{(2)} + Y_{n-i-1}^{(2)} + \cdots + Y_1^{(2)}}{2}.$$

Note then that conditioning on $Y_1^{(1)} + Y_2^{(1)} + \cdots + Y_n^{(1)} + Y_1^{(2)} + Y_2^{(2)} + \cdots + Y_n^{(2)} = z$ all the $Y_i^{(k)}$'s are identically distributed (they are exchangeable) and hence $\frac{X_n^{(2)}}{2} - X_{n-i}^{(2)}$ and $\frac{X_n^{(2)}}{2} - X_{n-i}^{(2)}$ have the same conditional distributions. Therefore $\max_{0\le i\le n}\left(X_i^{(1)} - \frac{X_n^{(1)}}{2}\right)$ and $\max_{0\le i\le n}\left(\frac{X_n^{(2)}}{2} - X_{n-i}^{(2)}\right)$ have the same conditional distributions and hence

$$E\left(\max_{0\le i\le n}\left(X_i^{(1)}\right) + \min_{0\le i\le n}\left(X_i^{(2)}\right) - \frac{X_n^{(1)}}{2} - \frac{X_n^{(2)}}{2}\middle| X_n^{(1)} + X_n^{(2)}\right) = 0.$$

From section 1.3 we can then see that $\max_{0\le i\le n}\left(X_i^{(1)}\right) + \min_{0\le i\le n}\left(X_i^{(2)}\right)$ is therefore stochastically more variable than $\frac{X_n^{(1)}}{2} + \frac{X_n^{(2)}}{2}$ and so $M_{n,2n}$ is stochastically more variable than $\frac{1}{2}X_{2n}$. Moreover,

$$E(M_{n,2n}) = E\left(\frac{1}{2}X_{2n}\right) = nm. \tag{39}$$

By considering the limiting process as in AD (1996a) we conclude that if $(X(t), t \ge 0)$ is a process with stationary and independent increments with $E(X(t)) = mt$, we then have that $M_X\left(\frac{1}{2}, t\right)$ is stochastically more variable than $\frac{1}{2}X(t)$ and

$$E\left(M_X\left(\frac{1}{2}, t\right)\right) = E\left(\frac{1}{2}X(t)\right) = \frac{1}{2}nmt. \tag{40}$$

Equations (39) and (40) appear rather intuitive. However, it is hard to see how one can prove them without the results of the previous section.

It is now natural to compare $M_{n,2n}$ with other functionals of the same expectation such as X_n and $\frac{1}{2n+1}\sum_{i=0}^{2n} X_i$ or in the continuous time case $\frac{1}{2}X(t)$ and $\frac{\int_0^t X(s)ds}{t}$.

It turns out that $M_{n,2n}$ is stochastically less variable than X_n. In order to prove this first prove the following result.

Proposition 13. *Let M $M_{n,n} = \max_{0\le i\le n}(X_i)$ and $M_{0,n} = \min_{0\le i\le n}(X_i)$. Let also Y be a random variable with the same distribution as Y_1 and independent of the sequence $Y_1, Y_2, \ldots$. Then, $M_{n+1,n+1}$ has the same distribution as $(M_{n,n} + Y)^+$ and $-M_{0,n+1}$ has the same distribution as $(-M_{0,n} - Y)^+$*

Let us now prove our stochastic variability result.

Proposition 14. *Let $Y_1, Y_2, \ldots$ be i.i.d. random variables. Define $X = (X_0, X_1, \ldots)$ by*

$$X_n = \begin{cases} \sum_{i=1}^{n} Y_i & n = 1, 2, \ldots \\ 0 & n = 0 \end{cases}.$$

Then X_n is stochastically more variable than $M_{n,2n}$.

Proof. Recall that the random variable Z is stochastically more variable than the random variable Y if and only if $E\Big((Z-b)^+\Big) \geq E\Big((Y-b)^+\Big)$ for all $b \in \mathbb{R}$.

Assume $Y_1^{(k)}, Y_2^{(k)}$ $k = 1, 2$ are independent copies of $Y_1, Y_2, \ldots$. We will now proceed by induction. For $n = 1$, $X_1 = Y_1$, $M_{1,2}$ has the same distribution as $\Big(Y_1^{(1)}\Big)^+ - \Big(-Y_1^{(2)}\Big)^+$ and for $b \geq 0$,

$$E\bigg(\Big(Y_1^{(1)}\Big)^+ - \Big(-Y_1^{(2)}\Big)^+ - b\bigg)^+ = E\bigg(Y_1^{(1)} - \Big(-Y_1^{(2)}\Big)^+ - b\bigg)^+ \leq E(Y_1 - b)^+.$$

For $b < 0$,

$$E(M_{1,2} - b)^+ = E(M_{1,2} - b) + E(-M_{1,2} + b)^+ =$$

$$E(X_1 - b) + E\bigg(-\Big(Y_1^{(1)}\Big)^+ + \Big(-Y_1^{(2)}\Big)^+ + b\bigg)^+ =$$

$$E(X_1 - b) + E\bigg(-\Big(Y_1^{(1)}\Big)^+ - Y_1^{(2)} + b\bigg)^+ \leq$$

$$E(X_1 - b) - E\Big(-Y_1^{(2)} + b\Big)^+ = E(X_1 - b) - E(-X_1 + b)^+ = E(X_1 - b)^+.$$

We now prove the induction step. Suppose the proposition statement is true for $n = k$. We will prove it is also true for $n = k + 1$. $M_{k+1,2k+2}$ has the same distribution as $M_{k+1,k+1} + M_{0,k+1}$. Using the results of the previous exercise, we see that for $b \geq 0$,

$$E(M_{k+1,k+1} + M_{0,k+1} - b)^+ = E\Big((M_{k,k} + Y)^+ + M_{0,k+1} - b\Big)^+ =$$

$$E(M_{k,k} + Y + M_{0,k+1} - b)^+ \leq E(M_{k,k} + Y + M_{0,k+1} - b)^+ \leq$$

$$E(M_{k,k} + Y + M_{0,k} - b)^+ \leq E(X_k - b)^+.$$

A similar argument proves the case $b < 0$.

□

By taking limits one can show the following.

Proposition 15. *Let $(X(t), t \geq 0)$ be a Lévy process. The median $M_X\Big(\frac{1}{2}, t\Big)$ is stochastically less variable than $X\Big(\frac{t}{2}\Big)$.*

Note that this result does not depend on the distribution of the increments of the process (as long as they have a finite mean). One can now compare median with European options.

Comparing the median of a Lévy process with its average is an open problem and probably needs some assumptions about the distribution of the increments in order to proceed. We can make some headway in the case of the the standard Brownian motion $(B(t), t \geq 0)$. Note that then $M_B\Big(\frac{1}{2}, 1\Big)$ has a symmetric distribution and of course $E\Big(M_B\Big(\frac{1}{2}, 1\Big)\Big) = 0$. One can compare it to other symmetric random variables, which happen to be normally distributed. We have already established that $M_B\Big(\frac{1}{2}, 1\Big)$ is stochastically larger than a normal random variable with mean 0 and variance $\frac{1}{4}$ (this is $\frac{1}{2}B(1)$) and stochastically smaller than a normal random variable with mean 0 and variance $\frac{1}{2}$ (this is $B\Big(\frac{1}{2}\Big)$). The "upper" bound can not be improved. as we can see from the fact that

$$\lim_{b \to \infty} \frac{E(Z-b)^+}{E\Big(M_B\Big(\frac{1}{2}, 1\Big) - b\Big)^+} = 0.$$

However, we can improve the "lower" bound.

Proposition 16. *Let $(B(t), t \geq 0)$ be a standard Brownian motion and let Z be a normal random variable with mean 0 and variance σ^2, with $\sigma^2 \leq 6 - 4\sqrt{2}$. We then have that $M_B\Big(\frac{1}{2}, 1\Big)$ is stochastically larger than Z.*

Proof. We only need to prove it for $\sigma^2 = 6 - 4\sqrt{2}$. By symmetry we need to prove $E\Big(M_B\Big(\frac{1}{2}, 1\Big) - b\Big)^+ \geq E(Z-b)^+$ for $b \geq 0$ only. Set

$$g(b) = E(Z-b)^+ - E\bigg(M_B\Big(\frac{1}{2}, 1\Big) - b\bigg)^+.$$

Differentiating we get

$$g'(b) = \Pr\bigg(M_B\Big(\frac{1}{2}, 1\Big) > b\bigg) - \Pr(Z > b)$$

and once more

$$g''(b) = -\frac{4}{\sqrt{2\pi}}\bar{\Phi}(b)\exp\Big(-\frac{b^2}{2}\Big) + \frac{1}{\sqrt{2\pi\sigma^2}}\exp\Big(-\frac{b^2}{2\sigma^2}\Big)$$

where $\bar{\Phi}(b) = 1 - \Phi(b)$ (see exercise 18). This has the same sign as

$$h(b) = -\bar{\Phi}(b) + \frac{1}{4\sigma}\exp\left(-b^2\frac{1-\sigma^2}{2\sigma^2}\right)$$

whose derivative

$$h'(b) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{b^2}{2}\right) - b\frac{1-\sigma^2}{4\sigma^3}\exp\left(-b^2\frac{1-\sigma^2}{2\sigma^2}\right)$$

has the same sign as

$$k(b) = \exp\left(b^2\frac{1-2\sigma^2}{2\sigma^2}\right) - b\sqrt{2\pi}\frac{1-\sigma^2}{4\sigma^3}$$

which is a convex function with $k(0) > 0$ and $\lim_{b\to\infty} k(b) = \infty$ so it is either positive for all $b \geq 0$ or it is first positive then negative and then positive again. This means that $h(b)$ is either increasing or first increasing then decreasing and then increasing again, We observe that for $\sigma^2 = 6 - 4\sqrt{2}$, $h(0) < 0$ and $\lim_{b\to\infty} h(b) = 0$, so $h(b)$ and therefore $g''(b)$ is either negative for all $b \geq 0$ or first negative then positive and then negative again. So $g'(b)$ is either decreasing or decreasing, increasing and then decreasing again. Since $g'(0) = 0$ and $\lim_{b\to\infty} g'(b) = 0$, it can not be decreasing, so it is first decreasing, then increasing and then decreasing again and also it is first negative and then positive changing sign only once. We then conclude that $g(b)$ is first decreasing and then increasing. Note again form exercise 18 that $E\left(M_B\left(\frac{1}{2},1\right)\right)^+ = \sqrt{\frac{2}{\pi}}\left(1 - \sqrt{\frac{1}{2}}\right)$ and for $\sigma^2 = 6 - 4\sqrt{2}$, $E(Z)^+ = \frac{\sigma}{2}\sqrt{\frac{2}{\pi}} = \frac{\sqrt{6-4\sqrt{2}}}{2}\sqrt{\frac{2}{\pi}} = \sqrt{\frac{2}{\pi}}\left(1 - \sqrt{\frac{1}{2}}\right)$ and so $g(0) = 0$. Note also that $\lim_{b\to\infty} g(b) = 0$ and hence $g(b)$ has to be negative. We therefore conclude that

$$E(Z-b)^+ < E\left(M_B\left(\frac{1}{2},1\right) - b\right)^+$$

for all $b > 0$.

□

An important corollary is the following.

Corollary 17. $M_B\left(\frac{1}{2},1\right)$ *is stochastically larger than* $\int_0^1 B(s)\mathrm{ds}$.

Proof. Observe that $\int_0^1 B(s)\mathrm{ds}$ is normally distributed with mean zero and variance $\frac{1}{3}$. Since $\frac{1}{3} < 6 - 4\sqrt{2}$, the corollary follows. □

Note that $\frac{1}{3}$ and $6 - 4\sqrt{2}$ are very close so the result might be a bit fortuitous. The corollary does not generalise to other processes. It does not even generalise to the case of a Brownian motion with drift as it is possible to find μ and b such that $X(t) = \mu t + B(t)$ and

$$E\left(\int_0^1 X(s)\mathrm{ds} - b\right)^+ > E\left(M_X\left(\frac{1}{2},1\right) - b\right)^+.$$

One can see this by calculating both quantities for $\mu = 0.1$ and $b = 0.2$.

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