Ruin by Delayed Claims*

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Abstract

In this paper, we introduce a simple risk model with delayed claims, an extension of the classical Poisson model. The arrival of claims is assumed to be a Poisson process, and each loss payment of the claims will be settled with a random period of delay. We obtain the asymptotic expressions for the ruin probability, and exploit a connection to the Poisson models that are not time-homogeneous. In particular, the exact ruin probability can be derived for the special case with exponentially delayed claims and exponentially distributed sizes.

Keywords: Delayed claims; Risk model; Ruin probability; Asymptotics; Generalised Cramér-Lundberg approximation; Non-homogeneous Poisson process.

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JEL Classification C10

1 Introduction

In a variety of real situations, claims could have already occurred but have not been settled or reported immediately. Many factors may lead to the delay of the actual loss payment of the claims. For instance, the acronyms, such as IBNR (Incurred But Not Reported) and IBNR (Reported But Not Settled) are typically used to classify the delayed claims by different reasonings.

In the literature, the issues of ruin problem involving delayed claim settlement have been studied. Waters and Papatriandafylou (1985) and Trufin, Albrecher and Denuit (2011) considered a discrete-time model for a risk process allowing claims being delayed. Boogaert and Haezendonck (1989) discussed a liability process with settling delay in the framework of economical environment. Yuen, Guo and Ng (2005) introduced a continuous-time model with one claim settled immediately and the other claim (named 'by-claim') settled with delay for the each time of claim occurrences. Delaying claims were also modelled by a Poisson shot noise process, see Klüppelberg and Mikosch (1995) and Brémaud (2000), or by a shot noise Cox process, see also Macci and Torrisi (2004) and Albrecher and Asmussen (2006).

This paper introduces a simple delayed-claim model. We assume claims arrive as a Poisson process, and each of the claims will be settled in a randomly delayed period of time. The loss of each claim

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payment only occurs at the settlement time, rather than at the arrival time. In particular, we consider the special case of exponential delay where the ultimate ruin probability and asymptotics can be exactly obtained by a power series, and this is also a simplified version of the model by Yuen, Guo and Ng (2005) without the immediate settled claims.

The paper is organised as follows. Section 2 introduces our model setting of the delayed-claim risk process and the underlying processes of claim arrival, delay and settlement. Section 3 derives the asymptotics of ruin probability for the general case of delay, and in particular, exploit a well known connection to the non-homogeneous Poisson models. For the special case of exponential delay, the Laplace transform of non-ruin probability and the asymptotics of ruin probability are obtained in Section 4. Section 5 derives an exact formula of ruin probability by assuming the claims are exponentially delayed and sizes are exponentially distributed.

2 Risk Process

Consider a surplus process $\{X_t\}_{t\geq 0}$ in continuous time on a probability space $(\Omega, \mathcal{F}, P)$,

$$X_t = x + ct - \sum_{i=1}^{N_t} Z_i, \quad t \geq 0,$$

where

- $x = X_0 \geq 0$ is the initial reserve at time $t = 0$;
- $c > 0$ is the constant rate of premium payment per time unit;
- N_t is the number of cumulative settled claims within the time interval $[0, t]$ and assume $N_0 = 0$;
- $\{Z_i\}_{i=1,2,\ldots}$ is a sequence of independent and identically distributed positive random variables (claims sizes), independent of N_t, with the cumulative distribution function $Z(z), z > 0$; the mean, Laplace transform and tail of Z are denoted respectively by

$$\mu_{1_Z} = \int_0^\infty z\mathrm{d}Z(z), \quad \hat{z}(w) = \int_0^\infty e^{-wz}\mathrm{d}Z(z), \quad \overline{Z}(x) = \int_x^\infty \mathrm{d}Z(s).$$

Assume the arrival of claims follows a Poisson process of rate ρ, and each of the claims will be settled with a random delay. Loss only occurs when claims are being settled. M_t is denoted as the number of cumulative unsettled claims within the time interval $[0, t]$ and assume the initial number $M_0 = 0$. $\left\{T_k\right\}_{k=1,2,\ldots}$, $\left\{L_k\right\}_{k=1,2,\ldots}$ and $\left\{T_k + L_k\right\}_{k=1,2,\ldots}$ are denoted as the (random) times of claim arrival, delay and settlement, respectively, and hence,

$$\begin{aligned} M_t &= \sum_k \mathbb{I}\left\{T_k \leq t\right\}, \\ N_t &= \sum_k \mathbb{I}\left\{T_k + L_k \leq t\right\}. \end{aligned}$$

$\left\{L_k\right\}_{k=1,2,\ldots}$ are independent and identically distributed non-negative random variables with the cumulative distribution function L. A sample path of the joint point processes of the cumulative settled and unsettled claims (N_t, M_t) is given by *Figure 1*.

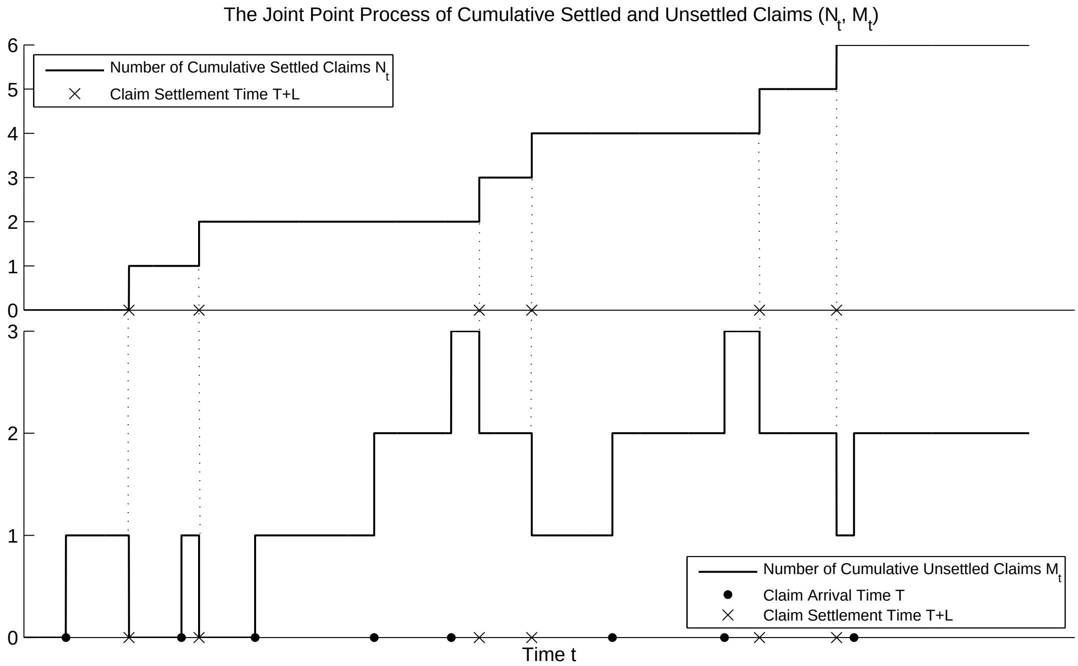

Figure 1: A Sample Path of the Joint Point Processes of Cumulative Settled and Unsettled Claims (N_t, M_t)

The ruin (stopping) time after time $t \geq 0$ is defined by

$$\tau_t^* =: \begin{cases} \inf\{s : s > t, X_s \leq 0\}, & \\ \inf\{\varnothing\} = \infty, & \text{if } X_s > 0 \text{ for all } t; \end{cases}$$

in particular, $\tau_t^* = \infty$ means ruin does not occur. We are interested in the ultimate ruin probability at time t, i.e.

$$\psi(x,t) =: P\left\{\tau_t^* < \infty \middle| X_t = x\right\}, \tag{1}$$

or, the ultimate non-ruin probability at time t, i.e.

$$\phi(x,t) =: 1 - \psi(x,t). \tag{2}$$

Note that, $\psi(x,t)$ defined by (1) is the ultimate ruin probability at the general time $t \geq 0$, rather than the conventionally defined ruin probability of finite-horizon time t.

3 Ruin with Randomly Delayed Claims

3.1 Preliminaries

The net profit condition remains the same as the classical Poisson model, i.e. $c > \rho\mu_{1_Z}$, since, obviously,

$$\lim_{t\to\infty} \frac{\int_0^t \overline{L}(s)\mathrm{d}s}{t} = 0,$$

and

$$\lim_{t\to\infty} \frac{\mathbb{E}\left[X_t\right]}{t} = \lim_{t\to\infty} \frac{x + ct - \mu_{1_Z}\mathbb{E}\left[N_t\right]}{t} = \lim_{t\to\infty} \frac{x + ct - \mu_{1_Z}\rho\left(t - \int_0^t \overline{L}(s)\mathrm{d}s\right)}{t} = c - \rho\mu_{1_Z} > 0.$$

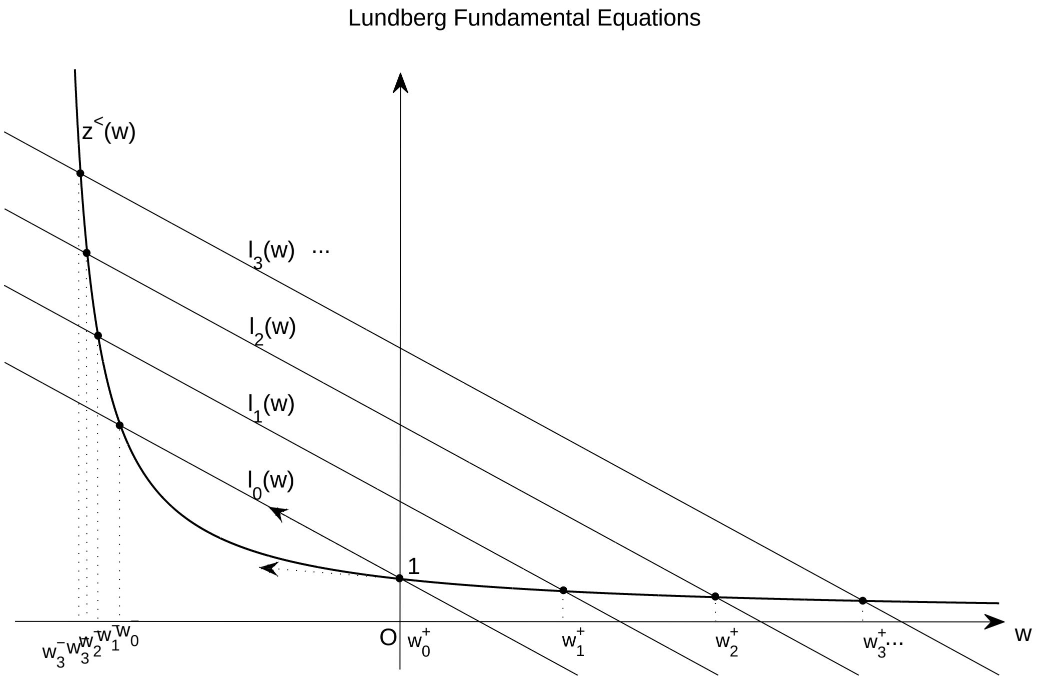

Figure 2: Lundberg Fundamental Equations

Lemma 3.1. *Assume $c > \rho\mu_{1_Z}$, we have a series of modified Lundberg fundamental equations*

$$cw - \rho\left[1 - \hat{z}(w)\right] - \delta j = 0, \quad j = 0, 1, ...; \tag{3}$$

- *for $j = 0$, (3) has solution zero and a unique negative solution (denoted by $W_0^+ = 0$ and $W_0^- < 0$);*
- *for $j = 1, 2, ...$, (3) has unique positive and negative solutions (denoted by $W_j^+ > 0$ and $W_j^- < 0$).*

Proof. Rewrite (3) as

$$\hat{z}(w) = l_j(w), \tag{4}$$

where

$$l_j(w) =: -\frac{c}{\rho}w + \left(1 + \frac{\delta}{\rho}j\right), \quad j = 0, 1, ...$$

Note that,

$$\left.\frac{\mathrm{d}\hat{z}(w)}{\mathrm{d}w}\right|_{w=0} = -\mu_{1_Z}, \quad \left.\frac{\mathrm{d}l_j(w)}{\mathrm{d}w}\right|_{w=0} = -\frac{c}{\rho},$$

by the net profit condition $c > \rho\mu_{1_Z}$, we have

$$\left.\frac{\mathrm{d}\hat{z}(w)}{\mathrm{d}w}\right|_{w=0} > \left.\frac{\mathrm{d}l_j(w)}{\partial w}\right|_{w=0}.$$

In particular, for $j = 0$, we have $l_0(0) = \hat{z}(0) = 1$. Then, further by the convexity of $\hat{z}(w)$ and the linearity of $l_j(w)$, the uniqueness of the positive and negative solutions to (3) follows immediately. It is more obvious by plotting (4) in *Figure 2*.

□

Denote the (modified) adjustment coefficients by

$$R_j =: -W_j^-, \quad j = 0, 1, ...,$$

note that,

$$0 < R_0 < R_1 < R_2 < ... < R_\infty,$$

where $R_\infty =: \inf\left\{R \middle| \hat{z}(-R) = \infty\right\}$.

Example 3.1. If $Z \sim \mathrm{Exp}(\gamma)$, then, we have a series of the modified Lundberg fundamental equations

$$cw^2 + (c\gamma - \rho - \delta j)w - \gamma\delta j = 0, \quad j = 0, 1, ...,$$

with explicit solutions

$$W_j^{\pm} = \frac{(\rho + \delta j - c\gamma) \pm \sqrt{(\rho + \delta j - c\gamma)^2 + 4c\gamma\delta j}}{2c}, \quad j = 0, 1, ...,$$

and

$$R_\infty = \lim_{j\to\infty} R_j = \gamma.$$

3.2 Asymptotics of Ruin Probability

By Mirasol (1963), we know that, a delayed (or displaced) Poisson process is still a (non-homogeneous) Poisson process, see also Newell (1966) and Lawrance and Lewis (1975). According to the model setting in Section 2, the settlement process N_t hence is a non-homogeneous Poisson process with rate $\rho L(t)$, and we can obtain the asymptotics of the ruin probability as below.

Theorem 3.1. *Assume $c > \rho\mu_{1_Z}$ and the first, second moments of L exist, we have the asymptotics of ruin probability*

$$\psi(x,t) \sim e^{-cR_0\int_t^\infty \overline{L}(s)\mathrm{d}s} \frac{c - \rho\mu_{1_Z}}{\rho\int_0^\infty ze^{R_0 z}\mathrm{d}Z(z) - c} e^{-R_0 x} + o\left(e^{-R_0 x}\right), \quad x \to \infty,$$

where $\overline{L}(t) =: 1 - L(t)$.

Proof. The integro-differential equation of the ruin probability $\psi(x,t)$ defined by (1) is given by

$$\frac{\partial\psi(x,t)}{\partial t} + c\frac{\partial\psi(x,t)}{\partial x} + \rho L(t)\left(\int_0^x \psi(x-z,t)\mathrm{d}Z(z) + \overline{Z}(x) - \psi(x,t)\right) = 0.$$

By the Laplace transform

$$\hat{\psi}(w,t) =: \mathcal{L}_w\left\{\psi(x,t)\right\} = \int_0^\infty e^{-wx}\psi(x,t)\mathrm{d}x,$$

we have

$$\frac{\partial\hat{\psi}(w,t)}{\partial t} + c\left(w\hat{\psi}(w,t) - \psi(0,t)\right) - \rho L(t)\left([1 - \hat{z}(w)]\,\hat{\psi}(w,t) - \frac{1 - \hat{z}(w)}{w}\right) = 0,$$

or,

$$\frac{\partial\hat{\psi}(w,t)}{\partial t} - c\psi(0,t) + \left(cw - \rho L(t)\,[1 - \hat{z}(w)]\right)\hat{\psi}(w,t) + \rho L(t)\frac{1 - \hat{z}(w)}{w} = 0. \tag{5}$$

Note that, the special case of $t \to \infty$ corresponds to the classical Poisson case as $L(t) \to 1$, i.e.

$$c\frac{\partial\psi(x,\infty)}{\partial x} + \rho\left(\int_0^x \psi(x-z,\infty)\mathrm{d}Z(z) + \overline{Z}(x) - \psi(x,\infty)\right) = 0,$$

and it is well known that the Laplace transform of the solution $\psi(x,\infty)$ is given by

$$\hat{\psi}(w,\infty) = \frac{\rho\left(\mu_{1_Z} - \frac{1-\hat{z}(w)}{w}\right)}{cw - \rho\,[1 - \hat{z}(w)]}.$$

Define

$$\hat{\psi}(w,t) =: \frac{\rho\left(\mu_{1_Z} - \frac{1-\hat{z}(w)}{w}\right)}{cw - \rho\left[1-\hat{z}(w)\right]} e^{\rho\int_t^\infty [1-\hat{z}(w)]\overline{L}(s)\mathrm{d}s} + \hat{k}(w,t), \tag{6}$$

where $\hat{k}(w,t)$ is the Laplace transform of a function $k(x,t)$ and satisfies

$$\lim_{t\to\infty} \hat{k}(w,t) = 0. \tag{7}$$

Plug (6) into (5), then, we have the ODE of $\hat{k}(w,t)$,

$$\begin{aligned}
&\frac{\partial \hat{k}(w,t)}{\partial t} + c\left(w\hat{k}(w,t) - \psi(0,t)\right) - \rho L(t)\left(\left[1-\hat{z}(w)\right]\hat{k}(w,t) - \frac{1-\hat{z}(w)}{w}\right) \\
&+\rho\left(\mu_{1_Z} - \frac{1-\hat{z}(w)}{w}\right) e^{\rho\int_t^\infty [1-\hat{z}(w)]\overline{L}(s)\mathrm{d}s} = 0,
\end{aligned}$$

or,

$$\begin{aligned}
&\frac{\partial \hat{k}(w,t)}{\partial t} + \left(cw - \rho\left[1-\hat{z}(w)\right] + \rho\overline{L}(t)\left[1-\hat{z}(w)\right]\right)\hat{k}(w,t) \\
= \;& c\left(\psi(0,t) - \frac{\rho\mu_{1_Z}}{c}\right) + \rho\left(\frac{1-\hat{z}(w)}{w} - \mu_{1_Z}\right)\left(e^{\rho\int_t^\infty [1-\hat{z}(w)]\overline{L}(s)\mathrm{d}s} - 1\right) + \rho\overline{L}(t)\frac{1-\hat{z}(w)}{w}.
\end{aligned}$$

By multiplying (multiplier factor) $e^{(cw-\rho[1-\hat{z}(w)])t}e^{-\rho\int_t^\infty \overline{L}(s)[1-\hat{z}(w)]\mathrm{d}s}$, we have

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\hat{k}(w,t)e^{(cw-\rho[1-\hat{z}(w)])t}e^{-\rho\int_t^\infty \overline{L}(s)[1-\hat{z}(w)]\mathrm{d}s}\right) \\
= \;& \left[c\left(\psi(0,t) - \frac{\rho\mu_{1_Z}}{c}\right) + \rho\left(\frac{1-\hat{z}(w)}{w} - \mu_{1_Z}\right)\left(e^{\rho\int_t^\infty [1-\hat{z}(w)]\overline{L}(s)\mathrm{d}s} - 1\right) + \rho\overline{L}(t)\frac{1-\hat{z}(w)}{w}\right] \\
&\times e^{(cw-\rho[1-\hat{z}(w)])t}e^{-\rho\int_t^\infty \overline{L}(s)[1-\hat{z}(w)]\mathrm{d}s},
\end{aligned}$$

with the boundary condition (7), and then the solution

$$\begin{aligned}
&\hat{k}(w,t) \\
= \;& e^{-(cw-\rho[1-\hat{z}(w)])t}e^{\rho\int_t^\infty \overline{L}(s)[1-\hat{z}(w)]\mathrm{d}s}\int_t^\infty e^{(cw-\rho[1-\hat{z}(w)])s}e^{-\rho\int_s^\infty \overline{L}(u)[1-\hat{z}(w)]\mathrm{d}u} \\
&\times\left[-c\left(\psi(0,s) - \frac{\rho\mu_{1_Z}}{c}\right) - \rho\left(\frac{1-\hat{z}(w)}{w} - \mu_{1_Z}\right)\left(e^{\rho\int_s^\infty [1-\hat{z}(w)]\overline{L}(u)\mathrm{d}u} - 1\right) - \rho\overline{L}(s)\frac{1-\hat{z}(w)}{w}\right]\mathrm{d}s.
\end{aligned} \tag{8}$$

Obviously, from *Figure 2*, for $-R_0 < w < 0$, we have $l_0(w) > \hat{z}(w)$, i.e.

$$cw - \rho\left[1-\hat{z}(w)\right] < 0, \quad -R_0 < w < 0.$$

Now, we discuss the three terms of $\hat{k}(w,t)$ given by (8), respectively.

1. It is well known that (see Gerber (1979) and Grandel (1991)), in the classical model when the claim settlement follows a Poisson process with a constant rate λ, the ruin probability with the initial reserve $x = 0$ is simply $\frac{\mu_{1_Z}}{c}\lambda$, whereas $\psi(0,t)$ here in the first term of (8) is based on the realisation of the rate $\{\rho L(s)\}_{t\le s\le\infty}$. Also, the cumulative function $L(s)$ is an increasing function of s, then, the ruin probability $\psi(0,t)$ should be greater than the case $\lambda = \rho L(t)$ and smaller than the case $\lambda = \rho L(\infty) = \rho$ of the classical model, i.e.

$$\frac{\mu_{1_Z}}{c}\rho L(t) < \psi(0,t) < \frac{\mu_{1_Z}}{c}\rho,$$

or,

$$0 < \frac{\rho\mu_{1_Z}}{c} - \psi(0,t) < \frac{\rho\mu_{1_Z}}{c}\overline{L}(t).$$

If the first moment of L exists, then, we have

$$\int_t^\infty \left|\psi(0,s)-\frac{\rho\mu_{1Z}}{c}\right|\mathrm{d}s<\frac{\rho\mu_{1Z}}{c}\int_t^\infty \overline{L}(s)\mathrm{d}s<\frac{\rho\mu_{1Z}}{c}\int_0^\infty \overline{L}(s)\mathrm{d}s<\infty.$$

2. For the second term of (8), if the second moment of L exists, then,

$$\begin{aligned}
& \int_t^\infty e^{-\rho\int_s^\infty[1-\hat{z}(w)]\overline{L}(u)\mathrm{d}u}\left(e^{\rho\int_s^\infty[1-\hat{z}(w)]\overline{L}(u)\mathrm{d}u}-1\right)\mathrm{d}s\\
= & \int_t^\infty\left(1-e^{-\rho\int_s^\infty[1-\hat{z}(w)]\overline{L}(u)\mathrm{d}u}\right)\mathrm{d}s\\
< & \int_t^\infty \rho\int_s^\infty[1-\hat{z}(w)]\,\overline{L}(u)\mathrm{d}u\mathrm{d}s\\
< & \rho\,[1-\hat{z}(w)]\int_0^\infty\int_s^\infty \overline{L}(u)\mathrm{d}u\mathrm{d}s<\infty.
\end{aligned}$$

3. For the third term of (8), if the first moment of L exists, then,

$$\int_t^\infty \rho\overline{L}(s)\frac{1-\hat{z}(w)}{w}\mathrm{d}s=\rho\frac{1-\hat{z}(w)}{w}\int_t^\infty\overline{L}(s)\mathrm{d}s<\rho\frac{1-\hat{z}(w)}{w}\int_0^\infty\overline{L}(s)\mathrm{d}s<\infty.$$

Therefore, for $-R_0<w<0$, we have

$$\hat{k}(w,t)<\infty,$$

and

$$\hat{k}(-R_0,t)=\lim_{w\downarrow -R_0}\hat{k}(w,t)=\int_0^\infty e^{R_0x}k(x,t)\mathrm{d}x<\infty,$$

hence,

$$k(x,t)=o\left(e^{-R_0x}\right).$$

By the *Final Value Theorem* and $\hat{\psi}(w,t)$ given by (6), we have

$$\begin{aligned}
& \lim_{x\to\infty}e^{R_0x}\psi(x,t)\\
= & \lim_{w\to 0}w\mathcal{L}_w\left\{e^{R_0x}\psi(x,t)\right\}\\
= & \lim_{w\to 0}w\hat{\psi}(w-R_0,t)\\
= & e^{\rho\int_t^\infty[1-\hat{z}(-R_0)]\overline{L}(s)\mathrm{d}s}\lim_{w\to 0}w\frac{\rho\left(\mu_{1Z}-\frac{1-\hat{z}(w-R_0)}{w-R_0}\right)}{c(w-R_0)-\rho\,[1-\hat{z}(w-R_0)]}+\lim_{w\to 0}w\hat{k}(w-R_0,t)\\
= & e^{\rho\int_t^\infty[1-\hat{z}(-R_0)]\overline{L}(s)\mathrm{d}s}\frac{c-\rho\mu_{1Z}}{\rho\int_0^\infty ze^{R_0z}\mathrm{d}Z(z)-c}+0\\
= & e^{-cR_0\int_t^\infty\overline{L}(s)\mathrm{d}s}\frac{c-\rho\mu_{1Z}}{\rho\int_0^\infty ze^{R_0z}\mathrm{d}Z(z)-c}.
\end{aligned}$$

Note that, by definition, $-R_0$ is the solution to $cw-\rho[1-\hat{z}(w)]=0$, and we have

$$1-\hat{z}(-R_0)=-\frac{cR_0}{\rho}.$$

□

4 Ruin with Exponentially Delayed Claims

By specifying the distribution of the period of delay L, we could improve the result in *Theorem 3.1* with higher order of asymptotics. Here, for instance, we consider the special case when the claims are exponentially delayed, in order to derive $o\left(e^{-R_0x}\right)$ with more details.

4.1 Laplace Transform of Non-ruin Probability

We derive the the Laplace transform of non-ruin probability in two different expressions as given by *Theorem 4.1* and *Theorem 4.2*, respectively, and then, they will be used to derive the asymptotics of ruin probability.

Theorem 4.1. *Assume $c > \rho\mu_{1_Z}$ and $L \sim \mathrm{Exp}(\delta)$, we have the Laplace transform of non-ruin probability*

$$\hat{\phi}(w,t) = e^{\vartheta e^{-\delta t}[1-\hat{z}(w)]}\left(\frac{c-\rho\mu_{1_Z}}{cw-\rho\left[1-\hat{z}(w)\right]} + c\sum_{j=1}^{\infty} e^{-j\delta t}\frac{\sum_{\ell=0}^{j} r_\ell \frac{[\vartheta\hat{z}(w)]^{j-\ell}}{(j-\ell)!}}{cw-\rho\left[1-\hat{z}(w)\right]-\delta j}\right), \tag{9}$$

where $\vartheta = \frac{\rho}{\delta}$,

$$r_0 = 1-\frac{\rho}{c}\mu_{1_Z}, \quad r_\ell = -\sum_{i=0}^{\ell-1}\frac{\left[\vartheta\hat{z}(W_\ell^+)\right]^{\ell-i}}{(\ell-i)!}r_i, \quad \ell = 1,2,.... \tag{10}$$

Proof. If $L \sim \mathrm{Exp}(\delta)$, then, $L(t) = 1-e^{-\delta t}$, and N_t is a non-homogeneous Poisson process with rate $\rho - \vartheta\delta e^{-\delta t}$, and the non-ruin probability $\phi(x,t)$ defined by (2) satisfies the integro-differential equation

$$\frac{\partial\phi(x,t)}{\partial t} + c\frac{\partial\phi(x,t)}{\partial x} + \left(\rho-\vartheta\delta e^{-\delta t}\right)\left(\int_0^x \phi(x-z,t)\mathrm{d}Z(z) - \phi(x,t)\right) = 0.$$

By the Laplace transform

$$\hat{\phi}(w,t) =: \mathcal{L}_w\left\{\phi(x,t)\right\} = \int_0^\infty e^{-wx}\phi(x,t)\mathrm{d}x, \tag{11}$$

we have

$$\frac{\partial\hat{\phi}(w,t)}{\partial t} + c\left(w\hat{\phi}(w,t)-\phi(0,t)\right) - \left(\rho-\vartheta\delta e^{-\delta t}\right)\left[1-\hat{z}(w)\right]\hat{\phi}(w,t) = 0. \tag{12}$$

Define

$$\hat{h}(w,t) =: \hat{\phi}(w,t)\exp\left(\int_0^t \delta\vartheta e^{-\delta s}\left[1-\hat{z}(w)\right]\mathrm{d}s\right),$$

where $\hat{h}(w,t)$ is the Laplace transform of a function $h(x,t)$, then,

$$\hat{\phi}(w,t) = \hat{h}(w,t)e^{-\vartheta\left(1-e^{-\delta t}\right)[1-\hat{z}(w)]}. \tag{13}$$

Plug (13) into (12), we have

$$\frac{\partial\hat{h}(w,t)}{\partial t} + c\left(w\hat{h}(w,t) - \phi(0,t)e^{\vartheta\left(1-e^{-\delta t}\right)[1-\hat{z}(w)]}\right) - \rho\left[1-\hat{z}(w)\right]\hat{h}(w,t) = 0. \tag{14}$$

Note that, by (13), we have

$$\begin{aligned}
\hat{\phi}(w,t) &= \hat{h}(w,t)e^{-\vartheta\left(1-e^{-\delta t}\right)}e^{\vartheta\left(1-e^{-\delta t}\right)\hat{z}(w)} \\
&= \hat{h}(w,t)e^{-\vartheta\left(1-e^{-\delta t}\right)}\left(1+\sum_{k=1}^{\infty}\frac{\left(\vartheta\left(1-e^{-\delta t}\right)\right)^k}{k!}\hat{z}^k(w)\right) \\
&= e^{-\vartheta\left(1-e^{-\delta t}\right)}\left(\hat{h}(w,t)+\sum_{k=1}^{\infty}\frac{\left(\vartheta\left(1-e^{-\delta t}\right)\right)^k}{k!}\hat{h}(w,t)\hat{z}^k(w)\right),
\end{aligned}$$

which is the Laplace transform of

$$\phi(x,t) = e^{-\vartheta\left(1-e^{-\delta t}\right)}\left(h(x,t)+\sum_{k=1}^{\infty}\frac{\left(\vartheta\left(1-e^{-\delta t}\right)\right)^k}{k!}\int_0^x h(x-z,t)\mathrm{d}Z^{(k)}(z)\right),$$

where $Z^{(k)}$ is the $k-$fold convolution of the distribution Z, i.e.

$$Z^{(k)} \stackrel{\mathcal{D}}{=} \sum_{i=1}^{k} Z_i,$$

then, we have

$$\phi(0,t) = h(0,t)e^{-\vartheta\left(1-e^{-\delta t}\right)}. \tag{15}$$

Plug (15) into (14), we have

$$\frac{\partial \hat{h}(w,t)}{\partial t} + \Big(cw - \rho\left[1-\hat{z}(w)\right]\Big)\hat{h}(w,t) - ce^{-\vartheta\hat{z}(w)}h(0,t)e^{\vartheta e^{-\delta t}\hat{z}(w)} = 0.$$

This equation of $\hat{h}(w,t)$ has a power series solution

$$\hat{h}(w,t) = \sum_{j=0}^{\infty} e^{-j\delta t}\hat{h}_j(w),$$

the Laplace transform of

$$h(x,t) = \sum_{j=0}^{\infty} e^{-j\delta t}h_j(x).$$

Since

$$\begin{aligned}
\frac{\partial \hat{h}(w,t)}{\partial t} &= -\delta\sum_{j=0}^{\infty} je^{-j\delta t}\hat{h}_j(w), \\
h(0,t)e^{\vartheta e^{-\delta t}\hat{z}(w)} &= \sum_{j=0}^{\infty} e^{-j\delta t}h_j(0) \times \sum_{k=0}^{\infty}\frac{e^{-k\delta t}\left[\vartheta\hat{z}(w)\right]^k}{k!} \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} e^{-(j+k)\delta t}h_j(0)\frac{\left[\vartheta\hat{z}(w)\right]^k}{k!} \qquad (j+k=i) \\
&= \sum_{i=0}^{\infty} e^{-i\delta t}\sum_{j=0}^{i} h_j(0)\frac{\left[\vartheta\hat{z}(w)\right]^{i-j}}{(i-j)!},
\end{aligned}$$

we have

$$\sum_{j=0}^{\infty} e^{-j\delta t}\left[\Big(-\delta j + cw - \rho\left[1-\hat{z}(w)\right]\Big)\hat{h}_j(w) - ce^{-\vartheta\hat{z}(w)}\sum_{\ell=0}^{j} h_\ell(0)\frac{\left[\vartheta\hat{z}(w)\right]^{j-\ell}}{(j-\ell)!}\right] = 0,$$

then, for any $j = 0, 1, ...,$

$$\Big(-\delta j + cw - \rho\left[1-\hat{z}(w)\right]\Big)\hat{h}_j(w) - ce^{-\vartheta\hat{z}(w)}\sum_{\ell=0}^{j} h_\ell(0)\frac{\left[\vartheta\hat{z}(w)\right]^{j-\ell}}{(j-\ell)!} = 0,$$

and hence,

$$\hat{h}_j(w) = \frac{ce^{-\vartheta\hat{z}(w)}}{cw - \rho\left[1-\hat{z}(w)\right] - \delta j}\sum_{\ell=0}^{j} h_\ell(0)\frac{\left[\vartheta\hat{z}(w)\right]^{j-\ell}}{(j-\ell)!}, \quad j = 0, 1, ... \tag{16}$$

Note that, the denominator of (16) is the modified Lundberg fundamental equation given by *Lemma 3.1*.

By (13), we have

$$\hat{\phi}(w,t) = e^{-\vartheta\left(1-e^{-\delta t}\right)\left[1-\hat{z}(w)\right]}\left(\hat{h}_0(w) + \sum_{j=1}^{\infty} e^{-j\delta t}\hat{h}_j(w)\right). \tag{17}$$

Note that, if $t \to \infty$, it recovers the classical Poisson model. By (17), we have

$$\hat{\phi}(w,\infty) = e^{-\vartheta[1-\hat{z}(w)]}\hat{h}_0(w), \tag{18}$$

$$\hat{\phi}(w,0) = \sum_{j=0}^{\infty}\hat{h}_j(w). \tag{19}$$

The series of constants $\{h_\ell(0)\}_{\ell=0,1,\ldots}$ in (16) can be obtained as follows.

For case $j=0$, by (16), we have

$$\hat{h}_0(w) = \frac{ce^{-\vartheta\hat{z}(w)}}{cw-\rho\left[1-\hat{z}(w)\right]}h_0(0).$$

By (15) and (18), we have

$$\begin{aligned}
\phi(0,\infty) &= h(0,\infty)e^{-\vartheta} = h_0(0)e^{-\vartheta},\\
\hat{\phi}(w,\infty) &= \hat{h}_0(w)e^{-\vartheta[1-\hat{z}(w)]} = \frac{ce^{-\vartheta}h_0(0)}{cw-\rho\left[1-\hat{z}(w)\right]} = \frac{c\phi(0,\infty)}{cw-\rho\left[1-\hat{z}(w)\right]}.
\end{aligned}$$

Since

$$\lim_{x\to\infty}\phi(x,\infty) = \lim_{w\to 0} w\hat{\phi}(w,\infty) = 1,$$

i.e.

$$\lim_{w\to 0} w\frac{c\phi(0,\infty)}{cw-\rho\left[1-\hat{z}(w)\right]} = \frac{c\phi(0,\infty)}{\lim_{w\to 0}\frac{1}{w}\left(cw-\rho\left[1-\hat{z}(w)\right]\right)} = \frac{c\phi(0,\infty)}{c-\rho\mu_{1_Z}} = 1,$$

we have

$$\phi(0,t) = \frac{c-\rho\mu_{1_Z}}{c},$$

$$h_0(0) = \frac{e^{\vartheta}\left(c-\rho\mu_{1_Z}\right)}{c}, \tag{20}$$

and

$$\hat{\phi}(w,t) = \frac{c-\rho\mu_{1_Z}}{cw-\rho\left[1-\hat{z}(w)\right]},$$

which is exactly the Laplace transform of ultimate non-ruin probability of the classical Poisson model. Hence, we have

$$\hat{h}_0(w) = e^{\vartheta[1-\hat{z}(w)]}\frac{c-\rho\mu_{1_Z}}{cw-\rho\left[1-\hat{z}(w)\right]}. \tag{21}$$

For case $j=1,2,\ldots$, since $\hat{h}_j(w)$ of (16) exists at $w=W_j^+$, we have

$$\lim_{w\to W_j^+}\left(ce^{-\vartheta\hat{z}(w)}\sum_{\ell=0}^{j}h_\ell(0)\frac{\left[\vartheta\hat{z}(w)\right]^{j-\ell}}{(j-\ell)!}\right) = 0, \quad j=1,2,\ldots,$$

or,

$$\sum_{\ell=0}^{j}\frac{\left[\vartheta\hat{z}(W_j^+)\right]^{j-\ell}}{(j-\ell)!}h_\ell(0) = 0, \quad j=1,2,\ldots.$$

Given the initial value $h_0(0)$ by (20), obviously, the series of constants $\{h_\ell(0)\}_{\ell=1,2,\ldots}$ can be solved uniquely and explicitly by recursion. Define the solution by

$$r_j =: e^{-\vartheta}h_j(0),$$

with the initial value $r_0 = 1 - \frac{\rho}{c}\mu_{1_Z}$, and we have

$$\hat{h}_j(w) = \frac{ce^{\vartheta[1-\hat{z}(w)]}}{cw - \rho\left[1 - \hat{z}(w)\right] - \delta j} \sum_{\ell=0}^{j} r_\ell \frac{\left[\vartheta \hat{z}(w)\right]^{j-\ell}}{(j-\ell)!}, \quad j = 1, 2, ..., \tag{22}$$

where

$$r_\ell = -\sum_{i=0}^{\ell-1} \frac{\left[\vartheta \hat{z}(W_\ell^+)\right]^{\ell-i}}{(\ell-i)!} r_i, \quad \ell = 1, 2, ...$$

Therefore, by (17), we have the Laplace transform of non-ruin probability

$$\hat{\phi}(w,t) = e^{-\vartheta\left(1-e^{-\delta t}\right)[1-\hat{z}(w)]} \left(\frac{e^{\vartheta[1-\hat{z}(w)]}\left(c - \rho\mu_{1_Z}\right)}{cw - \rho\left[1-\hat{z}(w)\right]} + \sum_{j=1}^{\infty} e^{-j\delta t} \frac{ce^{\vartheta[1-\hat{z}(w)]} \sum_{\ell=0}^{j} r_\ell \frac{[\vartheta \hat{z}(w)]^{j-\ell}}{(j-\ell)!}}{cw - \rho\left[1-\hat{z}(w)\right] - \delta j} \right).$$

□

Remark 4.1. In particular, for $t = 0$, we have

$$\hat{\phi}(w,0) = e^{\vartheta[1-\hat{z}(w)]} \left(\frac{c - \rho\mu_{1_Z}}{cw - \rho\left[1-\hat{z}(w)\right]} + c\sum_{j=1}^{\infty} \frac{\sum_{\ell=0}^{j} r_\ell \frac{[\vartheta \hat{z}(w)]^{j-\ell}}{(j-\ell)!}}{cw - \rho\left[1-\hat{z}(w)\right] - \delta j} \right),$$

and, for $t = \infty$,

$$\hat{\phi}(w,\infty) = \frac{c - \rho\mu_{1_Z}}{cw - \rho\left[1-\hat{z}(w)\right]},$$

which recovers the result of the classic Poisson model.

Alternatively, the Laplace transform of non-ruin probability can also be expressed by another power series as below.

Theorem 4.2. *Assume $c > \rho\mu_{1_Z}$ and $L \sim \mathrm{Exp}(\delta)$, we have the Laplace transform of the non-ruin probability*

$$\hat{\phi}(w,t) = \sum_{j=0}^{\infty} e^{-j\delta t} \hat{\phi}_j(w),$$

where $\left\{\hat{\phi}_j(w)\right\}_{j=0,1,...}$ follow the recurrence

$$\hat{\phi}_j(w) = \rho \frac{\left[1 - \hat{z}(W_j^+)\right] \hat{\phi}_{j-1}(W_j^+) - \left[1-\hat{z}(w)\right] \hat{\phi}_{j-1}(w)}{cw - \rho\left[1-\hat{z}(w)\right] - \delta j}, \quad j = 1, 2, ..., \tag{23}$$

$$\hat{\phi}_0(w) = \frac{c\left(1 - \frac{\rho}{c}\mu_{1_Z}\right)}{cw - \rho\left[1-\hat{z}(w)\right]}. \tag{24}$$

Proof. Rewrite (12) as

$$\frac{\partial \hat{\phi}(w,t)}{\partial t} + c\left(w\hat{\phi}(w,t) - \phi(0,t)\right) - \rho\left[1-\hat{z}(w)\right]\hat{\phi}(w,t) + \rho\left[1-\hat{z}(w)\right]e^{-\delta t}\hat{\phi}(w,t) = 0.$$

This equation has a power series solution

$$\hat{\phi}(w,t) = \sum_{j=0}^{\infty} e^{-j\delta t} \hat{\phi}_j(w),$$

the Laplace transform of the non-ruin probability

$$\phi(x,t) = \sum_{j=0}^{\infty} e^{-j\delta t} \phi_j(x).$$

Note that, by setting $\hat{\phi}_{-1}(w)=0$, we have

$$\begin{aligned}\frac{\partial\hat{\phi}(w,t)}{\partial t} &= -\delta\sum_{j=0}^{\infty} je^{-j\delta t}\hat{\phi}_j(w),\\ e^{-\delta t}\hat{\phi}(w,t) &= \sum_{j=0}^{\infty} e^{-(j+1)\delta t}\hat{\phi}_j(w)=\sum_{j=1}^{\infty} e^{-j\delta t}\hat{\phi}_{j-1}(w)=\sum_{j=0}^{\infty} e^{-j\delta t}\hat{\phi}_{j-1}(w),\end{aligned}$$

then,

$$\begin{aligned}&-\delta\sum_{j=0}^{\infty} je^{-j\delta t}\hat{\phi}_j(w)+c\left(w\sum_{j=0}^{\infty} e^{-j\delta t}\hat{\phi}_j(w)-\sum_{j=0}^{\infty} e^{-j\delta t}\phi_j(0)\right)-\rho\left[1-\hat{z}(w)\right]\sum_{j=0}^{\infty} e^{-j\delta t}\hat{\phi}_j(w)\\ &+\rho\left[1-\hat{z}(w)\right]\sum_{j=0}^{\infty} e^{-j\delta t}\hat{\phi}_{j-1}(w)=0,\end{aligned}$$

or,

$$\sum_{j=0}^{\infty} e^{-j\delta t}\Big[-\delta j\hat{\phi}_j(w)+c\left(w\hat{\phi}_j(w)-\phi_j(0)\right)-\rho\left[1-\hat{z}(w)\right]\hat{\phi}_j(w)+\rho\left[1-\hat{z}(w)\right]\hat{\phi}_{j-1}(w)\Big]=0,$$

and then, for any $j=0,1,...,$

$$-\delta j\hat{\phi}_j(w)+c\left(w\hat{\phi}_j(w)-\phi_j(0)\right)-\rho\left[1-\hat{z}(w)\right]\hat{\phi}_j(w)+\rho\left[1-\hat{z}(w)\right]\hat{\phi}_{j-1}(w)=0.$$

Hence, we have

$$\hat{\phi}_j(w)=\frac{c\phi_j(0)-\rho\left[1-\hat{z}(w)\right]\hat{\phi}_{j-1}(w)}{cw-\rho\left[1-\hat{z}(w)\right]-\delta j},\quad j=0,1,....$$

For the initial case $j=0$, note that $\hat{\phi}_{-1}(w)=0$, we have

$$\hat{\phi}_0(w)=\frac{c\phi_0(0)}{cw-\rho\left[1-\hat{z}(w)\right]}.$$

By the boundary condition

$$\lim_{w\to 0} w\hat{\phi}_0(w)=\lim_{x\to\infty}\phi_0(x)=1,$$

we have

$$\lim_{w\to 0} w\hat{\phi}_0(w)=\lim_{w\to 0}\frac{c\phi_0(0)}{c-\rho\frac{1-\hat{z}(w)}{w}}=\frac{c\phi_0(0)}{c-\rho\mu_{1Z}}=1,$$

then,

$$\phi_0(0)=1-\frac{\rho}{c}\mu_{1Z},$$

and $\hat{\phi}_0(w)$ as given by (24). Since $\hat{\phi}_j(w)$ exists at $w=W_j^+$ for any $j=1,2,...$, we have

$$\lim_{w\to W_j^+}\left(c\phi_j(0)-\rho\left[1-\hat{z}(w)\right]\hat{\phi}_{j-1}(w)\right)=0,$$

and

$$\phi_j(0)=\frac{\rho}{c}\left[1-\hat{z}(W_j^+)\right]\hat{\phi}_{j-1}(W_j^+),\quad j=1,2,....$$

Hence, we have the recurrence relation between $\hat{\phi}_j(w)$ and $\hat{\phi}_{j-1}(w)$ as given by (23).

$\square$

Remark 4.2. *Theorem 4.1* will be used to derive a general asymptotic formula (given by *Theorem 4.3*), whereas *Theorem 4.2* is more useful for obtaining an exact expression in the case of exponentially distributed claim sizes (given by *Theorem 5.1*).

4.2 Asymptotics of Ruin Probability

Theorem 4.3. *Assume* $c > \rho\mu_{1_Z}$ *and* $L \sim \mathrm{Exp}(\delta)$, *we have the asymptotics of the ruin probability*

$$\psi(x,t) \sim \sum_{j=0}^{\infty} \kappa_j(t) e^{-R_j x}, \quad x \to \infty, \tag{25}$$

where

$$\kappa_0(t) =: e^{-\frac{cR_0}{\rho}\vartheta e^{-\delta t}} \frac{c-\rho\mu_{1_Z}}{\rho\int_0^\infty z e^{R_0 z}\mathrm{d}Z(z) - c}, \tag{26}$$

$$\kappa_j(t) =: e^{-j\delta t} \frac{c e^{\vartheta e^{-\delta t}[1-\hat{z}(-R_j)]}}{\rho\int_0^\infty z e^{R_j z}\mathrm{d}Z(z) - c} \sum_{\ell=0}^{j} r_\ell \frac{[\vartheta\hat{z}(-R_j)]^{j-\ell}}{(j-\ell)!}, \quad j = 1,2,.... \tag{27}$$

Proof. Denote

$$\phi(x,t) =: \sum_{j=0}^{\infty} \phi_j(x,t),$$

then,

$$\hat{\phi}(w,t) = \sum_{j=0}^{\infty} \hat{\phi}_j(w,t),$$

where every term $\hat{\phi}_j(w,t)$ is specified by (9), i.e.

$$\hat{\phi}_0(w,t) =: e^{\vartheta e^{-\delta t}[1-\hat{z}(w)]} \frac{c-\rho\mu_{1_Z}}{cw - \rho[1-\hat{z}(w)]}, \tag{28}$$

$$\hat{\phi}_j(w,t) =: c e^{\vartheta e^{-\delta t}[1-\hat{z}(w)]} e^{-j\delta t} \frac{\sum_{\ell=0}^{j} r_\ell \frac{[\vartheta\hat{z}(w)]^{j-\ell}}{(j-\ell)!}}{cw - \rho[1-\hat{z}(w)] - \delta j}, \quad j = 1,2,.... \tag{29}$$

Now, we discuss the asymptotics of the terms $\phi_0(x,t)$ and $\{\phi_j(x,t)\}_{j=1,2,...}$, respectively.

For $\phi_0(x,t)$, we have the asymptotics

$$1 - \phi_0(x,t) \sim \kappa_0(t) e^{-R_0 x}, \quad x \to \infty,$$

since by *Final Value Theorem*,

$$\begin{aligned}
\kappa_0(t) &= \lim_{x\to\infty} e^{R_0 x}\Big(1-\phi_0(x,t)\Big) \\
&= \lim_{w\to 0} w\mathcal{L}_w\left\{e^{R_0 x}\Big(1-\phi_0(x,t)\Big)\right\} \\
&= \lim_{w\to 0} w\left(\frac{1}{w-R_0} - \hat{\phi}_0(w-R_0,t)\right) \\
&= -\lim_{w\to 0} w\hat{\phi}_0(w-R_0,t) \\
&= -\lim_{w\to 0} w\frac{e^{\vartheta e^{-\delta t}[1-\hat{z}(w-R_0)]}\left(c-\rho\mu_{1_Z}\right)}{c(w-R_0) - \rho[1-\hat{z}(w-R_0)]} \\
&= -\frac{e^{\vartheta e^{-\delta t}[1-\hat{z}(-R_0)]}\left(c-\rho\mu_{1_Z}\right)}{\frac{\mathrm{d}}{\mathrm{d}w}\Big(c(w-R_0) - \rho[1-\hat{z}(w-R_0)]\Big)\Big|_{w=0}} \\
&= \frac{e^{-\frac{cR_0}{\rho}\vartheta e^{-\delta t}}\left(c-\rho\mu_{1_Z}\right)}{\rho\int_0^\infty z e^{R_0 z}\mathrm{d}Z(z) - c}.
\end{aligned}$$

For $\phi_j(x,t)$, $j = 1,2,...$, we have the asymptotics

$$-\phi_j(x,t) \sim \kappa_j(t) e^{-R_j x}, \quad x \to \infty,$$

since, by *Final Value Theorem*,

$$
\begin{aligned}
\kappa_j(t) &= \lim_{x\to\infty} e^{R_j x}(-\phi_j(x,t)) \\
&= \lim_{w\to 0} w\mathcal{L}_w\left\{e^{R_j x}(-\phi_j(x,t))\right\} \\
&= -\lim_{w\to 0} w\hat{\phi}_j(w-R_j,t) \\
&= -\lim_{w\to 0}\left(w\frac{ce^{\vartheta e^{-\delta t}[1-\hat{z}(w-R_j)]}e^{-j\delta t}}{c(w-R_j)-\rho\left[1-\hat{z}(w-R_j)\right]-\delta j}\sum_{\ell=0}^{j} r_\ell \frac{[\vartheta\hat{z}(w-R_j)]^{j-\ell}}{(j-\ell)!}\right) \\
&= -\frac{ce^{\vartheta e^{-\delta t}[1-\hat{z}(-R_j)]}e^{-j\delta t}}{\frac{\mathrm{d}}{\mathrm{d}w}\left(c(w-R_j)-\rho\left[1-\hat{z}(w-R_j)\right]-\delta j\right)\Big|_{w=0}}\sum_{\ell=0}^{j} r_\ell \frac{[\vartheta\hat{z}(-R_j)]^{j-\ell}}{(j-\ell)!} \\
&= \frac{ce^{\vartheta e^{-\delta t}[1-\hat{z}(-R_j)]}e^{-j\delta t}}{\rho\int_0^\infty z e^{R_j z}\mathrm{d}Z(z)-c}\sum_{\ell=0}^{j} r_\ell \frac{[\vartheta\hat{z}(-R_j)]^{j-\ell}}{(j-\ell)!}.
\end{aligned}
$$

Therefore,

$$
\psi(x,t)=1-\phi(x,t)=1-\phi_0(x,t)+\sum_{j=1}^{\infty}-\phi_j(x,t),
$$

the result of asymptotics (25) follows immediately.

□

Remark 4.3. Set $L(t)=1-\frac{\vartheta}{\rho}\delta e^{-\delta t}$ and $t=0$ in *Theorem 3.1*, then, $\int_0^\infty \overline{L}(s)\mathrm{d}s=\frac{\vartheta}{\rho}$ and it recovers $\kappa_0(t)e^{-R_0 x}$, the first-order asymptotics of the ruin probability obtained by *Theorem 4.3*. The higher orders of asymptotics depend on the distributional property of the general distribution function L.

Remark 4.4. We can rewrite $\hat{\phi}_0(w,t)$ of (28) by

$$
\begin{aligned}
\hat{\phi}_0(w,t) &= e^{\vartheta e^{-\delta t}[1-\hat{z}(w)]}\frac{1}{w}\frac{p_0}{1-(1-p_0)\frac{1-\hat{z}(w)}{\mu_{1_Z} w}} \\
&= e^{\vartheta e^{-\delta t}[1-\hat{z}(w)]}\frac{1}{w}\sum_{i=0}^{\infty} p_0\left(1-p_0\right)^i\left(\frac{1-\hat{z}(w)}{\mu_{1_Z} w}\right)^i, \quad p_0=1-\frac{\rho\mu_{1_Z}}{c}.
\end{aligned}
$$

The third term of $\hat{\phi}_0(w,t)$ above is the Laplace transform of a compound geometric distribution

$$
\sum_{i=0}^{\infty} p_0\left(1-p_0\right)^i d_0^{(i)}(x),
$$

where $d_0^{(i)}(x)$ is the $i-$fold convolution of a proper density function

$$
d_0(x)=:\frac{\overline{Z}(x)}{\mu_{1_Z}},
$$

since $0<p_0<1$ and

$$
\begin{aligned}
\mathcal{L}_w\left\{d_0(x)\right\} &= \frac{1-\hat{z}(w)}{\mu_{1_Z} w}, \\
\int_0^\infty d_0(x)\mathrm{d}x &= \mathcal{L}_w\left\{d_0(x)\right\}\Big|_{w=0}=\frac{1}{\mu_{1_Z}}\lim_{w\to 0}\frac{1-\hat{z}(w)}{w}=\frac{1}{\mu_{1_Z}}\mu_{1_Z}=1.
\end{aligned}
$$

For, $j = 0, 1, ...$, we can also rewrite $\hat{\phi}_j(w,t)$ of (29) by

$$\begin{aligned}
&\hat{\phi}_j(w,t) \\
= &\frac{w - W_j^+}{cw - \rho\left[1-\hat{z}(w)\right] - \delta j - \left(cW_j^+ - \rho\left[1-\hat{z}(W_j^+)\right] - \delta j\right)} \frac{ce^{\vartheta e^{-\delta t}[1-\hat{z}(w)]}e^{-j\delta t}}{w - W_j^+} \sum_{\ell=0}^{j} r_\ell \frac{[\vartheta \hat{z}(w)]^{j-\ell}}{(j-\ell)!} \\
= &\frac{w - W_j^+}{c(w - W_j^+) - \rho\left[\hat{z}(W_j^+) - \hat{z}(w)\right]} \frac{ce^{\vartheta e^{-\delta t}[1-\hat{z}(w)]}e^{-j\delta t}}{w - W_j^+} \sum_{\ell=0}^{j} r_\ell \frac{[\vartheta \hat{z}(w)]^{j-\ell}}{(j-\ell)!} \\
= &\frac{p_j}{1 - (1-p_j)\frac{W_j^+}{1-\hat{z}(W_j^+)}\frac{\hat{z}(W_j^+)-\hat{z}(w)}{w-W_j^+}} \frac{1}{p_j} \frac{e^{\vartheta e^{-\delta t}[1-\hat{z}(w)]}e^{-j\delta t}}{w - W_j^+} \sum_{\ell=0}^{j} r_\ell \frac{[\vartheta \hat{z}(w)]^{j-\ell}}{(j-\ell)!}, \\
= &\sum_{i=0}^{\infty} p_j (1-p_j)^i \left(\frac{W_j^+}{1-\hat{z}(W_j^+)} \frac{\hat{z}(W_j^+) - \hat{z}(w)}{w - W_j^+}\right)^i \times \frac{1}{p_j} \frac{e^{\vartheta e^{-\delta t}[1-\hat{z}(w)]}e^{-j\delta t}}{w - W_j^+} \sum_{\ell=0}^{j} r_\ell \frac{[\vartheta \hat{z}(w)]^{j-\ell}}{(j-\ell)!},
\end{aligned}$$

where $p_j = \frac{\delta j}{cW_j^+}$. The first term of $\hat{\phi}_j(w,t)$ above is the Laplace transform of a compound geometric distribution

$$\sum_{i=0}^{\infty} p_j\,(1-p_j)^i\, d_j^{(i)}(x),$$

where $d_j^{(i)}(x)$ is the $i-$fold convolution of a proper density function

$$d_j(x) =: \frac{W_j^+}{1-\hat{z}(W_j^+)} e^{W_j^+ x} \int_x^{\infty} e^{-W_j^+ z} \mathrm{d}Z(z),$$

since

$$0 < p_j = 1 - \frac{\rho}{c}\frac{1-\hat{z}(W_j^+)}{W_j^+} = \frac{\delta j}{cW_j^+} = \frac{\delta j}{\rho\left[1-\hat{z}(W_j^+)\right] + \delta j} < 1,$$

and

$$\begin{aligned}
\mathcal{L}_w\left\{d_j(x)\right\} &= \frac{W_j^+}{1-\hat{z}(W_j^+)} \frac{\hat{z}(W_j^+) - \hat{z}(w)}{w - W_j^+}, \\
\int_0^{\infty} d_j(x)\mathrm{d}x &= \frac{W_j^+}{1-\hat{z}(W_j^+)} \frac{\hat{z}(W_j^+) - \hat{z}(w)}{w - W_j^+}\bigg|_{w=0} = 1.
\end{aligned}$$

Note that, for a constant ν, we have

$$\mathcal{L}_w\left\{e^{\nu x}\int_x^{\infty} e^{-\nu z}\mathrm{d}Z(z)\right\} = \frac{\hat{z}(\nu) - \hat{z}(w)}{w - \nu}.$$

5 Ruin with Exponentially Delayed Claims and Exponentially Distributed Sizes

The asymptotic formula of (25) becomes exact if the claim sizes follow an exponential distribution.

Theorem 5.1. *Assume $c > \rho\mu_{1_Z}$, $L \sim \mathrm{Exp}(\delta)$ and Z follows an exponential distribution, we have the ruin probability*

$$\psi(x,t) = \sum_{j=0}^{\infty} \kappa_j(t) e^{-R_j x}. \tag{30}$$

Proof. By *Theorem 4.2*, if $Z \sim \mathrm{Exp}(\gamma)$, then, for $j = 0$, we have

$$\hat{\phi}_0(w) = \frac{c - \frac{\rho}{\gamma}}{cw - \rho\frac{w}{\gamma+w}} = \left(1 - \frac{\rho}{c\gamma}\right)\frac{\gamma + w}{(w + R_0)w}. \tag{31}$$

For $j = 1, 2, ...$, we have

$$\begin{aligned}
\hat{\phi}_j(w) &= \rho\frac{\frac{W_j^+}{\gamma+W_j^+}\hat{\phi}_{j-1}(W_j^+) - \frac{w}{\gamma+w}\hat{\phi}_{j-1}(w)}{cw - \rho\frac{w}{\gamma+w} - \delta j} \\
&= \rho\frac{\frac{W_j^+}{\gamma+W_j^+}\hat{\phi}_{j-1}(W_j^+)(\gamma + w) - w\hat{\phi}_{j-1}(w)}{c(w + R_j)(w - W_j^+)} \\
&= \rho\frac{\frac{W_j^+}{\gamma+W_j^+}\hat{\phi}_{j-1}(W_j^+)(\gamma + W_j^+ + w - W_j^+) - w\hat{\phi}_{j-1}(w)}{c(w + R_j)(w - W_j^+)} \\
&= \rho\frac{W_j^+\hat{\phi}_{j-1}(W_j^+) - w\hat{\phi}_{j-1}(w) + \frac{W_j^+}{\gamma+W_j^+}\hat{\phi}_{j-1}(W_j^+)(w - W_j^+)}{c(w + R_j)(w - W_j^+)} \\
&= \rho\frac{\frac{W_j^+\hat{\phi}_{j-1}(W_j^+) - w\hat{\phi}_{j-1}(w)}{w - W_j^+} + \frac{W_j^+}{\gamma+W_j^+}\hat{\phi}_{j-1}(W_j^+)}{c(w + R_j)}.
\end{aligned}$$

In particular, for $j = 1$, we observe

$$\begin{aligned}
\hat{\phi}_1(w) &= \rho\frac{\frac{W_1^+\hat{\phi}_0(W_1^+) - w\hat{\phi}_0(w)}{w - W_1^+} + \frac{W_1^+}{\gamma+W_1^+}\hat{\phi}_0(W_1^+)}{c(w + R_1)} \\
&= \rho\frac{\left(1 - \frac{\rho}{c\gamma}\right)\frac{\gamma - R_0}{W_1^+ + R_0}\frac{1}{w + R_0} + \frac{W_1^+}{\gamma+W_1^+}\hat{\phi}_0(W_1^+)}{c(w + R_1)} \\
&= \rho\frac{\left(1 - \frac{\rho}{c\gamma}\right)\frac{\gamma - R_0}{W_1^+ + R_0} + \frac{W_1^+}{\gamma+W_1^+}\hat{\phi}_0(W_1^+)(w + R_0)}{c(w + R_0)(w + R_1)},
\end{aligned}$$

which is the Laplace transform of a linear combination of $e^{-R_0 x}$ and $e^{-R_1 x}$.

In general, for $j = 1, 2, ...$, assume

$$\hat{\phi}_j(w) = \frac{P_j(w)}{c\prod_{i=0}^{j}(w + R_i)}, \quad j = 1, 2, ...,$$

where $\{P_j(w)\}_{j=1,2,...}$ are functions of w, then,

$$\frac{P_j(w)}{c\prod_{i=0}^{j}(w + R_i)} = \rho\frac{\frac{W_j^+\frac{P_{j-1}(W_j^+)}{c\prod_{i=0}^{j-1}(W_j^+ + R_i)} - w\frac{P_{j-1}(w)}{c\prod_{i=0}^{j-1}(w + R_i)}}{w - W_j^+} + \frac{W_j^+}{\gamma+W_j^+}\frac{P_{j-1}(W_j^+)}{c\prod_{i=0}^{j-1}(W_j^+ + R_i)}}{c(w + R_j)},$$

or,

$$P_j(w) = \frac{\rho}{c}\prod_{i=0}^{j-1}(w + R_i)\left[\frac{W_j^+\frac{P_{j-1}(W_j^+)}{\prod_{i=0}^{j-1}(W_j^+ + R_i)} - w\frac{P_{j-1}(w)}{\prod_{i=0}^{j-1}(w + R_i)}}{w - W_j^+} + \frac{W_j^+}{\gamma + W_j^+}\frac{P_{j-1}(W_j^+)}{\prod_{i=0}^{j-1}(W_j^+ + R_i)}\right],$$

then, we have

$$P_j(w) = \frac{\rho}{c}\left[\frac{W_j^+ \frac{P_{j-1}(W_j^+)}{\prod_{i=0}^{j-1}(W_j^+ + R_i)}\prod_{i=0}^{j-1}(w+R_i) - wP_{j-1}(w)}{w - W_j^+} + \frac{W_j^+}{\gamma + W_j^+}\frac{P_{j-1}(W_j^+)}{\prod_{i=0}^{j-1}(W_j^+ + R_i)}\prod_{i=0}^{j-1}(w+R_i)\right], \quad j = 2, 3, ...,$$

$$P_1(w) = \rho\left[\left(1 - \frac{\rho}{c\gamma}\right)\frac{\gamma - R_0}{W_1^+ + R_0} + \frac{W_1^+}{\gamma + W_1^+}\hat{\phi}_0(W_1^+)(w + R_0)\right].$$

Note that, for $j = 2, 3, ...$, $w = W_j^+$ is one of the roots of the numerator of the first term, the denominator $w - W_j^+$ then is canceled. $P_1(w)$ is a polynomial function with degree of 1, and obviously, by the method of induction, $\{P_j(w)\}_{j=1,2,...}$ are polynomial functions of w with maximum degree of j. Hence, for any $j = 1, 2, ...$, we can have a partial fraction decomposition

$$\frac{P_j(w)}{c\prod_{i=0}^{j}(w + R_i)} = \sum_{i=0}^{j} b_{ji}\frac{1}{w + R_i},$$

where $\{b_{ji}\}_{i=0,1,...,j}$ are all constants. Since

$$\mathcal{L}_w\left\{e^{-R_i x}\right\} = \frac{1}{w + R_i}, \quad i = 0, 1, ..., j,$$

we have

$$\phi_j(x) = \sum_{i=0}^{j} b_{ji}e^{-R_i x}, \quad j = 1, 2, ...$$

For $j = 0$, we have $R_0 = \gamma - \frac{\rho}{c}$, and rewrite (31) as

$$\hat{\phi}_0(w) = \left(1 - \frac{\rho}{c\gamma}\right)\left[\frac{\gamma}{R_0}\frac{1}{w} + \left(1 - \frac{\gamma}{R_0}\right)\frac{1}{w + R_0}\right] = \frac{1}{w} - \frac{\rho}{c\gamma}\frac{1}{w + R_0},$$

which is the Laplace transform of

$$\phi_0(x) = 1 - \frac{\rho}{c\gamma}e^{-R_0 x}.$$

Then, the ruin probability $\psi(x, t)$ is a linear combination of $\left\{e^{-R_j x}\right\}_{j=0,1,...}$, since

$$\psi(x,t) = 1 - \phi(x,t) = 1 - \phi_0(x) - \sum_{j=1}^{\infty} e^{-j\delta t}\phi_j(x) = \frac{\rho}{c\gamma}e^{-R_0 x} - \sum_{j=1}^{\infty} e^{-j\delta t}\sum_{i=0}^{j} b_{ji}e^{-R_i x} = \sum_{j=0}^{\infty} B_j(t)e^{-R_j x},$$

where $\{B_j(t)\}_{j=0,1,...}$ are all deterministic functions of time t. Then, (5.1) should hold, because the asymptotic representation given by *Theorem 4.3* is also a linear combination of $\left\{e^{-R_j x}\right\}_{j=0,1,...}$.

□

References

Albrecher, H., Asmussen, S. (2006). Ruin Probabilities and Aggregate Claims Distributions for Shot Noise Cox Processes. *Scandinavian Actuarial Journal.* **2**: 86-110.

Boogaert, P., Haezendonck, J. (1989). Delay in Claim Settlement. *Insurance: Mathematics and Economics.* **8**, 321-330.

Brémaud, P. (2000). An Insensitivity Property of Lundberg's Estimate for Delayed Claims. *Journal of Applied Probability.* **37**, 914-917.

Dassios, A., Embrechts, P. (1989). Martingales and Insurance Risk. *Communications in Statistics-Stochastic Models.* **5(2)**, 181-217.

Davis, M.H.A. (1984). Piecewise Deterministic Markov Processes: A General Class of Nondiffusion Stochastic Models. *Journal of the Royal Statistical Society B.* **46**, 353-388.

Davis, M.H.A. (1993). *Markov Model and Optimization.* Chapman and Hall.

Feller, W. (1971). *Introduction to Probability Theory and its Applications, Vol.II, 2nd Edition.*Wiley.

Gerber, H.U. (1979). *An Introduction to Mathematical Risk Theory.* Huebner.

Grandel, J. (1991). *Aspects of Risk Theory.* Springer-Verlag.

Lawrance, A. J., Lewis, P. A. W. (1975). Properties of the Bivariate Delayed Poisson Process. *Journal of Applied Probability.* **12(2)**, 257-268.

Klüppelberg, C., Mikosch, T. (1995). Explosive Poisson Shot Noise Processes with Applications to Risk Reserves. *Bernoulli.* **1(1/2)**, 125-147.

Kaas, R., Goovaerts, M., Dhaene, J., Denuit, M. (2008). *Modern Actuarial Risk Theory: Using R.* Springer.

Macci, C., Torrisi, G. L. (2004). Asymptotic Results for Perturbed Risk Processes with Delayed Claims. *Insurance: Mathematics and Economics.* **34(2)**, 307-320.

Mirasol, N. M. (1963). The Output of an M/G/∞ Queuing System is Poisson. *Operations Research.* **11(2)**, 282-284.

Newell, G. F. (1966). M/G/∞ Queue. *SIAM Journal on Applied Mathematics.* **14(1)**, 86-88.

Rolski, T., Schmidli, H., Schmidt, V., Teugels, J. (1999). *Stochastic Processes for Insurance and Finance.* Wiley, Chichester.

Trufin, J., Albrecher, H., Denuit, M. (2011). Ruin Problems under IBNR Dynamics. *Applied Stochastic Models in Business and Industry (in press).*

Waters, H.R., Papatriandafylou, A. (1985). Ruin Probabilities Allowing for Delay in Claims Settlement. *Insurance: Mathematics and Economics.* **4**, 113-122.

Yuen, K.C., Guo, J., Ng, W. (2005). On Ultimate Ruin in a Delayed-Claims Risk Model. *Journal of Applied Probability.* **42**, 163-174.