Ruin by Dynamic Contagion Claims

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Abstract

In this paper, we consider a risk process with the arrival of claims modelled by a dynamic contagion process, a generalisation of the Cox process and Hawkes process introduced by Dassios and Zhao (2011). We derive results for the infinite horizon model that are generalisations of the Cramér-Lundberg approximation, Lundberg's fundamental equation, some asymptotics as well as bounds for the probability of ruin. Special attention is given to the case of exponential jumps and two numerical examples are provided.

Keywords: Dynamic contagion process, Ruin probability, Generalised Lundberg's fundamental equation, Cramér-Lundberg approximation, Change of measure, Martingale method
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1. Introduction

In the classical Cramér-Lundberg risk model, the arrival of claims is modelled by a Poisson process. As substantially discussed in the literature, this model is often not realistic in practice and hence a variety of extensions have been studied. Many researchers, such as Björk and Grandell (1988), Embrechts, Grandell and Schmidli (1993) had already suggested use the Cox process to model the arrival of claims, see also the book by Grandel (1991). Schmidli (1996) investigated the case for a Cox process with a piecewise constant intensity. More recently, Albrecher and Asmussen (2006) discussed a Cox process with shot noise intensity. On the other hand, only a few researchers have proposed risk models using self-excited processes, due to the observation of the clustering arrival of claims in reality, a similar pattern in the credit risk from the financial market, particularly during the current economic crisis. Stabile and Torrisi (2010) looked at the ruin problem in a model using the Hawkes process, a self-excited point process introduced by Hawkes (1971).

To capture the clustering phenomenon as well as some common external factors involved for the arrival of claims within one single consistent framework, in this paper, we extend further to use the dynamic contagion process introduced by Dassios and Zhao (2011), a generalisation of the externally excited Cox process with shot noise intensity (with exponential decay) and the self-excited Hawkes process (with exponential decay). It could be particularly useful for modelling the dependence structure of the underlying arriving events with dynamic contagion impact

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from both endogenous and exogenous factors. In this paper, we try to generalise results obtained for the classical model.

We organise our paper as follows. Section 2 provides distributional results we will use, mainly developed in Dassios and Zhao (2011). Section 3 formulates the problem. It also provides numerical examples and some asymptotics that are based on simulations. In Section 4, we use the martingale method and generalise Lundberg's fundamental equation. We derive bounds for the ruin probability in Section 5. In Section 6, we derive all results via a change of measure. This makes simulations more efficient as ruin is certain under the new measure. Section 7 concentrates on exponentially distributed claims. Our results are illustrated by two numerical examples.

2. Dynamic Contagion Process

The dynamic contagion process includes both the self-excited jumps (which are distributed according to the branching structure of a Hawkes process with exponential fertility rate) and the externally excited jumps (which are distributed according to a particular shot noise Cox process). We directly use the definition of the dynamic contagion process from Dassios and Zhao (2011).

Definition 2.1 (Dynamic Contagion Process)**.** *The* ***dynamic contagion process*** *is a cluster point process $\mathbb{D}$ on $\mathbb{R}_+$: The number of points in the time interval $(0, t]$ is defined by $N_t = N_{\mathbb{D}(0,t]}$. The cluster centers of $\mathbb{D}$ are the particular points called immigrants, and the other points are called offspring. They have the following structure:*

(a) *The immigrants are distributed according to a Cox process A with points $\{D_m\}_{m=1,2,...} \in (0, \infty)$ and shot noise stochastic intensity process*

$$a + (\lambda_0 - a)\, e^{-\delta t} + \sum_{i \geq 1} Y_i^{(1)} e^{-\delta(t - T_i^{(1)})} \mathbb{I}\left\{T_i^{(1)} \leq t\right\},$$

where

- *$a \geq 0$ is the constant reversion level;*
- *$\lambda_0 > 0$ is a constant as the initial value of the stochastic intensity process (defined later by (1));*
- *$\delta > 0$ is the constant rate of exponential decay;*
- *$\left\{Y_i^{(1)}\right\}_{i=1,2,...}$ is a sequence of independent identical distributed positive (externally excited) jumps with distribution function $H(y), y > 0$, at the corresponding random times $\left\{T_i^{(1)}\right\}_{i=1,2,...}$ following a homogeneous Poisson process M_t with constant intensity $\rho > 0$;*
- *$\mathbb{I}$ is the indicator function.*

(b) *Each immigrant D_m generates a cluster $C_m = C_{D_m}$, which is the random set formed by the points of generations $0, 1, 2, ...$ with the following branching structure:*

the immigrant D_m is said to be of generation 0. Given generations $0, 1, ..., j$ in C_m, each point $T^{(2)} \in C_m$ of generation j generates a Cox process on $(T^{(2)}, \infty)$ of offspring of generation $j + 1$ with the stochastic intensity $Y^{(2)} e^{-\delta(\cdot - T^{(2)})}$ where $Y^{(2)}$ is a positive (self-excited) jump at time $T^{(2)}$ with distribution function $G(y), y > 0$, independent of the points of generation $0, 1, ..., j$.

(c) *Given the immigrants, the centered clusters*

$$C_m - D_m = \left\{T^{(2)} - D_m : T^{(2)} \in C_m\right\}, \quad D_m \in A,$$

are independent identical distributed, and independent of A.

(d) $\mathbb{D}$ *consists of the union of all clusters, i.e.*

$$\mathbb{D} = \bigcup_{m=1,2,...} C_{D_m}.$$

Therefore, the dynamic contagion process can also be defined as a point process $N_t \equiv \left\{T_k^{(2)}\right\}_{k\geq 1}$ *on* $\mathbb{R}_+$*, with the non-negative* $\mathcal{F}_t$*–stochastic intensity process* λ_t *following the piecewise deterministic dynamics with positive jumps, i.e.*

$$\lambda_t = a + (\lambda_0 - a)\, e^{-\delta t} + \sum_{i\geq 1} Y_i^{(1)} e^{-\delta(t-T_i^{(1)})} \mathbb{I}\left\{T_i^{(1)} \leq t\right\} + \sum_{k\geq 1} Y_k^{(2)} e^{-\delta(t-T_k^{(2)})} \mathbb{I}\left\{T_k^{(2)} \leq t\right\}, \quad (1)$$

where

- $\{\mathcal{F}_t\}_{t\geq 0}$ *is a history of the process* N_t*, with respect to which* $\{\lambda_t\}_{t\geq 0}$ *is adapted,*
- $\left\{Y_k^{(2)}\right\}_{k=1,2,...}$ *is a sequence of independent identical distributed positive (self-excited) jumps with distribution function* $G(y), y > 0$*, at the corresponding random times* $\left\{T_k^{(2)}\right\}_{k=1,2,...}$*,*
- *the sequences* $\left\{Y_i^{(1)}\right\}_{i=1,2,...}$*,* $\left\{T_i^{(1)}\right\}_{i=1,2,...}$ *and* $\left\{Y_k^{(2)}\right\}_{k=1,2,...}$ *are assumed to be independent of each other.*

With the aid of the piecewise deterministic Markov process theory and using the results in Davis (1984), the infinitesimal generator of the dynamic contagion process (λ_t, N_t, t) acting on a function $f(\lambda, n, t) \in \Omega(\mathcal{A})$ is given by

$$\begin{aligned}\mathcal{A} f(\lambda, n, t) \quad = \quad & \frac{\partial f}{\partial t} - \delta\,(\lambda - a)\,\frac{\partial f}{\partial \lambda} + \rho \left(\int_0^\infty f(\lambda + y, n, t) \mathrm{d}H(y) - f(\lambda, n, t)\right) \\ & + \lambda \left(\int_0^\infty f(\lambda + y, n+1, t) \mathrm{d}G(y) - f(\lambda, n, t)\right),\end{aligned}$$

where $\Omega(\mathcal{A})$ is the domain of generator $\mathcal{A}$ such that $f(\lambda, n, t)$ is differentiable with respect to λ, t for all λ, n and t, and

$$\begin{aligned}\left|\int_0^\infty f(\lambda + y, n, t) \mathrm{d}H(y) - f(\lambda, n, t)\right| \quad & < \quad \infty, \\ \left|\int_0^\infty f(\lambda + y, n+1, t) \mathrm{d}G(y) - f(\lambda, n, t)\right| \quad & < \quad \infty.\end{aligned}$$

To give an intuitive picture of this new process by stochastic intensity representation, we present *Figure 1* for illustrating how the externally excited jumps $\left\{Y_i^{(1)}\right\}_{i=1,2,...}$ (marked by single arrow ↓) and self-excited jumps $\left\{Y_k^{(2)}\right\}_{k=1,2,...}$ (marked by double arrow ↕) in the intensity process λ_t interact with the point process N_t.

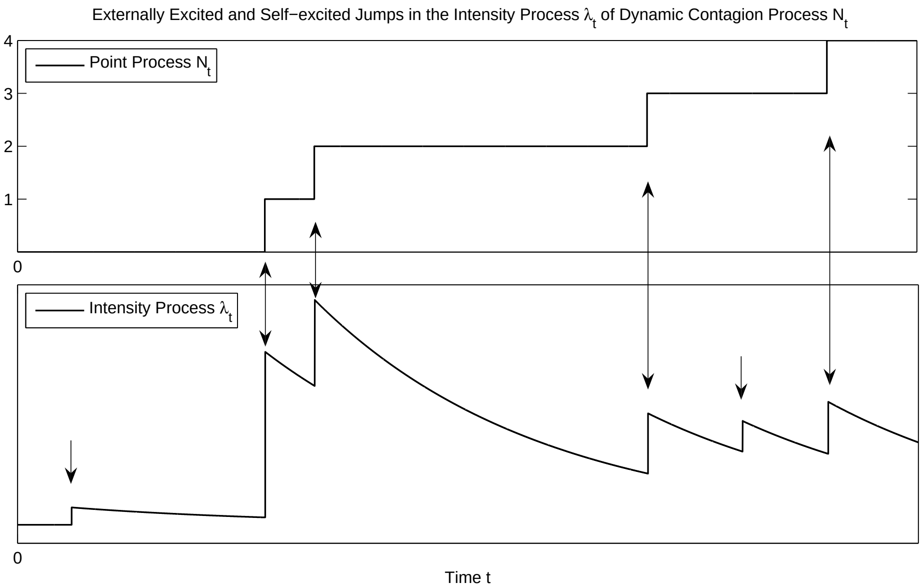

Figure 1: Externally Excited and Self-excited Jumps in Intensity Process λ_t of A Dynamic Contagion Process

The dynamic contagion process has some key distributional properties which will be used in this paper and are listed as below. The corresponding proofs have been given by Dassios and Zhao (2011) and we omit them here.

Proposition 2.1. *$\delta > \mu_{1_G}$ is the stationarity condition of the intensity process λ_t of a dynamic contagion process, where*

$$\mu_{1_G} =: \int_0^\infty y \mathrm{d}G(y).$$

Theorem 2.1. *If $\delta > \mu_{1_G}$, then the Laplace transform of the asymptotic distribution of λ_t is given by*

$$\hat{\Pi}(v) =: \lim_{t\to\infty} \mathbb{E}\left[e^{-v\lambda_t}\middle|\lambda_0\right] = \exp\left(-\int_0^v \frac{a\delta u + \rho[1-\hat{h}(u)]}{\delta u + \hat{g}(u) - 1}\mathrm{d}u\right), \tag{2}$$

and (2) is also the Laplace transform of the stationary distribution of the process $\{\lambda_t\}_{t\geq 0}$, where

$$\hat{h}(u) =: \int_0^\infty e^{-uy}\mathrm{d}H(y), \quad \hat{g}(u) =: \int_0^\infty e^{-uy}\mathrm{d}G(y).$$

Corollary 2.1. *If $\delta > \mu_{1_G}$, then,*

$$\lim_{t\to\infty} \mathbb{E}[\lambda_t|\lambda_0] = \frac{\mu_{1_H}\rho + a\delta}{\delta - \mu_{1_G}}, \tag{3}$$

where

$$\mu_{1_H} =: \int_0^\infty y \mathrm{d}H(y),$$

and (3) is also the mean of stationary distribution of the process $\{\lambda_t\}_{t\geq 0}$.

Theorem 2.2. *For any function* $f \in \Omega(\mathcal{A})$, *we have*

$$\int_E \mathcal{A} f(\lambda)\Pi(\lambda)\mathrm{d}\lambda = 0,$$

where $E = [a, \infty)$ *is the domain of* λ, $\mathcal{A} f(\lambda)$ *is the infinitesimal generator of the dynamic contagion process acting on* $f(\lambda)$, *i.e.*

$$\mathcal{A} f(\lambda) = -\delta(\lambda - a)\frac{\mathrm{d}f(\lambda)}{\mathrm{d}\lambda} + \rho\left(\int_0^\infty f(\lambda + y)\mathrm{d}H(y) - f(\lambda)\right) + \lambda\left(\int_0^\infty f(\lambda + z)\mathrm{d}G(z) - f(\lambda)\right),$$

and $\Pi(\lambda)$ *is the density function of* λ *with the Laplace transform specified by (2).*

Theorem 2.3. *If the externally excited and self-excited jumps follow exponential distributions, i.e.* $H \sim \mathrm{Exp}(\alpha)$, $G \sim \mathrm{Exp}(\beta)$ *and* $\delta\beta > 1$, *then, the stationary distribution of* λ_t *is given by*

$$\begin{cases} a + \Gamma_1 + \Gamma_2 & \text{for } \alpha \geq \beta \\ a + \Gamma_3 + \tilde{B} & \text{for } \alpha < \beta \text{ and } \alpha \neq \beta - \frac{1}{\delta} \\ a + \Gamma_4 + \tilde{P} & \text{for } \alpha = \beta - \frac{1}{\delta} \end{cases},$$

where independent random variables

$$\begin{aligned}
\Gamma_1 &\sim \mathrm{Gamma}\left(\frac{1}{\delta}\left(a + \frac{\rho}{\delta(\alpha-\beta)+1}\right), \frac{\delta\beta - 1}{\delta}\right); \\
\Gamma_2 &\sim \mathrm{Gamma}\left(\frac{\rho(\alpha-\beta)}{\delta(\alpha-\beta)+1}, \alpha\right); \\
\Gamma_3 &\sim \mathrm{Gamma}\left(\frac{a+\rho}{\delta}, \frac{\delta\beta-1}{\delta}\right); \\
\Gamma_4 &\sim \mathrm{Gamma}\left(\frac{a+\rho}{\delta}, \alpha\right); \\
\tilde{B} &\stackrel{\mathcal{D}}{=} \sum_{i=1}^{N_1} X_i^{(1)}, N_1 \sim \mathrm{NegBin}\left(\frac{\rho}{\delta}\frac{\beta-\alpha}{\gamma_1-\gamma_2}, \frac{\gamma_2}{\gamma_1}\right), X_i^{(1)} \sim \mathrm{Exp}(\gamma_1), \\
& \quad \gamma_1 = \max\left\{\alpha, \frac{\delta\beta-1}{\delta}\right\}, \gamma_2 = \min\left\{\alpha, \frac{\delta\beta-1}{\delta}\right\}; \\
\tilde{P} &\stackrel{\mathcal{D}}{=} \sum_{i=1}^{N_2} X_i^{(2)}, N_2 \sim \mathrm{Poisson}\left(\frac{\rho}{\delta^2\alpha}\right), X_i^{(2)} \sim \mathrm{Exp}(\alpha).
\end{aligned}$$

Remark 2.1. $\tilde{B}$ follows a compound negative binomial distribution with underlying exponential jumps, and $\tilde{P}$ follows a compound Poisson distribution with underlying exponential jumps. *Theorem 2.3* implies that the Laplace transform of λ_t is given by

$$\mathbb{E}\left[e^{-v\lambda_t}\right] = \begin{cases} e^{-va}\left(\frac{\alpha}{\alpha+v}\right)^{\frac{\rho(\alpha-\beta)}{\delta(\alpha-\beta)+1}}\left(\frac{\frac{\delta\beta-1}{\delta}}{v+\frac{\delta\beta-1}{\delta}}\right)^{\frac{1}{\delta}\left(a+\frac{\rho}{\delta(\alpha-\beta)+1}\right)} & \text{for } \alpha \geq \beta \\ e^{-va}\left(\frac{\frac{\delta\beta-1}{\delta}}{v+\frac{\delta\beta-1}{\delta}}\right)^{\frac{a+\rho}{\delta}}\left(\frac{\frac{\gamma_2}{\gamma_1}}{1-\left(1-\frac{\gamma_2}{\gamma_1}\right)\frac{\gamma_1}{\gamma_1+v}}\right)^{\frac{\rho}{\delta}\frac{\beta-\alpha}{\gamma_1-\gamma_2}} & \text{for } \alpha < \beta \text{ and } \alpha \neq \beta - \frac{1}{\delta} \\ e^{-va}\left(\frac{\alpha}{\alpha+v}\right)^{\frac{\rho+a}{\delta}}\exp\left[\frac{\rho}{\delta^2\alpha}\left(\frac{\alpha}{\alpha+v}-1\right)\right] & \text{for } \alpha = \beta - \frac{1}{\delta} \end{cases}. \tag{4}$$

3. Ruin Problem

We consider a company with its surplus process X_t in continuous time on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$,

$$X_t = x + ct - \sum_{i=1}^{N_t} Z_i, \quad t \geq 0, \tag{5}$$

where

- $X_0 = x \geq 0$ is the initial reserve at time $t = 0$;
- $c > 0$ is the constant rate of premium payment per time unit;
- N_t is a point process ($N_0 = 0$) counting the number of cumulative arrived claims in the time interval $(0, t]$, driven by a dynamic contagion process with its stochastic intensity process λ_t and the initial intensity $\lambda_0 = \lambda > 0$;
- $\{Z_i\}_{i=1,2,...}$ is a sequence of independent identical distributional positive random variables (claim sizes) with distribution function $Z(z), z > 0$, and also independent of N_t; the mean, Laplace transform of density function and tail are denoted respectively by

$$\mu_{1_Z} =: \int_0^\infty z\mathrm{d}Z(z), \quad \hat{z}(u) =: \int_0^\infty e^{-uz}\mathrm{d}Z(z), \quad \overline{Z}(x) =: \int_x^\infty \mathrm{d}Z(s).$$

The surplus process X_t is a right-continuous function of time t.

Definition 3.1 (Ruin Time). *The ruin (stopping) time τ^* is defined by*

$$\tau^* =: \begin{cases} \inf\left\{t > 0 \middle| X_t \leq 0\right\} & \\ \inf\{\varnothing\} = \infty & \text{if } X_t > 0 \text{ for all } t; \end{cases}$$

in particular, $\tau^* = \infty$ means ruin does not occur.

We are interested in the ruin probability in finite time,

$$P\left\{\tau^* < t \middle| X_0 = x, \lambda_0 = \lambda\right\};$$

in particular, the ultimate ruin probability in infinite time,

$$P\left\{\tau^* < \infty \middle| X_0 = x, \lambda_0 = \lambda\right\} =: \lim_{t\to\infty} P\left\{\tau^* < t \middle| X_0 = x, \lambda_0 = \lambda\right\};$$

and also the ultimate ruin probability in infinite time when the intensity process λ_t has stationary distribution,

$$P\left\{\tau^* < \infty \middle| X_0 = x, \lambda_0 \sim \Pi\right\},$$

where Π is the stationary distribution of λ_t given by *Theorem 2.1*.

3.1. Net Profit Condition

Theorem 3.1. *If $\delta > \mu_{1_G}$ and the arrival of claims is driven by a dynamic contagion process, then, the net profit condition is given by*

$$c > \frac{\mu_{1_H}\rho + a\delta}{\delta - \mu_{1_G}}\mu_{1_Z}. \tag{6}$$

Proof. Obviously, we have the expectation of surplus process X_t defined by (5),

$$\mathbb{E}[X_t] = x + ct - \mu_{1_Z}\mathbb{E}[\lambda_t]\, t.$$

If $\delta > \mu_{1_G}$ and the net profit condition holds, by *Corollary 2.1*, we have

$$\lim_{t\to\infty}\frac{\mathbb{E}[X_t]}{t} = c - \frac{\mu_{1_H}\rho + a\delta}{\delta - \mu_{1_G}}\mu_{1_Z} > 0.$$

□

3.2. Simulation Examples

Before giving mathematical proofs, we can have a first glance at this ruin problem via Monte Carlo simulation. Assume the stationarity condition for λ_t and net profit condition for X_t both hold, and the two types of jump sizes and claim sizes all follow exponential distributions, i.e. $H \sim \mathrm{Exp}(\alpha)$, $G \sim \mathrm{Exp}(\beta)$ and $Z \sim \mathrm{Exp}(\gamma)$. We implement the simulation algorithm for a dynamic contagion process provided by Dassios and Zhao (2011), with parameters set by

$$(a, \lambda_0, \rho, \delta; \alpha, \beta, \gamma; X_0, c) = (0.7, 0.7, 0.5, 2.0; 2.0, 1.5, 1.0; 10, 1.5).$$

In *Figure 2*, we plot the ruin probability $P\left\{\tau^* < t \middle| X_0 = x, \lambda_0 = \lambda\right\}$ against the time from $t = 0$ to $t = 400$. We can observe that the probability increases and converges around 30% when time t increases. Note that, each point is calculated based on 50,000 simulated paths of dynamic contagion processes. For instance, one example of simulated surplus process X_t with the underlying point process of claim arrival N_t and intensity process λ_t from time $t = 0$ to $t = 100$ is represented by *Figure 3*, and the pattern of clustering arrival of claims generated by a dynamic contagion process is also shown in the histogram. For comparison, the theoretical expectations of λ_t and N_t (given by *Corollary 2.1*) are plotted together with their simulated paths. More numerical examples are provided later by Section 7.3.

4. Exponential Martingales and Generalised Lundberg's Fundamental Equation

In this section, we find some useful exponential martingales which link to the generalised Lundberg's fundamental equation. More importantly, they are crucial for deriving some key results of the ruin problem in the later sections.

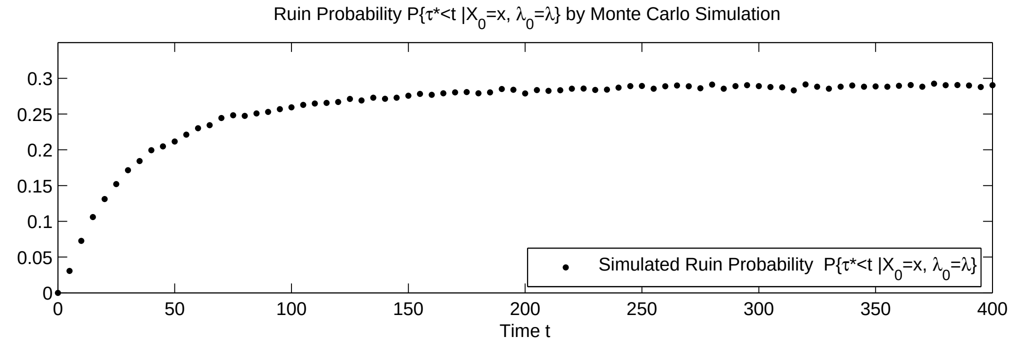

Figure 2: Ruin Probability $P\left\{\tau^* < t \middle| X_0 = x, \lambda_0 = \lambda\right\}$ by 50,000 Simulated Dynamic Contagion Processes

Theorem 4.1. *Assume $\delta > \mu_{1_G}$ and the net profit condition (6), we have a martingale*

$$e^{-v_r X_t} e^{\eta_r \lambda_t} e^{-rt}, \quad r \geq 0,$$

where constants r, v_r and η_r satisfy a generalised Lundberg's fundamental equation

$$\begin{cases} -\delta\eta_r + \hat{z}(-v_r)\hat{g}(-\eta_r) - 1 = 0 \\ \rho\left(\hat{h}(-\eta_r) - 1\right) - r + a\delta\eta_r - cv_r = 0 \end{cases} \qquad \left(c > \frac{\mu_{1_H}\rho + a\delta}{\delta - \mu_{1_G}}\mu_{1_Z}, \quad \delta > \mu_{1_G}\right). \tag{7}$$

If $0 \leq r < r^*$, then (7) has a unique positive solution $(v_r^+ > 0, \eta_r^+ > 0)$, where

$$r^* =: \rho\left(\hat{h}(-\eta^*) - 1\right) + a\delta\eta^*, \tag{8}$$

and η^* is the unique positive solution to

$$1 + \delta\eta_r = \hat{g}(-\eta_r). \tag{9}$$

Proof. The (Model-1 type) infinitesimal generator of the process (X_t, λ_t, t) acting on a function $f(x, \lambda, t) \in \Omega(\mathcal{A})$ is given by

$$\begin{aligned} \mathcal{A}f(x,\lambda,t) \quad = \quad & \frac{\partial f}{\partial t} - \delta(\lambda - a)\frac{\partial f}{\partial \lambda} + c\frac{\partial f}{\partial x} + \lambda\left(\int_{y=0}^{\infty}\int_{z=0}^{\infty} f(x-z, \lambda+y, t)\mathrm{d}Z(z)\mathrm{d}G(y) - f(x,\lambda,t)\right) \\ & +\rho\left(\int_0^{\infty} f(x, \lambda+y, t)\mathrm{d}H(y) - f(x,\lambda,t)\right). \end{aligned} \tag{10}$$

For the classification of Model-1 type and Model-2 type generators for ruin problem, see Dassios and Embrechts (1989).

Assume the form

$$f(x, \lambda, t) = e^{-v_r x} e^{\eta_r \lambda} e^{-rt},$$

and plug into the generator (10). To be a martingale, set $\mathcal{A}f(x, \lambda, t) = 0$, then,

$$-r - \delta(\lambda - a)\eta_r - cv_r + \lambda\left(\int_{y=0}^{\infty}\int_{z=0}^{\infty} e^{v_r z} e^{\eta_r y}\mathrm{d}Z(z)\mathrm{d}G(y) - 1\right) + \rho\left(\int_0^{\infty} e^{\eta_r y}\mathrm{d}H(y) - 1\right) = 0,$$

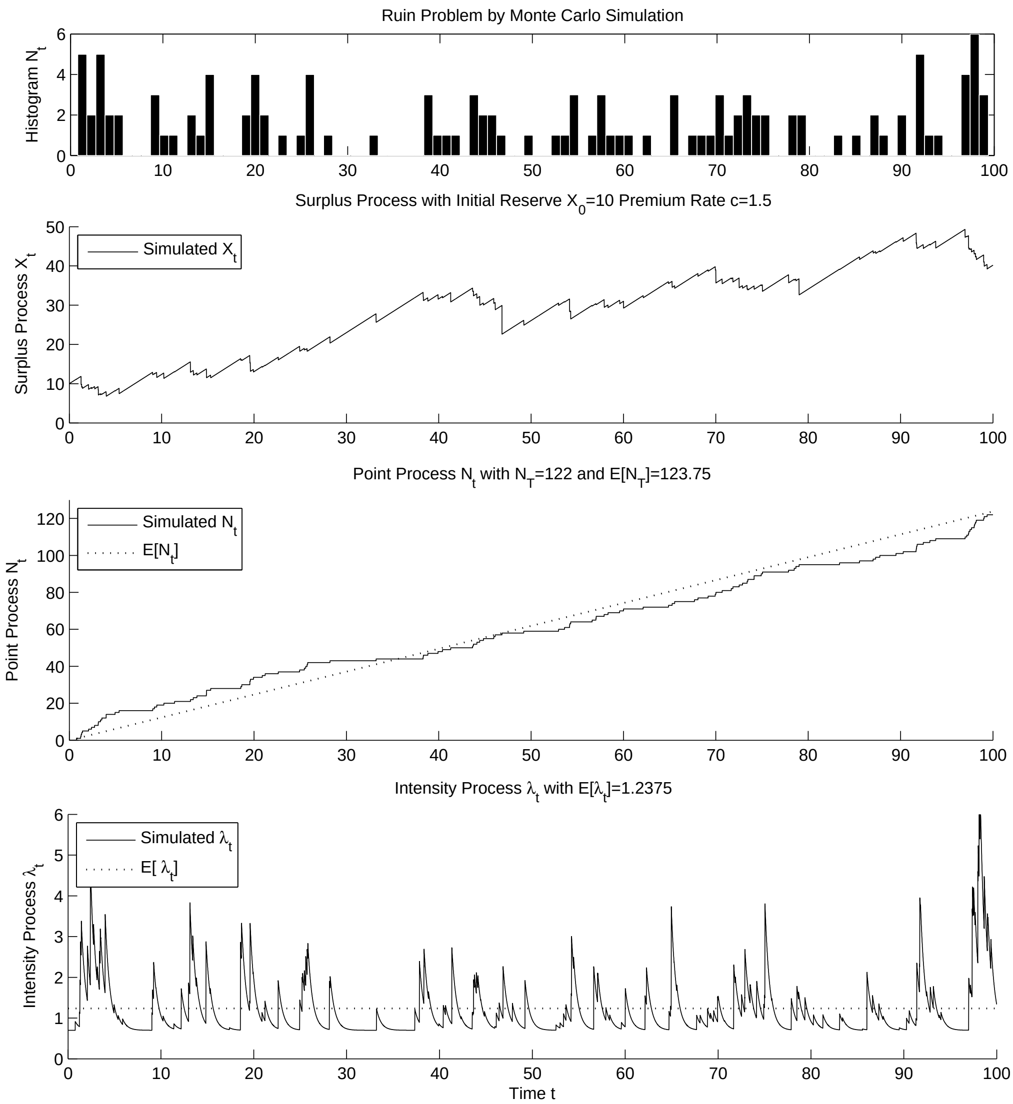

Figure 3: Example: Ruin Problem by One Simulated Dynamic Contagion Process

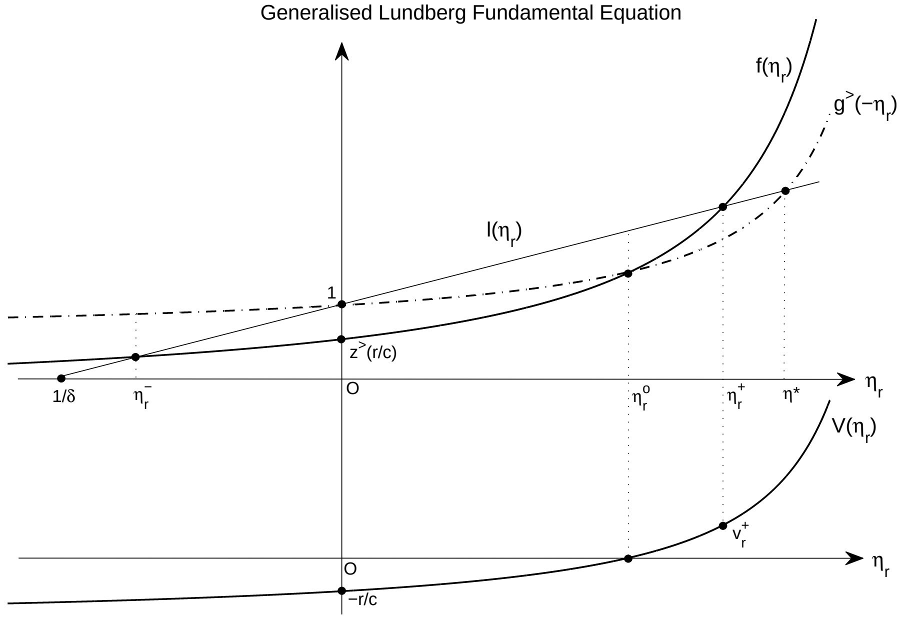

Figure 4: Generalised Lundberg Fundamental Equation

and rewrite as

$$\Big(- \delta\eta_r + \hat{z}(-v_r)\hat{g}(-\eta_r) - 1\Big)\lambda + \Big(\rho\left(\hat{h}(-\eta_r) - 1\right) - r + a\delta\eta_r - cv_r\Big) = 0,$$

holding for any λ. Hence, we have (7). The proofs of the uniqueness and the associated conditions for the solution to (7) are given by *Lemma 4.1* and *Lemma 4.2* as below. □

Lemma 4.1. *Under $\delta > \mu_{1_G}$ and the net profit condition (6), there are unique positive solution η_r^+ and unique negative solution η_r^- to η_r of the generalised Lundberg's fundamental equation (7); In particular, for $r = 0$, there are unique positive solution η_0^+ and solution zero.*

Proof. Rewrite the generalised Lundberg's fundamental equation (7) w.r.t. η_r,

$$\begin{cases} \hat{z}\left(\frac{r - a\delta\eta_r + \rho\left(1-\hat{h}(-\eta_r)\right)}{c}\right)\hat{g}(-\eta_r) = 1 + \delta\eta_r \\ -v_r = \frac{r - a\delta\eta_r + \rho\left(1-\hat{h}(-\eta_r)\right)}{c} \end{cases} \qquad \left(c > \frac{\mu_{1_H}\rho + a\delta}{\delta - \mu_{1_G}}\mu_{1_Z}, \quad \delta > \mu_{1_G}\right).$$

Consider the first equation above, i.e.

$$f(\eta_r) = l(\eta_r), \quad r > 0,$$

where

$$\begin{aligned} f(\eta_r) \quad &=: \quad \hat{z}\left(\frac{r - a\delta\eta_r + \rho(1 - \hat{h}(-\eta_r))}{c}\right)\hat{g}(-\eta_r), \\ l(\eta_r) \quad &=: \quad 1 + \delta\eta_r. \end{aligned}$$

Obviously, $f(\eta_r)$ is a strictly increasing and strictly convex function of η_r, since

$$\frac{\partial \hat{h}(-u)}{\partial u} > 0, \quad \frac{\partial \hat{g}(-u)}{\partial u} > 0, \quad \frac{\partial \hat{z}(u)}{\partial u} < 0,$$
$$\frac{\partial^2 \hat{h}(-u)}{\partial u^2} > 0, \quad \frac{\partial^2 \hat{g}(-u)}{\partial u^2} > 0, \quad \frac{\partial^2 \hat{z}(u)}{\partial u^2} > 0,$$

and

$$\begin{aligned}
\frac{\partial f(\eta_r)}{\partial \eta_r} &= \frac{-a\delta - \rho \frac{\partial \hat{h}(-\eta_r)}{\partial \eta_r}}{c} \frac{\partial \hat{z}(u)}{\partial u}\Big|_{u=\frac{r-a\delta\eta_r+\rho\left(1-\hat{h}(-\eta_r)\right)}{c}} \hat{g}(-\eta_r) \\
&+\hat{z}\left(\frac{r - a\delta\eta_r + \rho(1-\hat{h}(-\eta_r))}{c}\right) \frac{\partial \hat{g}(-\eta_r)}{\partial \eta_r} > 0, \\
\frac{\partial^2 f(\eta_r)}{\partial \eta_r^2} &= -\frac{\rho}{c} \frac{\partial^2 \hat{h}(-\eta_r)}{\partial \eta_r^2} \frac{\partial \hat{z}(u)}{\partial u}\Big|_{u=\frac{r-a\delta\eta_r+\rho\left(1-\hat{h}(-\eta_r)\right)}{c}} \hat{g}(-\eta_r) \\
&+ \frac{-a\delta - \rho \frac{\partial \hat{h}(-\eta_r)}{\partial \eta_r}}{c} \left[\frac{-a\delta - \rho \frac{\partial \hat{h}(-\eta_r)}{\partial \eta_r}}{c} \frac{\partial^2 \hat{z}(u)}{\partial u^2}\Big|_{u=\frac{r-a\delta\eta_r+\rho\left(1-\hat{h}(-\eta_r)\right)}{c}} \right. \\
&\left. +2\frac{\partial \hat{z}(u)}{\partial u}\Big|_{u=\frac{r-a\delta\eta_r+\rho\left(1-\hat{h}(-\eta_r)\right)}{c}} \frac{\partial \hat{g}(-\eta_r)}{\partial \eta_r} \right] \\
&+\hat{z}\left(\frac{r - a\delta\eta_r + \rho(1-\hat{h}(-\eta_r))}{c}\right) \frac{\partial^2 \hat{g}(-\eta_r)}{\partial \eta_r^2} > 0.
\end{aligned}$$

Also, $f(\eta_r) > 0$, $f(-\infty) = 0$, $f(+\infty) = +\infty$. $l(\eta_r)$ is a strictly linearly increasing function of η_r.

We discuss the solutions for the two cases $r > 0$ and $r = 0$ separately as below.

- For $r > 0$, we have
$$0 < f(0) = \hat{z}\left(\frac{r}{c}\right) < 1 = l(0),$$
and the slope of the tangent at $\eta_r = 0$,
$$\frac{\partial l(\eta_r)}{\partial \eta_r}\Big|_{\eta_r=0} > \frac{\partial f(\eta_r)}{\partial \eta_r}\Big|_{\eta_r=0} > 0.$$
By the stationarity condition $\delta > \mu_{1_G}$ and the net profit condition (6), we have
$$\begin{aligned}
\frac{\partial f(\eta_r)}{\partial \eta_r}\Big|_{\eta_r=0} &= \frac{-a\delta - \mu_{1_H}\rho}{c} \frac{\partial \hat{z}(u)}{\partial u}\Big|_{u=\frac{r}{c}} + \hat{z}\left(\frac{r}{c}\right)\mu_{1_G} \\
&< \frac{-a\delta - \mu_{1_H}\rho}{c} \frac{\partial \hat{z}(u)}{\partial u}\Big|_{u=0} + \hat{z}(0)\mu_{1_G} \\
&= \frac{a\delta + \mu_{1_H}\rho}{c}\mu_{1_Z} + \mu_{1_G} \\
&< \delta = \frac{\partial l(\eta_r)}{\partial \eta_r}\Big|_{\eta_r=0}.
\end{aligned}$$
It is clear that there are unique positive solution η_r^+ and unique negative solution η_r^- by plotting $f(\eta_r)$ and $l(\eta_r)$, see *Figure 4*.

- For $r = 0$, we have
$$0 < f(0) = \hat{z}(0) = 1 = l(0),$$
and the slope of the tangent at $\eta_r = 0$,
$$\frac{\partial l(\eta_r)}{\partial \eta_r}\bigg|_{\eta_r=0} > \frac{\partial f(\eta_r)}{\partial \eta_r}\bigg|_{\eta_r=0} > 0.$$
By the stationarity condition and the net profit condition, we have
$$\frac{\partial f(\eta_r)}{\partial \eta_r}\bigg|_{\eta_r=0} < \frac{a\delta + \mu_{1_H}\rho}{c}\mu_{1_Z} + \mu_{1_G} < \delta = \frac{\partial l(\eta_r)}{\partial \eta_r}\bigg|_{\eta_r=0}.$$
It is clear that there are unique positive solution η_0^+ and solution 0 by plotting $f(\eta_r)$ and $l(\eta_r)$.

□

In order to find the positive solution to v_r, we will only consider the unique positive solution η_r^+ for $r \geq 0$ in the sequel.

Lemma 4.2. *If* $0 \leq r < r^*$,
$$r^* =: \rho\left(\hat{h}(-\eta^*) - 1\right) + a\delta\eta^*, \tag{11}$$
where the constant η^* *is the unique positive solution to*
$$1 + \delta\eta_r = \hat{g}(-\eta_r), \quad \delta > \mu_{1_G},$$
then, there exists a unique positive solution v_r^+ *to* v_r *of the generalised Lundberg's fundamental equation (7),*
$$v_r^+ = -\frac{r - a\delta\eta_r^+ + \rho(1 - \hat{h}(-\eta_r^+))}{c}. \tag{12}$$

Proof. By substituting η_r^+ (from *Lemma 4.1*) into the second equation of the generalised Lundberg's fundamental equation (7), we have the solution to v_r, i.e. (12). Define
$$V(\eta_r) =: -\frac{r - a\delta\eta_r + \rho(1 - \hat{h}(-\eta_r))}{c}.$$
Obviously, $V(\eta_r)$ is a strictly increasing and strictly convex function of η_r, as $\frac{\partial V(\eta_r)}{\partial \eta_r} > 0$ and $\frac{\partial^2 V(\eta_r)}{\partial \eta_r^2} > 0$; also, $V(-\infty) = -\infty$, $V(+\infty) = +\infty$; $v(0) = -\frac{r}{c} < 0$; hence, there is unique (positive) root $\eta_r^o > 0$ such that $V(\eta_r^o) = 0$, also see *Figure 4*.

In order to find the unique positive solution v_r^+, such that $v_r^+ = V(\eta_r^+) > 0$, we have the condition $\eta_r^+ > \eta_r^o$, which also is equivalent to the condition
$$l(\eta_r^o) > f(\eta_r^o), \quad \eta_r^o > 0,$$
or,
$$1 - \delta\eta_r^o > \hat{g}(-\eta_r^o), \quad \eta_r^o > 0,$$

note that, $f(\eta_r^o) = \hat{g}(-\eta_r^o)$. Under the stationarity condition $\delta > \mu_{1_G}$, the equation $1+\delta\eta_r = \hat{g}(-\eta_r)$ has the unique positive solution η^* (independent from $r > 0$) and the solution 0. Therefore, we have the condition

$$0 < \eta_r^o < \eta^*,$$

such that

$$1 + \delta\eta_r^o > \hat{g}(-\eta_r^o), \quad \eta_r^o > 0.$$

We discuss the two cases $r > 0$ and $r = 0$ separately as below.

- If $r = 0$, we have $\eta_0^o = \eta_r^o\big|_{r=0} = 0$, and it is clear that $\eta_0^+ > \eta_0^o > 0$ holds, therefore, $v_0^+ > 0$ exists without any condition.
- If $r > 0$, then the condition $\eta^* > \eta_r^o > 0$ is also equivalent to the condition $V(\eta^*) > 0$ since $V(\cdot)$ is a strictly increasing function, i.e.

  $$V(\eta^*) = -\frac{r - a\delta\eta^* + \rho(1 - \hat{h}(-\eta^*))}{c} > 0.$$

  Hence, we can obtain the upper bound r^* for $r > 0$ explicitly, i.e. $0 < r < r^*$, where r^* is given by (11), note that, $r^* > 0$ as $\eta^* > 0$, also see *Figure 4*.

□

Remark 4.1. Given the existence and uniqueness of solution (η_r^+, v_r^+) to the generalised Lundberg's fundamental equation (7), we have $\eta^* > \eta_r^+$, since

$$1 + \delta\eta_r^+ = \hat{z}(-v_r^+)\,\hat{g}(-\eta_r^+) > \hat{g}(-\eta_r^+),$$

we know that, if $\delta > \mu_{1_G}$ the equation $1 + \delta\eta_r = \hat{g}(-\eta_r)$ has solution 0 and $\eta^* > 0$, then, η_r^+ should be between them, i.e. $\eta^* > \eta_r^+ > 0$, also see *Figure 4*. Therefore, we have the full ranking

$$0 < \eta_r^o < \eta_r^+ < \eta^*.$$

Remark 4.2. In particular, for $r = 0$, we have a martingale $e^{-v_0^+ X_t} e^{\eta_0^+ \lambda_t}$, where (v_0^+, η_0^+) is the unique positive solution to the equations

$$\begin{cases} \delta\eta_0^+ = \hat{z}\left(-v_0^+\right)\hat{g}\left(-\eta_0^+\right) - 1 \\ cv_0^+ = a\delta\eta_0^+ + \rho\left(\hat{h}(-\eta_0^+) - 1\right) \end{cases} \qquad \left(c > \frac{\mu_{1_H}\rho + a\delta}{\delta - \mu_{1_G}}\mu_{1_Z}, \quad \delta > \mu_{1_G}\right).$$

The martingales and generalised Lundberg's fundamental equation derived in this section are the building blocks of the martingale method and change of measure, two key approaches adopted in the following sections.

5. Ruin Probability via Original Measure

Theorem 5.1. *The ruin probability conditional on λ_0 and X_0 is given by*

$$P\left\{\tau^* < \infty \middle| X_0 = x, \lambda_0 = \lambda\right\} = \frac{e^{-v_0^+ x} e^{\eta_0^+ \lambda}}{\mathbb{E}\left[e^{-v_0^+ X_{\tau^*}} e^{\eta_0^+ \lambda_{\tau^*}} \middle| \tau^* < \infty; X_0 = x, \lambda_0 = \lambda\right]}. \tag{13}$$

Proof. By the optional stopping theorem, a bounded martingale stopped at a stopping time is still a martingale. Now we consider the martingale found by *Theorem 4.1* stopped at the ruin time, i.e.

$$e^{-v_r^+ X_{(\tau^*\wedge t)}} e^{\eta_r^+ \lambda_{(\tau^*\wedge t)}} e^{-r(\tau^*\wedge t)}, \quad 0 \le r < r^*.$$

By the martingale property, we have

$$\mathbb{E}\left[e^{-v_r^+ X_{(\tau^*\wedge t)}} e^{\eta_r^+ \lambda_{(\tau^*\wedge t)}} e^{-r(\tau^*\wedge t)}\right] = \mathbb{E}\left[e^{-v_r^+ X_{(\tau^*\wedge t)}} e^{\eta_r^+ \lambda_{(\tau^*\wedge t)}} e^{-r(\tau^*\wedge t)}\middle| X_0 = x, \lambda_0 = \lambda\right] = e^{-v_r^+ x} e^{\eta_r^+ \lambda},$$

and

$$\mathbb{E}\left[e^{-v_r^+ X_{\tau^*}} e^{\eta_r^+ \lambda_{\tau^*}} e^{-r\tau^*}\middle|\tau^* \le t\right] P\{\tau^* \le t\} + \mathbb{E}\left[e^{-v_r^+ X_t} e^{\eta_r^+ \lambda_t} e^{-rt}\middle|\tau^* > t\right] P\{\tau^* > t\} = e^{-v_r^+ x} e^{\eta_r^+ \lambda},$$

or,

$$\mathbb{E}\left[e^{-v_r^+ X_{\tau^*}} e^{\eta_r^+ \lambda_{\tau^*}} e^{-r\tau^*}\middle|\tau^* \le t\right] P\{\tau^* \le t\} + e^{-rt}\mathbb{E}\left[e^{-v_r^+ X_t} e^{\eta_r^+ \lambda_t}\middle|\tau^* > t\right] P\{\tau^* > t\} = e^{-v_r^+ x} e^{\eta_r^+ \lambda}, \tag{14}$$

where

$$\mathbb{E}\left[e^{-v_r^+ X_t} e^{\eta_r^+ \lambda_t}\middle|\tau^* > t\right] P\{\tau^* > t\} = \mathbb{E}\left[e^{-v_r^+ X_t} e^{\eta_r^+ \lambda_t}\mathbb{I}(\tau^* > t)\right] \le \mathbb{E}\left[e^{\eta_r^+ \lambda_t}\right].$$

Note that, by *Theorem 2.1*, we have

$$\lim_{t\to\infty}\mathbb{E}\left[e^{\eta_r^+ \lambda_t}\right] = \exp\left(\int_{-\eta_r^+}^{0} \frac{a\delta u + \rho[1-\hat{h}(u)]}{\delta u + \hat{g}(u) - 1}\mathrm{d}u\right) < \infty,$$

since by *Remark 4.1*, for $0 < r < r^*$, we have $-\eta^* < -\eta_r^+ < 0$ where $-\eta^*$ is the negative singular point of the integrand function above, i.e. the unique negative solution to $\delta u + \hat{g}(u) - 1 = 0$. Hence, for the second term in (14),

$$\lim_{t\to\infty} e^{-rt}\mathbb{E}\left[e^{-v_r^+ X_t} e^{\eta_r^+ \lambda_t}\middle|\tau^* > t\right] P\{\tau^* > t\} = 0.$$

Let $t \to \infty$ in (14), then, $\{\tau^* \le t\} \to \{\tau^* < \infty\}$, and

$$\mathbb{E}\left[e^{-v_r^+ X_{\tau^*}} e^{\eta_r^+ \lambda_{\tau^*}} e^{-r\tau^*}\middle|\tau^* < \infty\right] P\{\tau^* < \infty\} = e^{-v_r^+ x} e^{\eta_r^+ \lambda}.$$

Let $r \to 0$, we have

$$\mathbb{E}\left[e^{-v_0^+ X_{\tau^*}} e^{\eta_0^+ \lambda_{\tau^*}}\middle|\tau^* < \infty\right] P\{\tau^* < \infty\} = e^{-v_0^+ x} e^{\eta_0^+ \lambda},$$

then (13) follows. □

Corollary 5.1. *If* $Z \sim \mathrm{Exp}(\gamma)$*, then,*

$$P\left\{\tau^* < \infty \middle| X_0 = x, \lambda_0 = \lambda\right\} = \frac{\gamma - v_0^+}{\gamma} \frac{e^{\eta_0^+ \lambda} e^{-v_0^+ x}}{\mathbb{E}\left[e^{\eta_0^+ \lambda_{\tau^*}}\middle|\tau^* < \infty; X_0 = x, \lambda_0 = \lambda\right]}.$$

Proof. If $Z \sim \text{Exp}(\gamma)$, due to the memoryless property of the exponential distribution, the overshoot $-X_{\tau^*} > 0$ then follows the same exponential distribution, i.e. $-X_{\tau^*} \sim \text{Exp}(\gamma)$. Hence, for (13) we have

$$\mathbb{E}\left[e^{-\nu_0^+ X_{\tau^*}} e^{\eta_0^+ \lambda_{\tau^*}} \middle| \tau^* < \infty\right] = \mathbb{E}\left[e^{-\nu_0^+ X_{\tau^*}}\right] \mathbb{E}\left[e^{\eta_0^+ \lambda_{\tau^*}} \middle| \tau^* < \infty\right] = \frac{\gamma}{\gamma - \nu_0^+} \mathbb{E}\left[e^{\eta_0^+ \lambda_{\tau^*}} \middle| \tau^* < \infty\right].$$

□

Remark 5.1. Note that, the overshoot $-X_{\tau^*} > 0$, $\lambda_{\tau^*} > 0$, then, $e^{-\nu_0^+ X_{\tau^*}} > 1$, $e^{\eta_0^+ \lambda_{\tau^*}} > 1$, we have an inequality for the ruin probability,

$$P\left\{\tau^* < \infty \middle| X_0 = x, \lambda_0 = \lambda\right\} < \frac{e^{\eta_0^+ \lambda} e^{-\nu_0^+ x}}{\mathbb{E}\left[e^{\eta_0^+ \lambda_{\tau^*}} \middle| \tau^* < \infty; X_0 = x, \lambda_0 = \lambda\right]} < e^{\eta_0^+ \lambda} e^{-\nu_0^+ x}.$$

$e^{\eta_0^+ \lambda} e^{-\nu_0^+ x}$ is a rough up bound of ruin probability, as it could be greater than one when λ_0 is relatively large. In order to obtain a more precise upper bound, it is better to find the distribution property of $\mathbb{E}\left[e^{\eta_0^+ \lambda_{\tau^*}} \middle| \tau^* < \infty\right]$ but it would be not easy.

Example 5.1. If $Z \sim \text{Exp}(\gamma)$, then,

$$P\left\{\tau^* < \infty \middle| X_0 = x, \lambda_0 = \lambda\right\} < \frac{\gamma}{\gamma - \nu_0^+} e^{\eta_0^+ \lambda} e^{-\nu_0^+ x}.$$

For instance, the comparison between the boundaries and the ruin probability $P\left\{\tau^* < \infty \middle| X_0 = 10, \lambda_0 = \lambda\right\}$ simulated by $50,000$ sample paths with parameter setting

$$(a; \rho, \delta; \alpha, \beta, \gamma; X_0, c) = (0.7; 0.5, 2.0; 2.0, 1.5, 1.0; 10, 1.5), \quad (\eta_0^+, \nu_0^+) = (0.0842, 0.0932),$$

is given by *Table 1* and *Figure 5*.

Table 1: Example: The Comparison between the Boundaries and the Simulated Ruin Probability

$\lambda_0 = \lambda$	1	2	3	4	5	6	7	8	9	10	11	12
$P\left\{\tau^* < \infty \middle| X_0 = 10, \lambda_0 = \lambda\right\}$	28.83%	31.34%	34.39%	37.34%	40.01%	43.46%	46.67%	50.45%	53.34%	56.83%	60.56%	63.66%
Up Bound $e^{\eta_0^+ \lambda_0} e^{-\nu_0^+ X_0}$	42.84%	46.60%	50.69%	55.15%	59.99%	65.26%	70.99%	77.23%	84.01%	91.39%	99.42%	**108.16%**
Up Bound $\frac{\gamma - \nu_0^+}{\gamma} e^{\eta_0^+ \lambda_0} e^{-\nu_0^+ X_0}$	38.84%	42.26%	45.97%	50.01%	54.40%	59.18%	64.38%	70.03%	76.18%	82.88%	90.16%	98.08%

6. Ruin Probability via Change of Measure

In this section, we investigate the ruin probability and asymptotics by change of measure via the martingale derived by *Theorem 4.1.* We will find that under this new measure the ruin becomes certain, and this makes the simulation more efficient than under the original measure where the ruin is not certain and even rare. Similar ideas of improving simulation of rare events by a change of measure can also be found in Asmussen (1985).

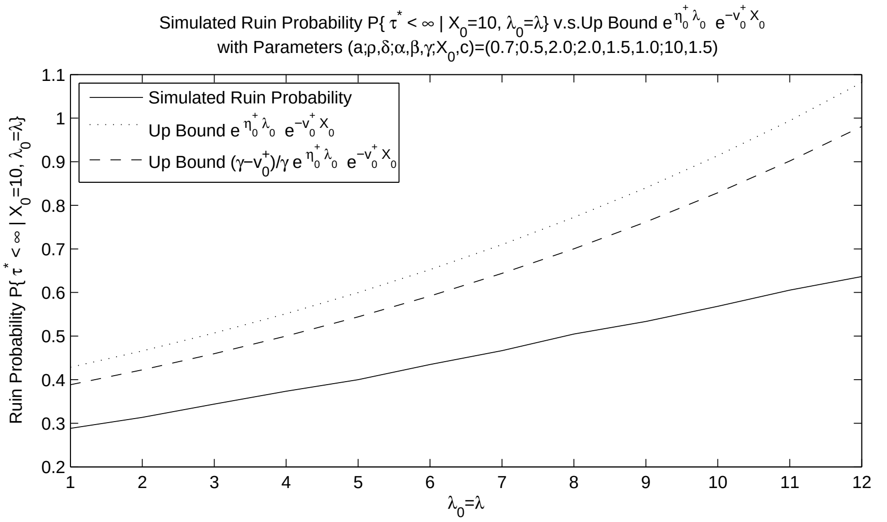

Figure 5: Simulated Ruin Probability $P\left\{\tau^* < \infty \middle| X_0 = 10, \lambda_0 = \lambda\right\}$ v.s. Up Bounds

6.1. Ruin Probability by Change of Measure

Theorem 6.1. *The ruin probability conditional on X_0 and λ_0 can be expressed under new measure $\widetilde{\mathbb{P}}$ by*

$$P\left\{\tau^* < \infty \middle| X_0 = x, \lambda_0 = \lambda\right\} = e^{-v_0^+ x} e^{m_0^+ \widetilde{\lambda}} \widetilde{\mathbb{E}}\left[\Psi\left(X_{\tau_-^*}\right) \frac{e^{-m_0^+ \widetilde{\lambda}_{\tau_-^*}}}{\hat{g}(-\eta_0^+)} \middle| X_0 = x, \widetilde{\lambda}_0 = \widetilde{\lambda}\right], \tag{15}$$

where $\widetilde{\lambda} =: (1 + \delta\eta_0^+)\lambda$, $m_0^+ =: \frac{\eta_0^+}{\delta\eta_0^+ + 1}$,

$$\Psi(x) =: \frac{\overline{Z}(x) e^{v_0^+ x}}{\int_x^\infty e^{v_0^+ z} \mathrm{d}Z(z)}, \tag{16}$$

assuming the net profit condition holds under the original measure $\mathbb{P}$, and the stationarity condition holds under both measures $\mathbb{P}$ and $\widetilde{\mathbb{P}}$. The parameter setting for the process (X_t, λ_t) under $\mathbb{P}$ transforms to the new parameter setting for the process $\left(X_t, \widetilde{\lambda}_t\right)$ under $\widetilde{\mathbb{P}}$ as follows:

- $a \nearrow \widetilde{a} =: \left(1 + \delta\eta_0^+\right) a$,
- $c \to \widetilde{c} =: c$,
- $\delta \to \widetilde{\delta} =: \delta$,
- $\rho \nearrow \widetilde{\rho} =: \hat{h}(-\eta_0^+)\rho$,
- $Z(z) \to \widetilde{Z}(z)$,
- $g(u) \to \widetilde{\widetilde{g}}(u) =: \frac{\widetilde{g}\left(\frac{u}{1+\delta\eta_0^+}\right)}{1+\delta\eta_0^+}$,

- $h(u) \to \widetilde{\widetilde{h}}(u) =: \frac{\widetilde{h}\left(\frac{u}{1+\delta\eta_0^+}\right)}{1+\delta\eta_0^+}$,

where

$$\mathrm{d}\widetilde{Z}(z) =: \frac{e^{v_0^+ z}\mathrm{d}Z(z)}{\hat{z}\left(-v_0^+\right)}, \quad \mathrm{d}\widetilde{G}(u) =: \frac{e^{\eta_0^+ u}\mathrm{d}G(u)}{\hat{g}\left(-\eta_0^+\right)}, \quad \mathrm{d}\widetilde{H}(u) =: \frac{e^{\eta_0^+ u}\mathrm{d}H(u)}{\hat{h}\left(-\eta_0^+\right)}, \tag{17}$$

and $\mathrm{d}\widetilde{H}(u) =: \widetilde{h}(u)\mathrm{d}u$*,* $\mathrm{d}\widetilde{G}(u) =: \widetilde{g}(u)\mathrm{d}u$*;* $\mathrm{d}\widetilde{\widetilde{H}}(u) =: \widetilde{\widetilde{h}}(u)\mathrm{d}u$*,* $\mathrm{d}\widetilde{\widetilde{G}}(u) =: \widetilde{\widetilde{g}}(u)\mathrm{d}u$*.*

Proof. We consider the (Model-2 type) generator

$$\begin{aligned}\mathcal{A}f(x,\lambda) &= -\delta(\lambda-a)\frac{\partial f}{\partial \lambda} + c\frac{\partial f}{\partial x} + \lambda\left(\int_{y=0}^{\infty}\int_{z=0}^{x} f(x-z,\lambda+y)\mathrm{d}Z(z)\mathrm{d}G(y) + \overline{Z}(x) - f(x,\lambda)\right) \\ &\quad +\rho\left(\int_0^{\infty} f(x,\lambda+y)\mathrm{d}H(y) - f(x,\lambda)\right), \quad x>0. \end{aligned}\tag{18}$$

The solution of the integro-differential equation $\mathcal{A}f(x,\lambda)=0$ is the ruin probability

$$f(x,\lambda) = P\left\{\tau^* < \infty \middle| X_0 = x, \lambda_0 = \lambda\right\}.$$

Change Measure from $\mathbb{P}$ *to* $\widetilde{\mathbb{P}}$*.* Substitute the function

$$f(x,\lambda) = e^{-v_0^+ x}e^{\eta_0^+\lambda}\widetilde{f}(x,\lambda)$$

into the generator (18), we have

$$\begin{aligned} &- \delta(\lambda-a)\left(\eta_0^+\widetilde{f} + \frac{\partial \widetilde{f}}{\partial \lambda}\right) + c\left(-v_0^+\widetilde{f} + \frac{\partial \widetilde{f}}{\partial x}\right) \\ &+ \lambda\left(\int_0^{\infty}\int_0^{x} \widetilde{f}(x-z,\lambda+y)e^{v_0^+ z}e^{\eta_0^+ y}\mathrm{d}Z(z)\mathrm{d}G(y) + \overline{Z}(x)e^{v_0^+ x}e^{-\eta_0^+\lambda} - \widetilde{f}\right) \\ &+ \rho\left(\int_0^{\infty} \widetilde{f}(x,\lambda+y)e^{\eta_0^+ y}\mathrm{d}H(y) - \widetilde{f}\right) = 0. \end{aligned}\tag{19}$$

Remind that, by *Theorem 4.1* for $r=0$, we have a $\mathcal{F}_t^{\mathbb{P}}$−martingale $e^{-v_0^+ X_t}e^{\eta_0^+\lambda_t}$ where (v_0^+, η_0^+) is the unique positive solution to the equations

$$\begin{cases} \delta\eta_0^+ = \hat{z}\left(-v_0^+\right)\hat{g}\left(-\eta_0^+\right) - 1 \\ cv_0^+ = a\delta\eta_0^+ + \rho\left(\hat{h}(-\eta_0^+) - 1\right) \end{cases} \qquad \left(c > \frac{\mu_{1_H}\rho + a\delta}{\delta - \mu_{1_G}}\mu_{1_Z}, \quad \delta > \mu_{1_G}\right).$$

Substitute $cv_0^+ = a\delta\eta_0^+ + \rho\left(\hat{h}(-\eta_0^+) - 1\right)$ and $\delta\eta_0^+ = \hat{z}\left(-v_0^+\right)\hat{g}\left(-\eta_0^+\right) - 1$ into (19), we have

$$\begin{aligned} &- \delta(\lambda-a)\frac{\partial \widetilde{f}}{\partial \lambda} + c\frac{\partial \widetilde{f}}{\partial x} \\ &+ \lambda\left(\int_0^{\infty}\int_0^{x} \widetilde{f}(x-z,\lambda+y)e^{v_0^+ z}e^{\eta_0^+ y}\mathrm{d}Z(z)\mathrm{d}G(y) + \overline{Z}(x)e^{v_0^+ x}e^{-\eta_0^+\lambda} - \hat{z}\left(-v_0^+\right)\hat{g}\left(-\eta_0^+\right)\widetilde{f}\right) \\ &+ \rho\left(\int_0^{\infty} \widetilde{f}(x,\lambda+y)e^{\eta_0^+ y}\mathrm{d}H(y) - \hat{h}(-\eta_0^+)\widetilde{f}\right) = 0. \end{aligned}$$

Change measure (*Esscher transform*) by (17), and rewrite as

$$
\begin{aligned}
- &\quad \delta(\lambda - a)\frac{\partial \widetilde{f}}{\partial \lambda} + c\frac{\partial \widetilde{f}}{\partial x} \\
+ &\quad \hat{z}\left(-v_0^+\right)\hat{g}\left(-\eta_0^+\right)\lambda\left(\int_0^\infty \int_0^x \widetilde{f}(x-z, \lambda+y)\mathrm{d}\widetilde{Z}(z)\mathrm{d}\widetilde{G}(y) + \overline{Z}(x)\frac{e^{v_0^+ x}e^{-\eta_0^+\lambda}}{\hat{z}\left(-v_0^+\right)\hat{g}\left(-\eta_0^+\right)} - \widetilde{f}\right) \\
+ &\quad \hat{h}(-\eta_0^+)\rho\left(\int_0^\infty \widetilde{f}(x, \lambda+y)\mathrm{d}\widetilde{H}(y) - \widetilde{f}\right) = 0.
\end{aligned}
$$

Since $\hat{z}\left(-v_0^+\right)\hat{g}\left(-\eta_0^+\right) = 1 + \delta\eta_0^+$, we have

$$
\begin{aligned}
- &\quad \delta(\lambda - a)\frac{\partial \widetilde{f}}{\partial \lambda} + c\frac{\partial \widetilde{f}}{\partial x} \\
+ &\quad (1+\delta\eta_0^+)\lambda\left(\int_0^\infty \int_0^x \widetilde{f}(x-z, \lambda+y)\mathrm{d}\widetilde{Z}(z)\mathrm{d}\widetilde{G}(y) + \overline{Z}(x)\frac{e^{v_0^+ x}e^{-\eta_0^+\lambda}}{\hat{z}\left(-v_0^+\right)\hat{g}\left(-\eta_0^+\right)} - \widetilde{f}\right) \\
+ &\quad \hat{h}(-\eta_0^+)\rho\left(\int_0^\infty \widetilde{f}(x, \lambda+y)\mathrm{d}\widetilde{H}(y) - \widetilde{f}\right) = 0.
\end{aligned}
$$

Note that,

$$
\overline{\overline{Z}}(x) =: \int_x^\infty \mathrm{d}\widetilde{Z}(z) = \int_x^\infty \frac{e^{v_0^+ z}\mathrm{d}Z(z)}{\hat{z}\left(-v_0^+\right)} = \frac{\int_x^\infty e^{v_0^+ z}\mathrm{d}Z(z)}{\hat{z}\left(-v_0^+\right)},
$$

we have

$$
\overline{Z}(x)\frac{e^{v_0^+ x}e^{-\eta_0^+\lambda}}{\hat{z}(-v_0^+)\hat{g}(-\eta_0^+)} = \frac{\overline{Z}(x)e^{v_0^+ x}}{\int_x^\infty e^{v_0^+ z}\mathrm{d}Z(z)}\frac{\int_x^\infty e^{v_0^+ z}\mathrm{d}Z(z)}{\hat{z}\left(-v_0^+\right)}\frac{e^{-\eta_0^+\lambda}}{\hat{g}(-\eta_0^+)} = \Psi(x)\frac{e^{-\eta_0^+\lambda}}{\hat{g}(-\eta_0^+)}\overline{\overline{Z}}(x),
$$

where $\Psi(x)$ is defined by (16). Hence, we have

$$
\begin{aligned}
- &\quad \delta(\lambda - a)\frac{\partial \widetilde{f}}{\partial \lambda} + c\frac{\partial \widetilde{f}}{\partial x} \\
+ &\quad (1+\delta\eta_0^+)\lambda\left(\int_0^\infty \int_0^x \widetilde{f}(x-z, \lambda+y)\mathrm{d}\widetilde{Z}(z)\mathrm{d}\widetilde{G}(y) + \Psi(x)\frac{e^{-\eta_0^+\lambda}}{\hat{g}(-\eta_0^+)}\overline{\overline{Z}}(x) - \widetilde{f}\right) \\
+ &\quad \hat{h}(-\eta_0^+)\rho\left(\int_0^\infty \widetilde{f}(x, \lambda+y)\mathrm{d}\widetilde{H}(y) - \widetilde{f}\right) = 0.
\end{aligned} \tag{20}
$$

This integro-differential equation has the solution

$$
\widetilde{f}(x, \lambda) = \widetilde{\mathbb{E}}\left[\Psi(X_{\tau_-^*})\frac{e^{-\eta_0^+\lambda_{\tau_-^*}}}{\hat{g}(-\eta_0^+)}\mathbb{I}(\tau^* < \infty)\middle|\lambda_0 = \lambda, X_0 = x\right].
$$

It is similar to the expectation of a Gerber-Shiu penalty function (see Gerber and Shiu (1998)). Therefore, by comparing (20) with (18), we have the parameters for the process (X_t, λ_t) under $\mathbb{P}$ transformed to the parameters for the process (X_t, λ_t) under $\widetilde{\mathbb{P}}$ as follows:

- $a \to \widetilde{a} = a$,

- $c \to \widetilde{c} = c$,
- $\delta \to \widetilde{\delta} = \delta$,
- $\rho \to \widetilde{\rho} = \hat{h}(-\eta_0^+)\rho$,
- $Z(z) \to \widetilde{Z}(z)$,
- $G(y) \to \widetilde{G}(y)$,
- $H(y) \to \widetilde{H}(y)$,

and the ruin probability is given by

$$P\left\{\tau^* < \infty \middle| X_0 = x, \lambda_0 = \lambda\right\} = e^{-\nu_0^+ x} e^{\eta_0^+ \lambda} \widetilde{\mathbb{E}}\left[\Psi(X_{\tau_-^*}) \frac{e^{-\eta_0^+ \lambda_{\tau_-^*}}}{\hat{g}(-\eta_0^+)} \mathbb{I}(\tau^* < \infty) \middle| X_0 = x, \lambda_0 = \lambda\right].$$

Expression by $\widetilde{\lambda}$. Alternatively, we can express the results above w.r.t. $\widetilde{\lambda}$ where $\widetilde{\lambda} = (1 + \delta\eta_0^+)\lambda$. Rewrite (20) as

$$\begin{aligned} &- \delta\left(\widetilde{\lambda} - (1 + \delta\eta_0^+)a\right) \frac{\partial \widetilde{f}}{\partial \widetilde{\lambda}} + c \frac{\partial \widetilde{f}}{\partial x} \\ &+ \widetilde{\lambda}\left(\int_0^\infty \int_0^x \widetilde{f}\left(x - z, \widetilde{\lambda} + (1 + \delta\eta_0^+)y\right) \mathrm{d}\widetilde{Z}(z) \mathrm{d}\widetilde{G}(y) + \Psi(x) \frac{e^{-\frac{\eta_0^+}{\delta\eta_0^+ + 1}\widetilde{\lambda}}}{\hat{g}(-\eta_0^+)} \overline{\widetilde{Z}}(x) - \widetilde{f}\right) \\ &+ \hat{h}(-\eta_0^+)\rho\left(\int_0^\infty \widetilde{f}\left(x, \widetilde{\lambda} + (1 + \delta\eta_0^+)y\right) \mathrm{d}\widetilde{H}(y) - \widetilde{f}\right) = 0. \end{aligned}$$

Given $\mathrm{d}\widetilde{H}(y) = \widetilde{h}(y)\mathrm{d}y$ and $\mathrm{d}\widetilde{G}(y) = \widetilde{g}(y)\mathrm{d}y$, change variable by $u = (1 + \delta\eta_0^+)y$, we have the equation of $\widetilde{f}\left(\widetilde{\lambda}, x\right)$,

$$\begin{aligned} &- \delta\left(\widetilde{\lambda} - (1 + \delta\eta_0^+)a\right) \frac{\partial \widetilde{f}}{\partial \widetilde{\lambda}} + c \frac{\partial \widetilde{f}}{\partial x} \\ &+ \widetilde{\lambda}\left(\int_0^\infty \int_0^x \widetilde{f}\left(x - z, \widetilde{\lambda} + u\right) \mathrm{d}\widetilde{Z}(z) \frac{\widetilde{g}\left(\frac{u}{1 + \delta\eta_0^+}\right)}{1 + \delta\eta_0^+} \mathrm{d}u + \Psi(x) \frac{e^{-\frac{\eta_0^+}{\delta\eta_0^+ + 1}\widetilde{\lambda}}}{\hat{g}(-\eta_0^+)} \overline{\widetilde{Z}}(x) - \widetilde{f}\right) \\ &+ \hat{h}(-\eta_0^+)\rho\left(\int_0^\infty \widetilde{f}\left(x, \widetilde{\lambda} + u\right) \frac{\widetilde{h}\left(\frac{u}{1 + \delta\eta_0^+}\right)}{1 + \delta\eta_0^+} \mathrm{d}u - \widetilde{f}\right) = 0. \end{aligned} \tag{21}$$

This integro-differential equation has the solution

$$\widetilde{f}(x, \widetilde{\lambda}) = \widetilde{\mathbb{E}}\left[\Psi\left(X_{\tau_-^*}\right) \frac{e^{-\frac{\eta_0^+}{\delta\eta_0^+ + 1}\widetilde{\lambda}_{\tau_-^*}}}{\hat{g}(-\eta_0^+)} \mathbb{I}(\tau^* < \infty) \middle| \lambda_0 = \lambda, X_0 = x\right].$$

Therefore, by comparing (21) with (18), we have the parameters for the process (X_t, λ_t) under $\mathbb{P}$ transformed to the parameters for the process $\left(X_t, \widetilde{\lambda}_t\right)$ under $\widetilde{\mathbb{P}}$ as follows:

- $a \nearrow \widetilde{a} = \left(1 + \delta\eta_0^+\right) a$,
- $c \to \widetilde{c} = c$,
- $\delta \to \widetilde{\delta} = \delta$,
- $\rho \nearrow \widetilde{\rho} = \hat{h}(-\eta_0^+)\rho$,
- $Z(z) \to \widetilde{Z}(z)$,
- $g(u) \to \widetilde{\widetilde{g}}(u) = \frac{\widetilde{g}\left(\frac{u}{1+\delta\eta_0^+}\right)}{1+\delta\eta_0^+}$,
- $h(u) \to \widetilde{\widetilde{h}}(u) = \frac{\widetilde{h}\left(\frac{u}{1+\delta\eta_0^+}\right)}{1+\delta\eta_0^+}$,

and the ruin probability is given by

$$\begin{aligned}
& P\left\{\tau^* < \infty \middle| X_0 = x, \lambda_0 = \lambda\right\} \\
= \quad & e^{-\nu_0^+ x} e^{\eta_0^+ \lambda} \widetilde{\mathbb{E}}\left[\Psi\left(X_{\tau_-^*}\right) \frac{e^{-m_0^+ \widetilde{\lambda}_{\tau_-^*}}}{\hat{g}(-\eta_0^+)} \mathbb{I}\left(\tau^* < \infty\right) \middle| X_0 = x, \lambda_0 = \lambda\right] \\
= \quad & e^{-\nu_0^+ x} e^{m_0^+ \widetilde{\lambda}} \widetilde{\widetilde{\mathbb{E}}}\left[\Psi\left(X_{\tau_-^*}\right) \frac{e^{-m_0^+ \widetilde{\lambda}_{\tau_-^*}}}{\hat{g}(-\eta_0^+)} \mathbb{I}\left(\tau^* < \infty\right) \middle| X_0 = x, \widetilde{\lambda}_0 = \widetilde{\lambda}\right], \quad m_0^+ = \frac{\eta_0^+}{\delta\eta_0^+ + 1}.
\end{aligned}$$

By *Theorem 6.3* (derived later in this section), if the net profit condition holds under $\mathbb{P}$ and the stationarity condition holds under $\mathbb{P}$ and $\widetilde{\mathbb{P}}$, then the net profit condition can not hold under $\widetilde{\widetilde{\mathbb{P}}}$, i.e. $\mathbb{I}\left(\tau^* < \infty\right) = 1$, hence, we have the ruin probability (15).

□

Remark 6.1. If $Z \sim \text{Exp}(\gamma)$, then, the expression of the ruin probability (15) can be much simplified, as $\Psi(x)$ is a constant, i.e.

$$\Psi(x) = \frac{e^{-\gamma x} e^{\nu_0^+ x}}{\int_x^\infty e^{\nu_0^+ z} \gamma e^{-\gamma z} \mathrm{d}z} = \frac{\gamma - \nu_0^+}{\gamma}.$$

6.2. Generalised Cramér-Lundberg Approximation for Exponentially Distributed Claims

Based on *Theorem 6.1*, if $Z \sim \text{Exp}(\gamma)$ and the initial intensity follows the stationary distribution under $\widetilde{\mathbb{P}}$, i.e. $\widetilde{\lambda} \sim \Pi$, then, the ruin probability is given by

$$P\left\{\tau^* < \infty \middle| X_0 = x\right\} = \frac{\gamma - \nu_0^+}{\gamma \hat{g}(-\eta_0^+)} \widetilde{\mathbb{E}}[e^{m_0^+ \widetilde{\lambda}}] \widetilde{\mathbb{E}}\left[e^{-m_0^+ \widetilde{\lambda}_{\tau_-^*}} \middle| X_0 = x\right] e^{-\nu_0^+ x}.$$

Now, we further generalise the Cramér-Lundberg approximation.

Theorem 6.2. *If the claim sizes follows exponential distribution and the initial intensity follows the stationary distribution under* $\widetilde{\mathbb{P}}$*, i.e.* $\widetilde{Z} \sim \text{Exp}(\widetilde{\gamma})$ *and* $\widetilde{\lambda} \sim \Pi$*, then, the generalised Cramér-Lundberg approximation is given by*

$$P\left\{\tau^* < \infty \middle| X_0 = x\right\} \sim C e^{-\nu_0^+ x}, \quad x \to \infty,$$

where

$$C =: \frac{\gamma - v_0^+}{\gamma \hat{g}(-\eta_0^+)} \widetilde{\mathbb{E}}[e^{m_0^+ \widetilde{\lambda}}] \frac{\frac{1}{\overline{\gamma}} \widetilde{\mathbb{E}}\left[e^{-m_0^+ \widetilde{\lambda}} \widetilde{\lambda}\right] - \widetilde{c} \widetilde{\mathbb{E}}\left[e^{-m_0^+ \widetilde{\lambda}_{\tau_-^*}} \middle| X_0 = 0\right]}{\frac{1}{\overline{\gamma}} \widetilde{\mathbb{E}}[\widetilde{\lambda}] - \widetilde{c}}. \tag{22}$$

Proof. Use the new set of parameters under $\widetilde{\mathbb{P}}$ given by *Theorem 6.1*, and rewrite (21) as

$$\begin{aligned}
&- \delta\left(\widetilde{\lambda} - \widetilde{a}\right) \frac{\partial \widetilde{f}}{\partial \widetilde{\lambda}} + \widetilde{c} \frac{\partial \widetilde{f}}{\partial x} \\
&+ \widetilde{\lambda}\left(\int_0^\infty \int_0^x \widetilde{f}\left(x - z, \widetilde{\lambda} + u\right) \mathrm{d}\widetilde{Z}(z) \mathrm{d}\widetilde{\widetilde{G}}(u) + \Psi(x) \frac{e^{-m_0^+ \widetilde{\lambda}}}{\hat{g}(-\eta_0^+)} \overline{\overline{Z}}(x) - \widetilde{f}\right) \\
&+ \widetilde{\rho}\left(\int_0^\infty \widetilde{f}\left(x, \widetilde{\lambda} + u\right) \mathrm{d}\widetilde{\widetilde{H}}(u) - \widetilde{f}\right) = 0.
\end{aligned}$$

If $\widetilde{Z} \sim \mathrm{Exp}(\widetilde{\gamma})$, $\widetilde{\gamma} = \gamma - v_0^+$ under $\widetilde{\mathbb{P}}$ (equivalent to $Z \sim \mathrm{Exp}(\gamma)$ under $\mathbb{P}$), then, by *Remark 6.1*, we have

$$\begin{aligned}
&- \delta\left(\widetilde{\lambda} - \widetilde{a}\right) \frac{\partial \widetilde{f}}{\partial \widetilde{\lambda}} + \widetilde{c} \frac{\partial \widetilde{f}}{\partial x} \\
&+ \widetilde{\lambda}\left(\int_0^\infty \int_0^x \widetilde{f}\left(x - z, \widetilde{\lambda} + u\right) \widetilde{\gamma} e^{-\widetilde{\gamma} z} \mathrm{d}z \mathrm{d}\widetilde{\widetilde{G}}(u) + \frac{\gamma - v_0^+}{\gamma} \frac{e^{-m_0^+ \widetilde{\lambda}}}{\hat{g}(-\eta_0^+)} e^{-\widetilde{\gamma} x} - \widetilde{f}\right) \\
&+ \widetilde{\rho}\left(\int_0^\infty \widetilde{f}\left(x, \widetilde{\lambda} + u\right) \mathrm{d}\widetilde{\widetilde{H}}(u) - \widetilde{f}\right) = 0.
\end{aligned}$$

Take Laplace transform w.r.t. x, i.e.

$$\hat{\widetilde{f}}(w, \widetilde{\lambda}) =: \mathcal{L}\left\{\widetilde{f}(x, \widetilde{\lambda})\right\} = \int_0^\infty \widetilde{f}(u, \widetilde{\lambda}) e^{-wu} \mathrm{d}u,$$

we have

$$\begin{aligned}
\mathcal{L}\left\{\frac{\partial \widetilde{f}(x, \widetilde{\lambda})}{\partial x}\right\} &= w \hat{\widetilde{f}}(w, \widetilde{\lambda}) - \widetilde{f}(0, \widetilde{\lambda}), \\
\mathcal{L}\left\{\int_0^x \widetilde{f}(x - z, \widetilde{\lambda} + u) \widetilde{\gamma} e^{-\widetilde{\gamma} z} \mathrm{d}z\right\} &= \frac{\widetilde{\gamma}}{\widetilde{\gamma} + w} \hat{\widetilde{f}}(w, \widetilde{\lambda} + u), \\
\mathcal{L}\left\{e^{-\widetilde{\gamma} x}\right\} &= \frac{1}{\widetilde{\gamma} + w},
\end{aligned}$$

then,

$$\begin{aligned}
&- \delta\left(\widetilde{\lambda} - \widetilde{a}\right) \frac{\partial \hat{\widetilde{f}}(w, \widetilde{\lambda})}{\partial \widetilde{\lambda}} + \widetilde{c}\left(w \hat{\widetilde{f}}(w, \widetilde{\lambda}) - \widetilde{f}(0, \widetilde{\lambda})\right) \\
&+ \widetilde{\lambda}\left(\frac{\widetilde{\gamma}}{\widetilde{\gamma} + w} \int_0^\infty \hat{\widetilde{f}}(w, \widetilde{\lambda} + u) \mathrm{d}\widetilde{\widetilde{G}}(u) + \frac{\gamma - v_0^+}{\gamma} \frac{e^{-m_0^+ \widetilde{\lambda}}}{\hat{g}(-\eta_0^+)} \frac{1}{\widetilde{\gamma} + w} - \hat{\widetilde{f}}(w, \widetilde{\lambda})\right) \\
&+ \widetilde{\rho}\left(\int_0^\infty \hat{\widetilde{f}}(w, \widetilde{\lambda} + u) \mathrm{d}\widetilde{\widetilde{H}}(u) - \hat{\widetilde{f}}(w, \widetilde{\lambda})\right) = 0,
\end{aligned}$$

or,

$$\widetilde{\mathcal{A}}\hat{\widetilde{f}}(w,\widetilde{\lambda})+\widetilde{c}\left(w\hat{\widetilde{f}}(w,\widetilde{\lambda})-\widetilde{f}(0,\widetilde{\lambda})\right)+\widetilde{\lambda}\left(-\frac{w}{\widetilde{\gamma}+w}\int_0^\infty\hat{\widetilde{f}}(w,\widetilde{\lambda}+u)\mathrm{d}\widetilde{\widetilde{G}}(u)+\frac{\gamma-v_0^+}{\gamma}\frac{e^{-m_0^+\widetilde{\lambda}}}{\hat{g}(-\eta_0^+)}\frac{1}{\widetilde{\gamma}+w}\right)=0.$$

If $\widetilde{\lambda}\sim\Pi$, then,

$$\widetilde{\mathbb{E}}\left[\widetilde{\mathcal{A}}\hat{\widetilde{f}}(w,\widetilde{\lambda})+\widetilde{c}\left(w\hat{\widetilde{f}}(w,\widetilde{\lambda})-\widetilde{f}(0,\widetilde{\lambda})\right)+\widetilde{\lambda}\left(-\frac{w}{\widetilde{\gamma}+w}\int_0^\infty\hat{\widetilde{f}}(w,\widetilde{\lambda}+u)\mathrm{d}\widetilde{\widetilde{G}}(u)+\frac{\gamma-v_0^+}{\gamma}\frac{e^{-m_0^+\widetilde{\lambda}}}{\hat{g}(-\eta_0^+)}\frac{1}{\widetilde{\gamma}+w}\right)\right]=0,$$

and

$$\lim_{w\to 0}\widetilde{\mathbb{E}}\left[\widetilde{\mathcal{A}}\hat{\widetilde{f}}(w,\widetilde{\lambda})+\widetilde{c}\left(w\hat{\widetilde{f}}(w,\widetilde{\lambda})-\widetilde{f}(0,\widetilde{\lambda})\right)+\widetilde{\lambda}\left(-\frac{w}{\widetilde{\gamma}+w}\int_0^\infty\hat{\widetilde{f}}(w,\widetilde{\lambda}+u)\mathrm{d}\widetilde{\widetilde{G}}(u)+\frac{\gamma-v_0^+}{\gamma}\frac{e^{-m_0^+\widetilde{\lambda}}}{\hat{g}(-\eta_0^+)}\frac{1}{\widetilde{\gamma}+w}\right)\right]=0.$$

Since

$$\widetilde{C}=:\lim_{x\to\infty}\widetilde{f}(x,\widetilde{\lambda})=\lim_{w\to 0}w\hat{\widetilde{f}}(w,\widetilde{\lambda}),$$

$$\lim_{w\to 0}\frac{w}{\widetilde{\gamma}+w}\int_0^\infty\hat{\widetilde{f}}(w,\widetilde{\lambda}+u)\mathrm{d}\widetilde{\widetilde{G}}(u)=\int_0^\infty\lim_{w\to 0}\frac{w}{\widetilde{\gamma}+w}\hat{\widetilde{f}}(w,\widetilde{\lambda}+u)\mathrm{d}\widetilde{\widetilde{G}}(u)=\int_0^\infty\frac{1}{\widetilde{\gamma}}\widetilde{C}\mathrm{d}\widetilde{\widetilde{G}}(u)=\frac{\widetilde{C}}{\widetilde{\gamma}},$$

and by *Theorem 2.2*, we also have $\mathbb{E}\left[\widetilde{\mathcal{A}}\hat{\widetilde{f}}(0,\widetilde{\lambda})\right]=0$, then,

$$\widetilde{\mathbb{E}}\left[\widetilde{c}\left(\widetilde{C}-\widetilde{f}(0,\widetilde{\lambda})\right)+\widetilde{\lambda}\left(-\frac{\widetilde{C}}{\widetilde{\gamma}}+\frac{\gamma-v_0^+}{\gamma}\frac{e^{-m_0^+\widetilde{\lambda}}}{\hat{g}(-\eta_0^+)}\frac{1}{\widetilde{\gamma}}\right)\right]=0,$$

and

$$\widetilde{C}=\frac{\gamma-v_0^+}{\gamma\hat{g}(-\eta_0^+)}\frac{\frac{1}{\widetilde{\gamma}}\widetilde{\mathbb{E}}\left[e^{-m_0^+\widetilde{\lambda}}\widetilde{\lambda}\right]-\widetilde{c}\widetilde{\mathbb{E}}\left[e^{-m_0^+\widetilde{\lambda}_{\tau_-^*}}\middle|\widetilde{\lambda}_0=\widetilde{\lambda}\sim\Pi,X_0=0\right]}{\frac{1}{\widetilde{\gamma}}\widetilde{\mathbb{E}}[\widetilde{\lambda}]-\widetilde{c}},\tag{23}$$

note that, by definition,

$$\widetilde{\mathbb{E}}\left[\widetilde{f}(0,\widetilde{\lambda})\right]=\frac{\gamma-v_0^+}{\gamma\hat{g}(-\eta_0^+)}\widetilde{\mathbb{E}}\left[e^{-m_0^+\widetilde{\lambda}_{\tau_-^*}}\middle|\widetilde{\lambda}_0=\widetilde{\lambda}\sim\Pi,X_0=0\right].$$

Hence, we have the generalised Cramér-Lundberg constant (22) for $\widetilde{\lambda}\sim\Pi$, as

$$C=:\lim_{x\to\infty}\frac{P\left\{\tau^*<\infty\middle|X_0=x\right\}}{e^{-v_0x}}=\lim_{x\to\infty}\widetilde{\mathbb{E}}\left[e^{m_0^+\widetilde{\lambda}}\widetilde{f}(x,\widetilde{\lambda})\right]=\widetilde{\mathbb{E}}[e^{m_0^+\widetilde{\lambda}}]\widetilde{C}.$$

□

Remark 6.2. For the Cramér-Lundberg constant (22), by *Theorem 2.1* and *Corollary 2.1*, we can explicitly calculate the terms

$$\begin{aligned}
\widetilde{\mathbb{E}}[\widetilde{\lambda}] &= \frac{\mu_{1_{\widetilde{\widetilde{H}}}}\widetilde{\rho} + \widetilde{a\delta}}{\widetilde{\delta} - \mu_{1_{\widetilde{\widetilde{G}}}}}, \\
\widetilde{\mathbb{E}}[e^{m_0^+\widetilde{\lambda}}] &= \exp\left(\int_{-m_0^+}^{0} \frac{\widetilde{a\delta}u + \widetilde{\rho}[1 - \hat{\widetilde{\widetilde{h}}}(u)]}{\widetilde{\delta}u + \hat{\widetilde{\widetilde{g}}}(u) - 1}\mathrm{d}u\right), \\
\widetilde{\mathbb{E}}\left[\widetilde{\lambda}e^{-m_0^+\widetilde{\lambda}}\right] &= -\frac{\mathrm{d}}{\mathrm{d}m}\widetilde{\mathbb{E}}[e^{-m\widetilde{\lambda}}]\Big|_{m=m_0^+} = \frac{\widetilde{a\delta}m_0^+ + \widetilde{\rho}[1 - \hat{\widetilde{\widetilde{h}}}(m_0^+)]}{\widetilde{\delta}m_0^+ + \hat{\widetilde{\widetilde{g}}}(m_0^+) - 1}\exp\left(-\int_0^{m_0^+} \frac{\widetilde{a\delta}u + \widetilde{\rho}[1 - \hat{\widetilde{\widetilde{h}}}(u)]}{\widetilde{\delta}u + \hat{\widetilde{\widetilde{g}}}(u) - 1}\mathrm{d}u\right).
\end{aligned}$$

Also, by *Theorem 6.3* for the net profit condition under the measure $\widetilde{\mathbb{P}}$, we have

$$\frac{1}{\widetilde{\widetilde{\gamma}}}\widetilde{\mathbb{E}}[\widetilde{\lambda}] - \widetilde{c} > 0.$$

6.3. Net Profit Condition under $\mathbb{P}$ and $\widetilde{\mathbb{P}}$

Theorem 6.3. *If the net profit condition and the stationarity condition both hold under $\mathbb{P}$, i.e.*

$$c > \frac{\mu_{1_H}\rho + a\delta}{\delta - \mu_{1_G}}\mu_{1_Z}, \quad \delta > \mu_{1_G},$$

and the stationarity condition also holds under the new measure $\widetilde{\mathbb{P}}$, i.e. $\widetilde{\delta} > \mu_{1_{\widetilde{\widetilde{G}}}}$, then, under $\widetilde{\mathbb{P}}$, we have

$$\frac{\mu_{1_{\widetilde{\widetilde{H}}}}\widetilde{\rho} + \widetilde{a\delta}}{\widetilde{\delta} - \mu_{1_{\widetilde{\widetilde{G}}}}}\mu_{1_{\widetilde{Z}}} > \widetilde{c}, \tag{24}$$

and the ruin becomes certain (almost surely), i.e.

$$\widetilde{\mathbb{P}}\{\tau^* < \infty\} =: \lim_{t\to\infty}\widetilde{\mathbb{P}}\{\tau^* \le t\} = 1.$$

Proof. By the transformation between two measures from *Theorem 6.1*, we have

$$\mu_{1_{\widetilde{Z}}} =: \widetilde{\mathbb{E}}[Z_i] = \int_0^\infty z\mathrm{d}\widetilde{Z}(z) = \int_0^\infty z\frac{e^{v_0^+z}\mathrm{d}Z(z)}{\hat{z}\left(-v_0^+\right)} = \frac{1}{\hat{z}\left(-v_0^+\right)}\int_0^\infty ze^{v_0^+z}\mathrm{d}Z(z) = \frac{\hat{z}'\left(-v_0^+\right)}{\hat{z}\left(-v_0^+\right)}.$$

Change variable $y = \frac{1}{1+\delta\eta_0^+}u$, then,

$$\begin{aligned}
\mu_{1_{\widetilde{\widetilde{H}}}} &= \widetilde{\mathbb{E}}\left[Y^{(1)}\right] = \int_0^\infty u\frac{\widetilde{h}\left(\frac{u}{1+\delta\eta_0^+}\right)}{1+\delta\eta_0^+}\mathrm{d}u = \frac{\int_0^\infty ue^{\frac{\eta_0^+}{1+\delta\eta_0^+}u}h\left(\frac{1}{1+\delta\eta_0^+}u\right)\mathrm{d}u}{\left(1+\delta\eta_0^+\right)\hat{h}(-\eta_0^+)} = \frac{1+\delta\eta_0^+}{\hat{h}(-\eta_0^+)}\int_0^\infty ye^{\eta_0^+y}\mathrm{d}H(y); \\
\mu_{1_{\widetilde{\widetilde{G}}}} &= \widetilde{\mathbb{E}}\left[Y^{(2)}\right] = \frac{1+\delta\eta_0^+}{\hat{g}(-\eta_0^+)}\int_0^\infty ye^{\eta_0^+y}\mathrm{d}G(y) = \hat{z}(-v_0^+)\hat{g}'(-\eta_0^+). \qquad \left(\because \hat{z}(-v_0^+)\hat{g}(-\eta_0^+) = 1+\delta\eta_0^+\right)
\end{aligned}$$

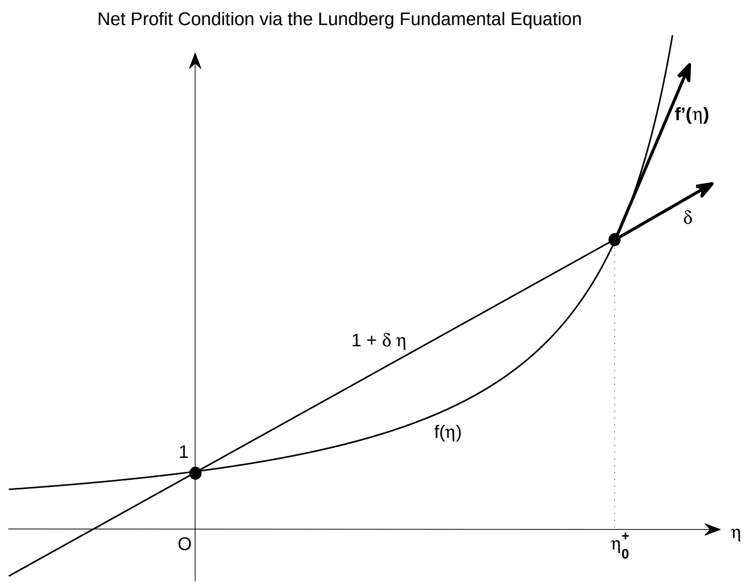

Figure 6: Net Profit Condition via the Generalised Lundberg Fundamental Equation

The mean of self-excited jump sizes under $\widetilde{\mathbb{P}}$ is greater than the one under $\mathbb{P}$, since

$$\mu_{1_{\widetilde{G}}} > \hat{g}'(-\eta_0^+) = \int_0^\infty y e^{\eta_0^+ y} \mathrm{d}G(y) > \int_0^\infty y \mathrm{d}G(y) = \mu_{1_G}.$$

Hence,

$$\begin{aligned} & \frac{\mu_{1_{\widetilde{H}}}\widetilde{\rho} + \widetilde{a}\widetilde{\delta}}{\widetilde{\delta} - \mu_{1_{\widetilde{G}}}} \mu_{1_{\widetilde{Z}}} \\ = & \frac{\rho \int_0^\infty y e^{\eta_0^+ y} \mathrm{d}H(y) + a\delta}{\delta - \hat{z}(-v_0^+)\hat{g}'(-\eta_0^+)} \frac{1 + \delta\eta_0^+}{\hat{z}(-v_0^+)} \int_0^\infty z e^{v_0^+ z} \mathrm{d}Z(z) \qquad \left(\because \hat{z}(-v_0^+)\hat{g}(-\eta_0^+) = 1 + \delta\eta_0^+ \right) \\ = & \hat{z}'(-v_0^+)\hat{g}(-\eta_0^+) \frac{\hat{h}'(-\eta_0^+)\rho + a\delta}{\delta - \hat{z}(-v_0^+)\hat{g}'(-\eta_0^+)}. \end{aligned} \tag{25}$$

From the generalised Lundberg's fundamental equation, we have

$$1 + \delta\eta_0^+ = \hat{z}\left(\frac{-a\delta\eta_0^+ + \rho\left(1 - \hat{h}(-\eta_0^+)\right)}{c} \right) \hat{g}(-\eta_0^+).$$

If the net profit condition and stationarity condition both holds under $\mathbb{P}$, the right-hand-side function is a strictly increasing and convex function of η_0^+ as obviously a convex function of a function convex function is still a convex function; it was also proved formally in the proof of *Lemma 4.1*. Hence, as shown in *Figure 6*, at the point η_0^+ the slope of the left-hand-side function is greater than the slope of the right-hand-side function, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left(1+\delta\eta\right)\Big|_{\eta=\eta_0^+} < \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\hat{z}\left(\frac{-a\delta\eta+\rho\left(1-\hat{h}(-\eta)\right)}{c}\right)\hat{g}(-\eta)\right)\Big|_{\eta=\eta_0^+},$$

or,

$$\begin{aligned}
\delta \quad < \quad & -\left(\frac{a\delta+\rho\frac{\mathrm{d}\hat{h}(-\eta_0^+)}{\mathrm{d}\eta_0^+}}{c}\right)\frac{\mathrm{d}\hat{z}(u)}{\mathrm{d}u}\Big|_{u=\frac{-a\delta\eta_0^+ +\rho\left(1-\hat{h}(-\eta_0^+)\right)}{c}}\hat{g}(-\eta_0^+)+\hat{z}(-v_0^+)\frac{\mathrm{d}\hat{g}(-\eta_0^+)}{\mathrm{d}\eta_0^+} \\
= & -\left(\frac{a\delta+\rho\frac{\mathrm{d}\hat{h}(-\eta_0^+)}{\mathrm{d}\eta_0^+}}{c}\right)\frac{\mathrm{d}\hat{z}(u)}{\mathrm{d}u}\Big|_{u=-v_0^+}\hat{g}(-\eta_0^+)+\hat{z}(-v_0^+)\frac{\mathrm{d}\hat{g}(-\eta_0^+)}{\mathrm{d}\eta_0^+} \\
= & \left(\frac{a\delta+\rho\frac{\mathrm{d}\hat{h}(-\eta_0^+)}{\mathrm{d}\eta_0^+}}{c}\right)\frac{\mathrm{d}\hat{z}(-v_0^+)}{\mathrm{d}v_0^+}\hat{g}(-\eta_0^+)+\hat{z}(-v_0^+)\frac{\mathrm{d}\hat{g}(-\eta_0^+)}{\mathrm{d}\eta_0^+},
\end{aligned}$$

and

$$c\left(\delta-\hat{z}(-v_0^+)\frac{\mathrm{d}\hat{g}(-\eta_0^+)}{\mathrm{d}\eta_0^+}\right) < \left(a\delta+\rho\frac{\mathrm{d}\hat{h}(-\eta_0^+)}{\mathrm{d}\eta_0^+}\right)\frac{\mathrm{d}\hat{z}(-v_0^+)}{\mathrm{d}v_0^+}\hat{g}(-\eta_0^+).$$

Since the stationarity condition also holds under $\widetilde{\mathbb{P}}$, i.e.

$$\delta > \hat{z}(-v_0^+)\frac{\mathrm{d}\hat{g}(-\eta_0^+)}{\mathrm{d}\eta_0^+},$$

then,

$$c < \frac{a\delta+\rho\frac{\mathrm{d}\hat{h}(-\eta_0^+)}{\mathrm{d}\eta_0^+}}{\delta-\hat{z}(-v_0^+)\frac{\mathrm{d}\hat{g}(-\eta_0^+)}{\mathrm{d}\eta_0^+}}\hat{g}(-\eta_0^+)\frac{\mathrm{d}\hat{z}(-v_0^+)}{\mathrm{d}v_0^+},$$

and by (25), we have (24).

□

Remark 6.3. If the net profit condition and the stationarity condition hold under $\mathbb{P}$, but the stationarity condition does not hold under $\widetilde{\mathbb{P}}$, i.e. $\widetilde{\delta} < \mu_{1_{\widetilde{\widetilde{G}}}}$, then, the intensity $\widetilde{\lambda}_t$ under $\widetilde{\mathbb{P}}$ will increase arbitrarily. It does not mean the measures are not equivalent, as we are only considering them till a fixed time T anyway in the optional stopping theorem; also, ruin does occur with probability one and pretty fast (which will manifest itself in the simulation).

In particular, for the special case of shot noise intensity, interestingly , we find a conjugate relationship between the expected loss rates under the two measures.

Corollary 6.1. *For the shot noise case with $H \sim \mathrm{Exp}(\alpha)$ and $Z \sim \mathrm{Exp}(\gamma)$, if the net profit condition holds under the original measure $\mathbb{P}$, i.e.*

$$c > \frac{\rho}{\delta\alpha\gamma},$$

then, under the new measure $\widetilde{\mathbb{P}}$, we have

$$\widetilde{c} < \frac{\widetilde{\rho}}{\widetilde{\delta\alpha\gamma}},$$

and

$$\frac{\rho}{\delta\alpha\gamma}\frac{\widetilde{\rho}}{\widetilde{\delta\alpha\gamma}} = c^2. \tag{26}$$

Proof. In particular, for the shot noise case with jump-size distributions $H \sim \text{Exp}(\alpha)$ and $Z \sim \text{Exp}(\gamma)$ (by setting $a = 0$ and $\hat{g}(\cdot) = 1$ in *Theorem 6.3*), we have the parameters transformed by

- $c \to \widetilde{c} = c$,
- $\delta \to \widetilde{\delta} = \delta$,
- $\rho \nearrow \widetilde{\rho} = \frac{\alpha}{\alpha - \eta_0^+}\rho$,
- $\gamma \searrow \widetilde{\gamma} = \gamma - v_0^+$,
- $\alpha \searrow \widetilde{\alpha} = \frac{\alpha - \eta_0^+}{1+\delta\eta_0^+}$,

where the constants are restricted by the generalised Lundberg's fundamental equation

$$\begin{cases} \delta\eta_0^+ = \frac{\gamma}{\gamma - v_0^+} - 1 \\ cv_0^+ = \rho\left(\frac{\alpha}{\alpha - \eta_0^+} - 1\right) \end{cases} \qquad \left(c > \frac{\rho}{\delta\alpha\gamma}\right).$$

The net profit condition holds under $\mathbb{P}$, i.e. $c > \frac{\rho}{\delta\alpha\gamma}$, but under $\widetilde{\mathbb{P}}$ we have $\frac{\widetilde{\rho}}{\widetilde{\delta\alpha\gamma}} > \widetilde{c}$, since

$$\begin{aligned}
\frac{\widetilde{\rho}}{\widetilde{\delta\alpha\gamma}} &= \frac{\frac{\alpha}{\alpha-\eta_0^+}\rho}{\frac{\alpha-\eta_0^+}{1+\delta\eta_0^+}\left(\gamma - v_0^+\right)\delta} \\
&= \frac{\alpha\rho}{\delta}\frac{1+\delta\eta_0^+}{(\alpha-\eta_0^+)^2\frac{\gamma}{\delta\eta_0^+ + 1}} \qquad \left(\because \gamma - v_0^+ = \frac{\gamma}{\delta\eta_0^+ + 1}\right) \\
&= \frac{\alpha\rho}{\delta\gamma}\left(\frac{1+\delta\eta_0^+}{\alpha - \eta_0^+}\right)^2 \\
&= \frac{\alpha\rho}{\delta\gamma}\left(\frac{c\delta\gamma}{\rho}\right)^2 \\
&= \frac{\delta\alpha\gamma}{\rho}c^2 \qquad \left(\because c = \frac{1+\delta\eta_0^+}{\alpha-\eta_0^+}\frac{\rho}{\delta\gamma}\right) \\
&> \frac{\delta\alpha\gamma}{\rho}\frac{\rho}{\delta\alpha\gamma}c = \widetilde{c}.
\end{aligned}$$

Hence, we also find (26).

□

7. Example: Jumps with Exponential Distributions

To represent the previous results in explicit forms, in this section, we further assume the externally excited and self-excited jumps in the intensity process λ_t and the claim sizes all follow exponential distributions, i.e. $H \sim \text{Exp}(\alpha)$, $G \sim \text{Exp}(\beta)$ and $Z \sim \text{Exp}(\gamma)$, with the density functions

$$h(y) = \alpha e^{-\alpha y}, \quad g(y) = \beta e^{-\beta y}, \quad z(z) = \gamma e^{-\gamma z}, \quad y, z; \alpha, \beta, \gamma > 0,$$

and the Laplace transforms

$$\hat{h}(u) = \frac{\alpha}{\alpha + u}, \quad \hat{g}(u) = \frac{\beta}{\beta + u}, \quad \hat{z}(u) = \frac{\gamma}{\gamma + u}.$$

7.1. Generalised Lundberg's Fundamental Equation

We discuss the general case $0 \leq r < r^*$ and the special case $r = 0$ for the generalised Lundberg's fundamental equation (from *Theorem 4.1*) respectively.

Case $0 \leq r < r^*$. By *Theorem 4.1*, we have the generalised Lundberg's fundamental equation for $0 \leq r < r^*$,

$$\begin{cases} \frac{\gamma}{\gamma - v_r} \frac{\beta}{\beta - \eta_r} = 1 + \delta\eta_r \\ -v_r = \frac{r - a\delta\eta_r + \rho\left(1 - \frac{\alpha}{\alpha - \eta_r}\right)}{c} \end{cases} \qquad \left(v_r < \gamma, \eta_r < (\alpha \wedge \beta); c > \frac{\beta(\rho + a\alpha\delta)}{\alpha\gamma(\delta\beta - 1)}, \delta\beta > 1 \right),$$

or, rewrite it w.r.t. η_r as

$$\begin{aligned} 1 + \delta\eta_r &= \frac{c\gamma\beta(\alpha - \eta_r)}{(a\delta\eta_r^2 - (\gamma c + \rho + a\delta\alpha + r)\eta_r + \gamma c\alpha + \alpha r)(\beta - \eta_r)}, \quad \eta_r < (\alpha \wedge \beta), \\ v_r &= \frac{\eta_r}{c}\left(\frac{\rho}{\alpha - \eta_r} + a\delta\right) - \frac{r}{c}, \quad v_r < \gamma, \end{aligned}$$

with parameters restricted by

$$c > \frac{\beta(\rho + a\alpha\delta)}{\alpha\gamma(\delta\beta - 1)}, \quad \delta\beta > 1.$$

Solve (9) of *Lemma 4.2* and substitute the unique negative solution $\eta^* = \frac{\delta\beta - 1}{\delta}$ into (8), we obtain the constant r^*,

$$r^* = (\delta\beta - 1)\left(a + \frac{\rho}{\delta(\alpha - \beta) + 1}\right).$$

Case $r = 0$. Set $r \to 0$, we have the generalised Lundberg's fundamental equation for $r = 0$,

$$\begin{cases} \frac{\gamma}{\gamma - v_0} \frac{\beta}{\beta - \eta_0} = 1 + \delta\eta_0 \\ -v_0 = \frac{-a\delta\eta_0 + \rho\left(1 - \frac{\alpha}{\alpha - \eta_0}\right)}{c} \end{cases} \qquad \left(v_0 < \gamma, \eta_0 < (\alpha \wedge \beta); c > \frac{\beta(\rho + a\alpha\delta)}{\alpha\gamma(\delta\beta - 1)}, \delta\beta > 1 \right),$$

or, rewrite w.r.t. η_0 as

$$\begin{aligned} 1 + \delta\eta_0 &= \frac{c\gamma\beta(\alpha - \eta_0)}{(a\delta\eta_0^2 - (\gamma c + \rho + a\delta\alpha)\eta_0 + \gamma c\alpha)(\beta - \eta_0)}, \quad \eta_0 < (\alpha \wedge \beta), \\ v_0 &= \frac{\eta_0}{c}\left(\frac{\rho}{\alpha - \eta_0} + a\delta\right), \quad v_0 < \gamma, \end{aligned}$$

with parameters restricted by

$$c > \frac{\beta(\rho + a\alpha\delta)}{\alpha\gamma(\delta\beta - 1)}, \quad \delta\beta > 1.$$

The results of case $r = 0$ here will be used later in Section 7.3 for numerical calculations.

7.2. *Ruin Probability and Generalised Cramér-Lundberg Approximation via Measure* $\widetilde{\mathbb{P}}$

The *Corollary 7.1* below is an example of *Theorem 6.1* and *Theorem 6.2* by additionally assuming the exponential distributions.

Corollary 7.1. *If* $H \sim \text{Exp}(\alpha)$*,* $G \sim \text{Exp}(\beta)$*,* $Z \sim \text{Exp}(\gamma)$*,* $\alpha \geq \beta$*, the net profit condition holds under* $\mathbb{P}$*, and stationarity condition holds under* $\mathbb{P}$ *and* $\widetilde{\mathbb{P}}$*, and the initial intensity follows the stationary distribution under* $\widetilde{\mathbb{P}}$*, i.e.* $\widetilde{\lambda} \overset{\mathcal{D}}{=} \widetilde{a} + \widetilde{\Gamma}_1 + \widetilde{\Gamma}_2$ *where*

$$\widetilde{\Gamma}_1 \sim \text{Gamma}\left(\frac{1}{\widetilde{\delta}}\left(\widetilde{a} + \frac{\widetilde{\rho}}{\widetilde{\delta}(\widetilde{\alpha} - \widetilde{\beta}) + 1}\right), \frac{\widetilde{\delta}\widetilde{\beta} - 1}{\widetilde{\delta}}\right), \quad \widetilde{\Gamma}_2 \sim \text{Gamma}\left(\frac{\widetilde{\rho}(\widetilde{\alpha} - \widetilde{\beta})}{\widetilde{\delta}(\widetilde{\alpha} - \widetilde{\beta}) + 1}, \widetilde{\alpha}\right),$$

then, we have the ruin probability

$$P\left\{\tau^* < \infty \middle| X_0 = x\right\} = \frac{\gamma - v_0^+}{\gamma}\frac{\beta - \eta_0^+}{\beta}\widetilde{\mathbb{E}}[e^{m_0^+\widetilde{\lambda}}]\widetilde{\mathbb{E}}\left[e^{-m_0^+\widetilde{\lambda}_{\tau_-^*}}\middle| X_0 = x\right]e^{-v_0^+ x}, \tag{27}$$

and the generalised Cramér-Lundberg approximation

$$P\left\{\tau^* < \infty \middle| X_0 = x\right\} \sim Ce^{-v_0^+ x}, \quad x \to \infty,$$

where

$$C =: \frac{\gamma - v_0^+}{\gamma}\frac{\beta - \eta_0^+}{\beta}\widetilde{\mathbb{E}}[e^{m_0^+\widetilde{\lambda}}]\frac{\frac{1}{\widetilde{\gamma}}\widetilde{\mathbb{E}}\left[e^{-m_0^+\widetilde{\lambda}}\widetilde{\lambda}\right] - \widetilde{c}\widetilde{\mathbb{E}}\left[e^{-m_0^+\widetilde{\lambda}_{\tau_-^*}}\middle| X_0 = 0\right]}{\frac{1}{\widetilde{\gamma}}\widetilde{\mathbb{E}}[\widetilde{\lambda}] - \widetilde{c}}. \tag{28}$$

The transformation from $\mathbb{P}$ *to* $\widetilde{\mathbb{P}}$ *is given by*

- $a \nearrow \widetilde{a} =: \left(1 + \delta\eta_0^+\right) a$,
- $c \to \widetilde{c} =: c$,
- $\delta \to \widetilde{\delta} =: \delta$,
- $\rho \nearrow \widetilde{\rho} =: \frac{\alpha}{\alpha - \eta_0^+}\rho$,
- $\gamma \searrow \widetilde{\gamma} =: \gamma - v_0^+$,
- $\beta \searrow \widetilde{\beta} =: \frac{\beta - \eta_0^+}{1 + \delta\eta_0^+}$,
- $\alpha \searrow \widetilde{\alpha} =: \frac{\alpha - \eta_0^+}{1 + \delta\eta_0^+}$.

Proof. If $H \sim \text{Exp}(\alpha)$, $G \sim \text{Exp}(\beta)$, $Z \sim \text{Exp}(\gamma)$, by *Theorem 2.3* for the case when $\alpha \geq \beta$, we have the Laplace transform

$$\widetilde{\mathbb{E}}\left[e^{-m_0^+\widetilde{\lambda}}\right] = e^{-m_0^+\widetilde{a}}\left(\frac{\widetilde{\alpha}}{\widetilde{\alpha}+m_0^+}\right)^{\frac{\widetilde{\rho}(\widetilde{\alpha}-\widetilde{\beta})}{\widetilde{\delta}(\widetilde{\alpha}-\widetilde{\beta})+1}}\left(\frac{\frac{\widetilde{\delta}\widetilde{\beta}-1}{\widetilde{\delta}}}{m_0^+ + \frac{\widetilde{\delta}\widetilde{\beta}-1}{\widetilde{\delta}}}\right)^{\frac{1}{\widetilde{\delta}}\left(\widetilde{a}+\frac{\widetilde{\rho}}{\widetilde{\delta}(\widetilde{\alpha}-\widetilde{\beta})+1}\right)}.$$

Use *Theorem 6.1* and *Theorem 6.2*, the ruin probability and generalised Cramér-Lundberg approximation can be derived immediately. ☐

We only discuss the case when $\alpha \geq \beta$ for instance. It is similar to derive the corresponding results for other cases when $\alpha < \beta$ and we omit them here.

Remark 7.1. We can calculate explicitly for the terms in (27) and (28) of *Corollary 7.1*,

$$\begin{aligned}
\widetilde{\mathbb{E}}\left[e^{m_0^+\widetilde{\lambda}}\right] &= e^{m_0^+\widetilde{a}}\left(\frac{\widetilde{\alpha}}{\widetilde{\alpha}-m_0^+}\right)^{\frac{\widetilde{\rho}(\widetilde{\alpha}-\widetilde{\beta})}{\widetilde{\delta}(\widetilde{\alpha}-\widetilde{\beta})+1}}\left(\frac{\frac{\widetilde{\delta}\widetilde{\beta}-1}{\widetilde{\delta}}}{\frac{\widetilde{\delta}\widetilde{\beta}-1}{\widetilde{\delta}} - m_0^+}\right)^{\frac{1}{\widetilde{\delta}}\left(\widetilde{a}+\frac{\widetilde{\rho}}{\widetilde{\delta}(\widetilde{\alpha}-\widetilde{\beta})+1}\right)}, \\
\widetilde{\mathbb{E}}\left[e^{-m_0^+\widetilde{\lambda}}\widetilde{\lambda}\right] &= e^{-m_0^+\widetilde{a}}\left(\frac{\widetilde{\alpha}}{\widetilde{\alpha}+m_0^+}\right)^{\frac{\widetilde{\rho}(\widetilde{\alpha}-\widetilde{\beta})}{\widetilde{\delta}(\widetilde{\alpha}-\widetilde{\beta})+1}}\left(\frac{\frac{\widetilde{\delta}\widetilde{\beta}-1}{\widetilde{\delta}}}{m_0^+ + \frac{\widetilde{\delta}\widetilde{\beta}-1}{\widetilde{\delta}}}\right)^{\frac{1}{\widetilde{\delta}}\left(\widetilde{a}+\frac{\widetilde{\rho}}{\widetilde{\delta}(\widetilde{\alpha}-\widetilde{\beta})+1}\right)}\frac{\widetilde{a}\widetilde{\delta}+\frac{\widetilde{\rho}}{\widetilde{\alpha}+m_0^+}}{\widetilde{\delta}-\frac{1}{\widetilde{\beta}+m_0^+}}, \\
\widetilde{\mathbb{E}}[\widetilde{\lambda}] &= \frac{\frac{\widetilde{\rho}}{\widetilde{\alpha}}+\widetilde{a}\widetilde{\delta}}{\widetilde{\delta}-\frac{1}{\widetilde{\beta}}},
\end{aligned}$$

except the term $\widetilde{\mathbb{E}}\left[e^{-m_0^+\widetilde{\lambda}_{\tau^*_-}}\middle|\lambda_0 = \widetilde{\lambda} \sim \Pi, X_0 = x\right]$. However, this term can be easily estimated by simulation under $\widetilde{\mathbb{P}}$ where ruin becomes certain. The procedure of estimation is discussed in Section 7.3.

7.3. Numerical Examples

For the purpose of simulation, the event of ruin is indicated by comparing the loss with the initial reserve $X_0 = x$, namely, ruin occurs if

$$\sup_{t>0}\left\{ct - \sum_{i=i}^{N_t} Z_i\right\} \geq X_0.$$

Hence, the ruin probability is rewritten as

$$P\left\{\tau^* < \infty\middle|X_0 = x\right\} = P\left\{\sup_{t>0}\left\{ct - \sum_{i=i}^{N_t} Z_i\right\} \geq x\right\}.$$

As discussed in Remark 7.1, for exponential distribution case when $\alpha \geq \beta$ of *Corollary 7.1*, all terms have explicit formulas except the one below that relies on simulation

$$\widetilde{\mathbb{E}}\left[e^{-m_0^+\widetilde{\lambda}_{\tau^*_-}}\middle|\widetilde{\lambda}_0 = \widetilde{\lambda} \sim \Pi, X_0 = x\right] = \lim_{k\to\infty}\frac{e^{-m_0^+\widetilde{\lambda}_{\tau^*_{1-}}} + e^{-m_0^+\widetilde{\lambda}_{\tau^*_{2-}}} + \dots + e^{-m_0^+\widetilde{\lambda}_{\tau^*_{k-}}}}{k},$$

where k is the number of simulations.

Remark 7.2. Under the original measure $\mathbb{P}$, the event of ruin is rare and particularly it is hard to simulate an infinitely long path ($t = \infty$) for estimating $P\{\tau^* < t = \infty \big| X_0 = x\}$ precisely. Thus we alternatively implement the simulation under the measure $\widetilde{\mathbb{P}}$ where ruin becomes certain and hence the simulation is much faster, particularly for $X_0 = 0$.

We provide two numerical examples based on 500,000 simulations with different parameter settings in Section 7.3.1 and Section 7.3.2, respectively. For each example, we compare the simulated ruin probability and the estimated Cramér constant under $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ based on *Corollary 7.1*.

7.3.1. Numerical Example 1

Simulation under $\mathbb{P}$ The parameters under original measure $\mathbb{P}$ are set by

$$(a, \rho, \delta; \alpha, \beta, \gamma; c) = (0.7, 0.5, 3; 2.5, 1, 1; 1.5).$$

Then, we can obtain $(\eta_0^+, v_0^+) = (0.0811, 0.1247)$, the unique solution of the generalised Lundberg's fundamental equation (given by Case $r = 0$ of Section 7.1). It is easy to check that $\alpha \geq \beta$, the stationarity condition and the net profit condition all hold, as

$$\delta = 3 > \frac{1}{\beta} = 1, \quad c = 1.5 > \frac{\frac{\rho}{\alpha} + a\delta}{\delta - \frac{1}{\beta}} \frac{1}{\gamma} = 1.15, \quad \Rightarrow \quad \mathbb{I}(\tau^* < \infty) < 1.$$

Calculate the ruin probability $P\{\tau^* < \infty \big| X_0 = x\}$ based on the simulation under $\mathbb{P}$ with $\lambda_0 \sim \Pi$, i.e. $\lambda_0 \stackrel{\mathcal{D}}{=} a + \Gamma_1 + \Gamma_2$, where

$$\Gamma_1 \sim \text{Gamma}\left(\frac{1}{\delta}\left(a + \frac{\rho}{\delta(\alpha - \beta) + 1}\right), \frac{\delta\beta - 1}{\delta}\right), \quad \Gamma_2 \sim \text{Gamma}\left(\frac{\rho(\alpha - \beta)}{\delta(\alpha - \beta) + 1}, \alpha\right).$$

Since $P\{\tau^* < \infty \big| X_0 = x\} \sim Ce^{-v_0 x}$, the Cramér constant C can be estimated by the ratio $P\{\tau^* < \infty \big| X_0 = x\}/e^{-v_0^+ x}$ for a large $X_0 = x$. The probability $P\{\tau^* < \infty \big| X_0 = x\}$ and the ratio for C estimation are given by the first and second rows of *Table 2* and the plotted by the first and second graphs of *Figure 7*. Alternatively, it could be more convenient to look at the results by taking logarithm as

$$\ln\Big(P\{\tau^* < \infty \big| X_0 = x\}\Big) \sim \ln C - v_0 x, \quad x \to \infty.$$

The results are given by the fourth and fifth rows of *Table 2* and plotted by the third graph of *Figure 7*.

Table 2: *Example 1*: Numerical Results

X_0	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
$P\left\{\tau^* < \infty \big\| X_0\right\}$	76.90%	56.56%	42.85%	32.93%	25.45%	19.75%	15.35%	11.93%	9.29%	7.24%	5.63%	4.38%	3.42%	2.66%	2.07%	1.62%
C Estimation	76.90%	72.58%	70.57%	69.59%	69.03%	68.75%	68.56%	68.40%	68.34%	68.30%	68.14%	68.14%	68.13%	68.10%	68.04%	68.10%
$P_{\widetilde{\mathbb{P}}}\left\{\tau^* < \infty \big\| X_0\right\}$	78.71%	57.76%	43.60%	33.42%	25.84%	20.06%	15.56%	12.10%	9.42%	7.33%	5.71%	4.44%	3.47%	2.70%	2.10%	1.64%
$\ln P\left\{\tau^* < \infty \big\| X_0\right\}$	-0.263	-0.570	-0.847	-1.111	-1.368	-1.622	-1.874	-2.126	-2.376	-2.626	-2.878	-3.127	-3.377	-3.627	-3.877	-4.126
$\ln(66.93\%) - v_0 X_0$	-0.402	-0.651	-0.900	-1.150	-1.399	-1.649	-1.898	-2.148	-2.397	-2.647	-2.896	-3.145	-3.395	-3.644	-3.894	-4.143

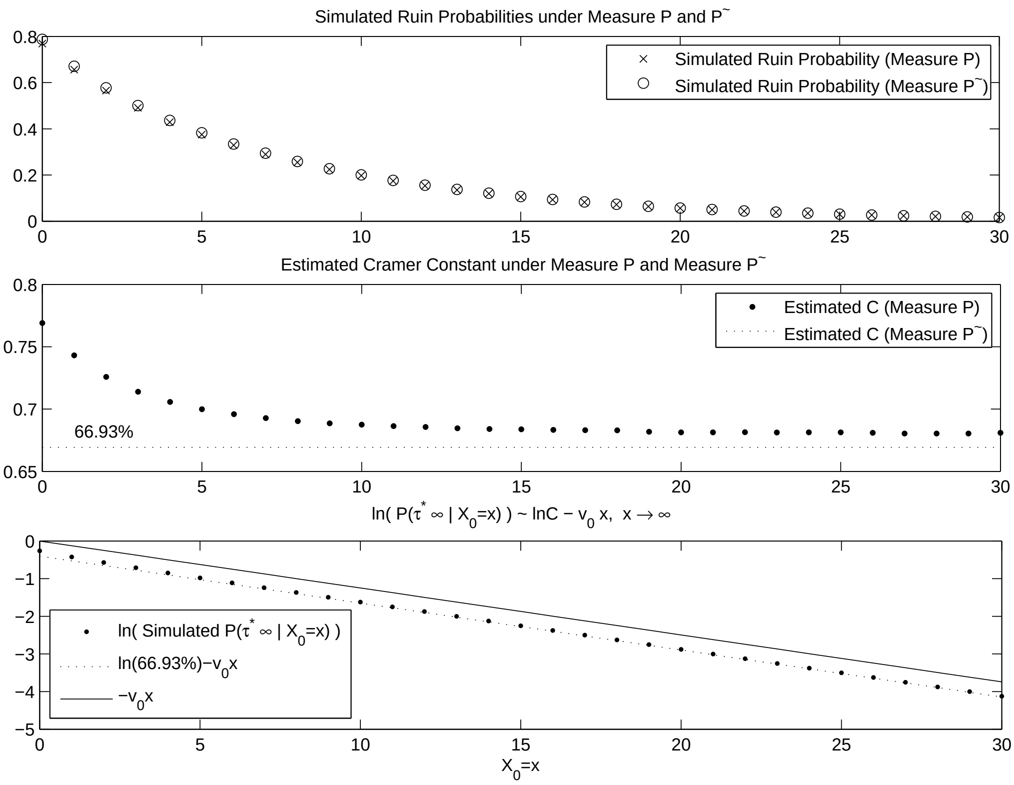

Figure 7: *Example 1*: Simulated Ruin Probabilities and Estimation for the Cramér Constant C under Measure $\mathbb{P}$ and $\widetilde{\mathbb{P}}$

Remark 7.3. The accuracy deceases when $X_0 = x$ increases, as both of the numerator the ruin probability $P\{\tau^* < \infty \big| X_0 = x\}$ and the denominator $e^{-v_0^+ x}$ are approaching very closely to 0. The ruin probability $P\{\tau^* < \infty \big| X_0 = x\}$ with a high initial reserve becomes harder to be estimated precisely.

Simulation under $\widetilde{\mathbb{P}}$ Under the new measure $\widetilde{\mathbb{P}}$, by the transformation from *Corollary 7.1*, the new parameter setting is given by

$$(\widetilde{a}, \widetilde{\rho}, \widetilde{\delta}; \widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}; \widetilde{c}) = (0.8703, 0.5168, 3; 1.9455, 0.7391, 0.8753; 1.5000).$$

Then, we can obtain $m_0^+ = 0.0652$. It is easy to check the stationarity condition holds but the net profit condition does not hold and ruin is certain, as

$$\widetilde{\delta} = 3 > \frac{1}{\widetilde{\beta}} = 1.3530, \quad \widetilde{c} = 1.5 < \frac{\frac{\widetilde{\rho}}{\widetilde{\alpha}} + \widetilde{a}\widetilde{\delta}}{\widetilde{\delta} - \frac{1}{\widetilde{\beta}}} \frac{1}{\widetilde{\gamma}} = 1.9954, \quad \Rightarrow \quad \mathbb{I}(\tau^* < \infty) = 1.$$

We can also calculate

$$\widetilde{\mathbb{E}}[e^{m_0^+ \widetilde{\lambda}}] = 1.1261, \quad \widetilde{\mathbb{E}}\left[e^{-m_0^+ \widetilde{\lambda}} \widetilde{\lambda}\right] = 1.4625,$$

explicitly, and estimate

$$\widetilde{\mathbb{E}}\left[e^{-m_0^+ \widetilde{\lambda}_{\tau_-^*}} \middle| \widetilde{\lambda}_0 = \widetilde{\lambda} \sim \Pi, X_0 = 0\right] \approx 86.98\%$$

from the simulation under $\widetilde{\mathbb{P}}$ given by third row of *Table 3*. Note that, under $\widetilde{\mathbb{P}}$, ruin is certain, i.e. $\widetilde{P}\{\tau^* < \infty\} = 1$ and $\widetilde{E}[\widetilde{\lambda}_t] = 1.7466$.

Table 3: *Example 1*: Ruin Simulation under $\widetilde{\mathbb{P}}$ with $X_0 = 0$

Time T	5	10	20	40	80	100	150	200	250	300	350	400
$\widetilde{P}\{\tau^* < T\}$	78.23%	87.23%	93.22%	97.01%	99.17%	99.44%	99.84%	99.94%	99.98%	99.99%	100.00%	100.00%
$\widetilde{\mathbb{E}}[\widetilde{\lambda}_{\tau^*}]$	1.1195	1.7354	2.5672	3.5823	4.6133	4.8776	5.2838	5.4970	5.4613	5.4577	5.6493	5.5630
$\widetilde{\mathbb{E}}[e^{-m_0^+ \widetilde{\lambda}_{\tau^*}}]$	0.8913	0.8813	0.8767	0.8723	0.8710	0.8709	0.8701	0.8696	0.8709	0.8698	0.8703	0.8699

Hence, $\widetilde{C} \approx 59.4328\%$ (defined by (23)), and the estimated Cramér constant $C = 1.1261 \times 59.4328\% = 66.93\%$, then,

$$P\left\{\tau^* < \infty \middle| X_0 = x\right\} \sim 66.93\% e^{-0.1247x}, \quad x \to \infty.$$

Here C is consistent with the result (round 68% in *Table 2*) obtained earlier by simulation under the original measure $\mathbb{P}$.

The comparison between the ruin probability $P_{\widetilde{\mathbb{P}}}\left\{\tau^* < \infty \middle| X_0\right\}$ (calculated by (27) of *Corollary 7.1*) simulated under $\widetilde{\mathbb{P}}$ and the ruin probability $P\left\{\tau^* < \infty \middle| X_0\right\}$ simulated directly under $\mathbb{P}$ is given by the first and third rows of *Table 2* and the first graph of *Figure 7*, and the results are very close.

Remark 7.4. To compare with the classical Poisson model, given $\mathbb{E}[\lambda_t] = 1.15$ for the dynamic contagion case in *Example 1*, we also set the constant intensity $\bar{\lambda} = 1.15$ for the Poisson model, and the corresponding Cramér constant can be then calculated exactly by

$$C = \frac{\bar{\lambda}}{c\gamma} = \frac{1.15}{1.5 \times 1} = 76.67\%.$$

7.3.2. Numerical Example 2

Similarly, we provide another numerical example by using a different set of parameters. The results are given by *Table 4*, *Table 5*, *Table 6* and *Figure 8*.

Remark 7.5. By comparing the simulation of ruin under the original measure $\mathbb{P}$ (given by *Figure 2*) and under the alternative measure $\widetilde{\mathbb{P}}$ (given by the first row of *Table 3* or *Table 6*), it becomes evident that the simulation is more efficient under $\widetilde{\mathbb{P}}$ as much more events of ruin are realised. For instance, for time $T = 100$ in *Table 6*, ruin is almost certain as $\widetilde{P}\{\tau^* < T\} \approx 1$.

Table 4: *Example 2*: Parameters under Measure $\mathbb{P}$ and $\widetilde{\mathbb{P}}$

	a	ρ	δ	α	β	γ	c	Stationarity	Net Profit
Measure $\mathbb{P}$	0.7	0.5	3	2	1.5	1	1.5	Yes	Yes
Measure $\widetilde{\mathbb{P}}$	1.0026	0.5388	3	1.2957	0.9467	0.7724	1.5	Yes	No
	η_0^+	v_0^+	m_0^+	$\widetilde{\mathbb{E}}[e^{m_0^+\widetilde{\lambda}}]$	$\widetilde{\mathbb{E}}[e^{-m_0^+\widetilde{\lambda}}\widetilde{\lambda}]$	$\widetilde{\mathbb{E}}[e^{-m_0^+\widetilde{\lambda}_{\tau^*_-}} \lvert \widetilde{\lambda}_0 = \widetilde{\lambda} \sim \Pi, X_0 = 0]$	$\widetilde{E}[\widetilde{\lambda}_t]$	$\widetilde{C}$	C
	0.1441	0.2276	0.1006	1.2019	1.3974	83.30%	1.7615	50.06%	60.17%

Table 5: *Example 2*: Numerical Results

X_0	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
$P\left\{\tau^* < \infty \middle\vert X_0\right\}$	68.65%	41.52%	25.72%	16.12%	10.16%	6.42%	4.05%	2.58%	1.64%	1.03%	0.65%	0.41%	0.26%	0.17%	0.11%	0.07%
C Estimation	68.65%	65.45%	63.93%	63.17%	62.78%	62.50%	62.23%	62.43%	62.42%	62.14%	61.95%	62.02%	62.28%	63.23%	62.74%	62.44%
$P_{\widetilde{\mathbb{P}}}\left\{\tau^* < \infty \middle\vert X_0\right\}$	69.89%	41.79%	25.76%	16.09%	10.13%	6.40%	4.05%	2.56%	1.62%	1.03%	0.65%	0.41%	0.26%	0.17%	0.11%	0.07%
$\ln P\left\{\tau^* < \infty \middle\vert X_0\right\}$	-0.376	-0.879	-1.358	-1.825	-2.286	-2.746	-3.206	-3.658	-4.113	-4.573	-5.031	-5.485	-5.936	-6.376	-6.839	-7.299
$\ln(60.17\%) - v_0 X_0$	-0.508	-0.963	-1.418	-1.874	-2.329	-2.784	-3.239	-3.694	-4.150	-4.605	-5.060	-5.515	-5.970	-6.426	-6.881	-7.336

Table 6: *Example 2*: Ruin Simulation under $\widetilde{\mathbb{P}}$ with $X_0 = 0$

Time T	5	10	20	40	80	100	150	200	250	300	350	400
$\widetilde{P}\{\tau^* < T\}$	85.23%	92.94%	97.50%	99.36%	99.93%	99.97%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%
$\widetilde{\mathbb{E}}[\widetilde{\lambda}_{\tau^*}]$	1.0488	1.5517	2.1127	2.6248	2.9161	2.9408	3.0033	2.9541	3.0213	3.0096	2.9674	3.0206
$\widetilde{\mathbb{E}}[e^{-m\widetilde{\lambda}_{\tau^*}}]$	0.8455	0.8396	0.8357	0.8335	0.8332	0.8329	0.8327	0.8330	0.8329	0.8331	0.8331	0.8330

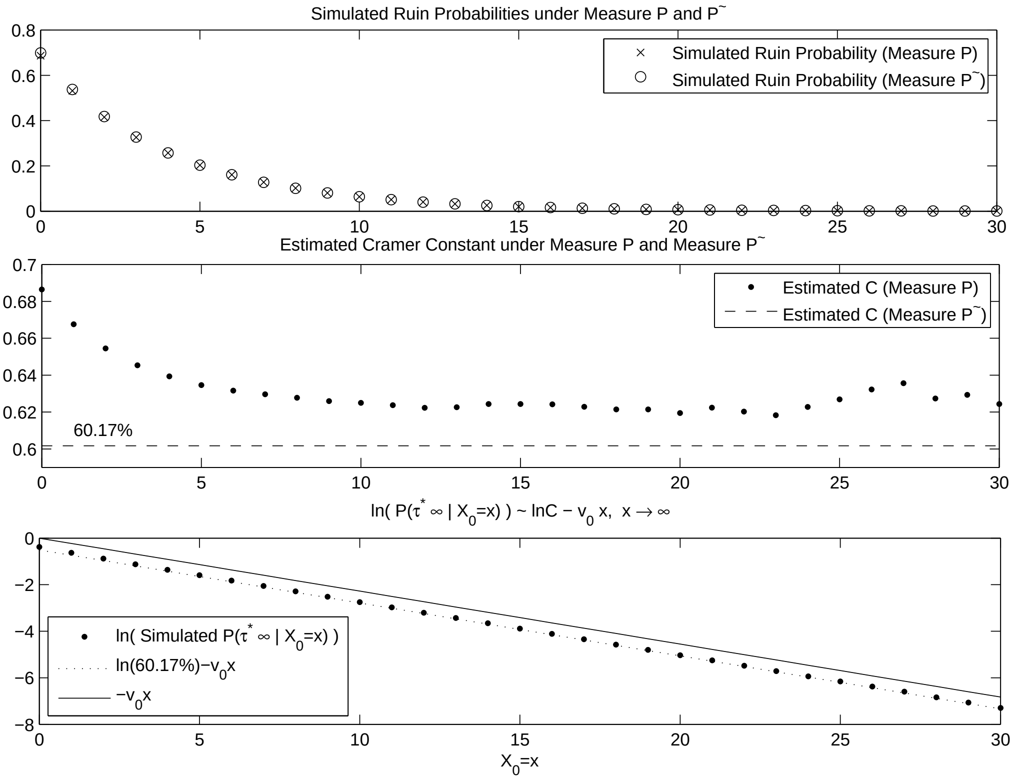

Figure 8: *Example 2*: Simulated Ruin Probabilities and Estimation for the Cramér Constant C under Measure $\mathbb{P}$ and $\widetilde{\mathbb{P}}$

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