Semi-Markov Model for Excursions and Occupation time of Markov Processes

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Abstract

In this paper, we study the excursion time and occupation time of a Markov process below or above a given level by using a simple two states semi-Markov model. In mathematical finance, these results have an important application in the valuation of path dependent options such as Parisian options and cumulative Parisian options. We introduce a new type of Parisian option, single barrier two-sided Parisian option and extend the concept of a ruin probability in ruin theory to a Parisian type of ruin probability.

Keywords: excursion time, occupation time, semi-Markov model, path dependent options, Parisian options, cumulative Parisian options, ruin probability.

1 Introduction

The concept of Parisian options was first introduced by Chesney, Jeanblanc-Picque and Yor [6]. A Parisian option is a special case of path dependent option. Its payoff does not only depend on the final price of the underlying asset, but also its price trajectory during the whole life span of the option. More precisely, a Parisian option will be either initiated or exterminated upon reaching a predetermined barrier level L and staying above or below the level for a predetermined time D before the maturity date T.

There are two different ways of measuring the time spent above or below the barrier corresponding to the excursion time and the occupation time defined below. The excursion time below (above) the barrier starts counting from 0 each time the process crosses the barrier from above (below) and stops counting when the process crosses the barrier from below (above). The occupation time up to a specific time t adds up all the time the process spend below (above) the barrier; it is therefore the summation of all excursion time intervals before time t. In [6] the Parisian options related to the occupation time are called cumulative Parisian options. Therefore the owner of a *Parisian Down-and-out option* loses the option if the underlying asset price S reaches the level L and remains constantly below this level for a time interval longer than D. For a *Parisian Down-and-in option* the same event gives the owner the right to exercise the option. The owner of a *cumulative Parisian Down-and-out option* loses the option if the total time the underlying asset price S stays below L up to the end of the contract for longer than D. For details on the pricing of Parisian options see [6], [11], [13] and [10]. For cumulative Parisian options see [6] and since these are related to the occupation time and hence the quantiles of the process, also see [1], [8] and [12].

From the description above, we can see that the key for pricing a Parisian option (a cumulative Parisian option) is the derivation of the distribution of the excursion time (the occupation time). As in [6], we reduce the problem to finding the Laplace transform of the first time the length of the excursion reaches level D. In [6] this was obtained by using the Brownian meander and the Azema martingale (see [2]). A restriction of this technique is that it relies heavily on the properties of Brownian motion; therefore the result cannot be extended to other processes easily. It is also hard to see how it can be used for the pricing of slightly more complicated options that we will introduce.

In this paper, we are going to study the excursion and occupation times in a more general framework using a simple semi-Markov model consisting of two states indicating whether the process is above or below a fixed level L . By applying the model to a Brownian motion, we can get the Laplace transform which is used in pricing Parisian options defined in [6]. One can then invert using techniques as in [11].

Furthermore, we introduce a new type of Parisian option, named *single barrier two-sided Parisian options.* In contrast to the Parisian options mentioned above, we consider the excursions both below and above the barrier. For example, the owner of this type of *Parisian Out option* loses the option if the underlying asset process S has either an excursion above the barrier for longer than d_1 or below barrier for longer than d_2 before the maturity of the option. And the owner of a *Parisian In option* gains the right to exercise the option if the same event happens. Later on, we will give the Laplace transform of the first time this event happens which can be used to price this type of options.

We also obtain a result which we call *Parisian type ruin probability* for a Brownian motion with a positive drift. For a stochastic process S, we define $T_0 = \inf\{t > 0 \,|\, S_t < 0\}$. Suppose that S is a Brownian motion with a positive drift, which can be used as an approximation to the surplus of an insurer (see for example [9] Chapter 1). In risk theory, the ruin probability is defined as $P(T_0 < \infty)$, i.e. the probability that the event of ruin, $\{\exists t > 0 \,|\, S_t < 0\}$, happens. We extend the concept of ruin to a Parisian type of ruin, which refers to the event that S falls below 0 and stays below 0 constantly for at least a time D. The Parisian type ruin probability is the probability that this event ever happens. From a regulatory point of view this might be a more reliable measure of insolvency than a very short lived cash flow problem.

In §2 we give the mathematical definitions and model setting. In §3 we present an important lemma together with its proof, which will be used in the following sections. We will give our main results applied to Brownian motion in §4, as well as introduce our newly-defined Parisian options and Parisian type ruin probabilities.

2 Definitions

We are going to use the same definition for the excursion as in [6] and [7]. Let S be a stochastic process and L be the level of the barrier. As in [6], we define

$$g^S_{L,t} = \sup\{s \le t \mid S_s = L\}, \qquad d^S_{L,t} = \inf\{s \ge t \mid S_s = L\} \tag{1}$$

with the usual convention, $\sup\{\emptyset\} = 0$ and $\inf\{\emptyset\} = \infty$. The trajectory between $g^S_{L,t}$ and $d^S_{L,t}$ is the excursion of process S which straddles time t. Assuming $d_1 > 0$, $d_2 > 0$, we now define

$$\tau^S_1 = \inf\{t > 0 \mid \mathbf{1}_{\{S_t > L\}}(t - g^S_{L,t}) \ge d_1\}, \tag{2}$$

$$\tau^S_2 = \inf\{t > 0 \mid \mathbf{1}_{\{S_t < L\}}(t - g^S_{L,t}) \ge d_2\}, \tag{3}$$

$$\tau^S = \tau^S_1 \wedge \tau^S_2. \tag{4}$$

τ^S_1 is therefore the first time that the length of the excursion of process S above the barrier L reaches given level d_1; τ^S_2 corresponds to the one below the barrier L; and τ^S is the smaller of τ^S_1 and τ^S_2.

Assume r is the risk-free rate, T is the term of the option, S_t is the price of its underlying asset, K is the strike price. If we have an up-out Parisian call option with barrier L, its price can be expressed as:

$$P = e^{-rT} E\left(\mathbf{1}_{\{\tau^S_1 > T\}} (S_T - K)^+\right);$$

and the price of a down-in Parisian put option with barrier L is:

$$P = e^{-rT} E\left(\mathbf{1}_{\{\tau^S_2 < T\}} (K - S_T)^+\right).$$

Furthermore, we define

$$L^S_{1,t} = \int_0^t \mathbf{1}_{\{S_u > L\}} \mathrm{d}u, \qquad L^S_{2,t} = \int_0^t \mathbf{1}_{\{S_u < L\}} \mathrm{d}u.$$

$L^S_{1,t}$ is the total time that the process spends above level L up to time t, i.e. the occupation time above level L by time t; and $L^S_{2,t}$ corresponds to the one below L.

From the description above, it is clear that we are actually considering two states, the state when the process is above the barrier and the state when it is below. For each state, we are interested in the time the process spends in it. Based on this point of view, we introduce the semi-Markov model.

First of all, we define

$$Z_t^S = \begin{cases} 1, & \text{if } S_t > L \\ 2, & \text{if } S_t < L \end{cases} .$$

We can now express the variables defined above in terms of Z_t:

$$g_{L,t}^S = \sup\{s \le t \mid Z_s^S \neq Z_t\}, \tag{5}$$

$$d_{L,t}^S = \inf\{s \ge t \mid Z_s^S \neq Z_t\}, \tag{6}$$

$$\tau_1^S = \inf\{t > 0 \mid \mathbf{1}_{\{Z_t^S=1\}}(t - g_{L,t}^S) \ge d_1\}, \tag{7}$$

$$\tau_2^S = \inf\{t > 0 \mid \mathbf{1}_{\{Z_t^S=2\}}(t - g_{L,t}^S) \ge d_2\}, \tag{8}$$

$$L_{1,t}^S = \int_0^t \mathbf{1}_{\{Z_u^S=1\}} \mathrm{d}u, \tag{9}$$

$$L_{2,t}^S = \int_0^t \mathbf{1}_{\{Z_u^S=2\}} \mathrm{d}u. \tag{10}$$

We then define

$$V_t^S = t - g_{L,t}^S,$$

the time Z_t^S has spent in the current state. It is easy to prove that (Z_t^S, V_t^S) is a Markov process. Z_t^S is therefore a semi-Markov process with the state space $\{1, 2\}$, where 1 stands for the state when the stochastic process S is above the barrier and 2 corresponds to the state below the barrier.

Furthermore, we set $U_{i,k}^S$, $i = 1, 2$ and $k = 1, 2, \cdots$ to be the time Z^S spends in state i when it visits i for the kth time. And we have, for each given i and k,

$$U_{i,k}^S = V_{d_{L,t}^S}^S = d_{L,t}^S - g_{L,t}^S, \quad \text{for some } t.$$

Notice that given i, $U_{i,k}^S$, $k = 1, 2, \cdots$, are i.i.d. We therefore define the transition density for Z^S:

$$p_{ij}(t) = \lim_{\Delta t \to 0} P(t < U_{i,k}^S < t + \Delta t),$$

$$P_{ij}(t) = P(U_{i,k}^S < t), \quad \bar{P}_{ij}(t) = P(U_{i,k}^S > t).$$

We have

$$P_{ij}(t) = \int_0^t p_{ij}(s)\mathrm{d}s = 1 - \bar{P}_{ij}(t),$$

which is actually the probability that the process will stay in state i for no more than time t. Notice that for some stochastic processes S and certain k, we have $P(U_{i,k}^S = \infty) > 0$ (we adopt the convention $U_{i,k}^S = \infty$ if the process never leaves state i at its kth excursion); therefore $\int_0^{+\infty} p_{ij}(s)\mathrm{d}s < 1$, i.e. with a positive probability, the process will stay in state i forever. Hence, it is not necessary the case that $\bar{P}_{ij}(t) = \int_t^{+\infty} p_{ij}(s)\mathrm{d}s$.

Moreover, in the definition of Z^S, we deliberately ignore the situation when $S_t = L$. The reason is that we only consider the processes, which

$$\int_0^t \mathbf{1}_{\{S_u=L\}}\mathrm{d}u = 0.$$

Also, when L is the regular point of the process (see [5] for definition), we have to deal with the degeneration of p_{ij}. Take Brownian Motion as an example. Assume $W_t^\mu = \mu t + W_t$ with $\mu \geq 0$, where W_t is a standard Brownian Motion. Setting x_0 to be its starting point, we know its density for the first hitting time of level L is

$$p_{x_0} = \frac{|L - x_0|}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(L - x_0 - \mu t)^2}{2t}\right\}$$

(see [4]). According to the definition of the transition density, $p_{12}(t) = p_{21}(t) = p_L(t) = 0$. Obviously, when $\mu = 0$ the same happens to the standard Brownian Motion W_t.

Without loss of generality, from now on, we assume $L = 0$. In order to solve the above problem, we introduce a new processe $X_t^{(\epsilon)}$, $\epsilon > 0$ as follow. Assume $W_0^\mu = \epsilon$. Define a sequence of stopping times

$$\begin{aligned}
\delta_0 &= 0, \\
\sigma_n &= \inf\{t > \delta_n \,|\, X_t^{(\epsilon)} = 0\}, \\
\delta_{n+1} &= \inf\{t > \sigma_n \,|\, X_t^{(\epsilon)} = \epsilon\},
\end{aligned}$$

where $n = 0, 1, \cdots$. The new process is given by

$$\begin{cases} X_t^{(\epsilon)} = W_t^\mu & \text{if} \quad \delta_n \leq t < \sigma_n \\ X_t^{(\epsilon)} = W_t^\mu - \epsilon & \text{if} \quad \sigma_n \leq t < \delta_{n+1} \end{cases}.$$

It is actually a process which starts from ϵ and has the same behavior as the related Brownian Motion expect that each time when it hits the barrier 0, it will have a jump towards the opposite side of the barrier with size ϵ (see Figure 1). Its excursions above L and below L alternate.

From the definition, it is clear that 0 becomes an irregular point for $X_t^{(\epsilon)}$, which converges to W_t^μ with $W_0^\mu = 0$ almost surely for all t. We prove in the appendix that the Laplace transforms of the variables defined based on $X_t^{(\epsilon)}$ converge to those based on W_t^μ. As a result, we can obtain the results for the Brownian Motion by carrying out the calculation for $X_t^{(\epsilon)}$ and take the limit $\epsilon \to 0$.

For $X_t^{(\epsilon)}$, we can define the Z^X, τ_1^X, τ_2^X, τ^X $L_{1,t}^X$ and $L_{2,t}^X$ as in (5)-(10). For Z^X, we have

$$\begin{aligned}
p_{12}(t) &= \frac{\epsilon}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(\epsilon + \mu t)^2}{2t}\right\}, \\
p_{21}(t) &= \frac{\epsilon}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(\epsilon - \mu t)^2}{2t}\right\}.
\end{aligned}$$

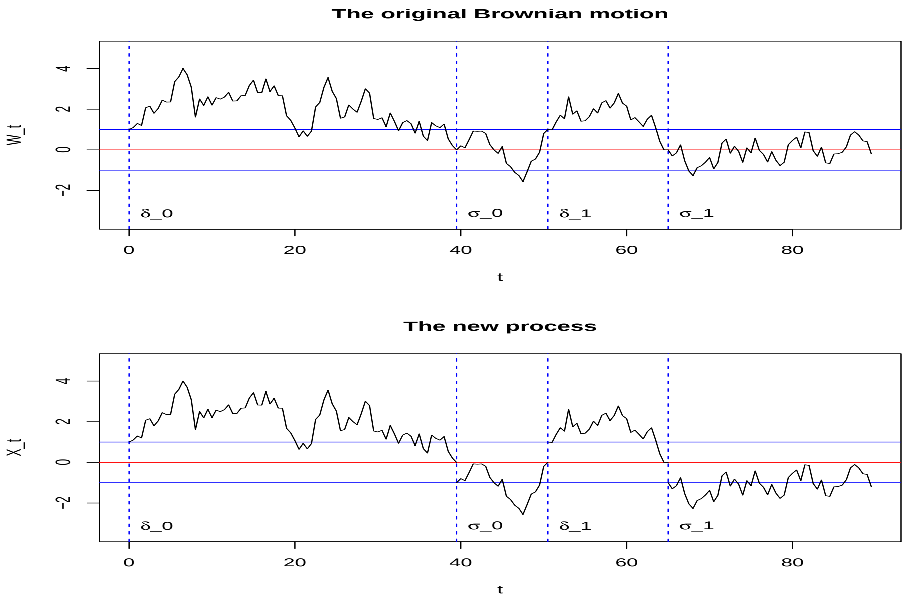

Figure 1: A sample path of $X_t^{(\epsilon)}$

Clearly, all the arguments above apply to the standard Brownian motion, which is a special case of W_t^{μ} when $\mu = 0$.

3 An Important Lemma

In this section, we will present an important lemma regarding to the excursion and occupation times, together with its proof.

Lemma 1 *For $X_t^{(\epsilon)}$, the joint Laplace transform for the occupation time above and below 0 up to τ^X is given by*

$$E\left(e^{-\alpha L_{1,\tau^X}^X - \beta L_{2,\tau^X}^X}\right) = \frac{e^{-\alpha d_1} \bar{P}_{12}(d_1) + e^{-\beta d_2} \bar{P}_{21}(d_2) \int_0^{d_1} e^{-\beta u} p_{12}(u) \mathrm{d}u}{1 - \int_0^{d_1} e^{-\alpha s} p_{12}(s) \mathrm{d}s \int_0^{d_2} e^{-\beta s} p_{21}(s) \mathrm{d}s}. \quad (11)$$

Proof: We are going to consider the case when $\tau_1^S < \tau_2^S$ and the case when $\tau_1^S > \tau_2^S$ respectively. We have

$$E\left(e^{-\alpha L_{1,\tau^S}^S - \beta L_{2,\tau^S}^S}\right) = E\left(e^{-\alpha L_{1,\tau^S}^S - \beta L_{2,\tau^S}^S} \mathbf{1}_{\{\tau_1^S < \tau_2^S\}}\right) + E\left(e^{-\alpha L_{1,\tau^S}^S - \beta L_{2,\tau^S}^S} \mathbf{1}_{\{\tau_1^S > \tau_2^S\}}\right).$$

A_k^i denotes the event that the first time the length of the excursion in state i reaches d_i happens during the kth excursion in this state, and it happens before the length of the excursion in other states reaches the required levels, i.e.

$$\left\{A_k^1\right\} = \left\{\tau_1^S < \tau_2^S, \quad \tau_1^S \text{ is achieved in the } k\text{th excursion in state } 1\right\},$$

$$\left\{A_k^2\right\} = \left\{\tau_1^S > \tau_2^S, \quad \tau_2^S \text{ is achieved in the } k\text{th excursion in state } 2\right\}.$$

So we have, for example,

$$\begin{aligned} E\left(e^{-\alpha L_{1,\tau^S}^S - \beta L_{2,\tau^S}^S} \mathbf{1}_{\{\tau_1^S < \tau_2^S\}}\right) &= \sum_{k=1}^{\infty} E\left(e^{-\alpha L_{1,\tau^S}^S - \beta L_{2,\tau^S}^S} \middle| A_k^1\right) P\left(A_k^1\right) \\ &= \sum_{k=1}^{\infty} E\left(e^{-\alpha L_{1,\tau_1^S}^S - \beta L_{2,\tau_1^S}^S} \middle| A_k^1\right) P\left(A_k^1\right). \end{aligned}$$

Notice that given A_k^1, $L_{1,\tau_1^S}^S$ is comprised of $k-1$ full excursions above barrier L with the length less than d_1 and last one with the length d_1; and $L_{2,\tau_1^S}^S$ is comprised of k full excursions below L with the length less than d_2, i.e.

$$L_{1,\tau_1^S}^S | A_k^1 = U_{1,1}^S + U_{1,2}^S + \cdots + U_{1,k-1}^S + d_1, \quad U_{1,1}^S < d_1, \cdots, U_{1,k-1}^S < d_1;$$

$$L_{2,\tau_1^S}^S | A_k^1 = U_{2,1}^S + U_{2,2}^S + \cdots + U_{2,k-1}^S, \quad U_{2,1}^S < d_2, \cdots, U_{2,k-1}^S < d_2.$$

More importantly, $U_{1,n}^S$'s have distribution P_{12}; $U_{2,n}^S$'s have distribution P_{21} and all these variables are independent of each other. As a result,

$$\begin{aligned} & E\left(e^{-\alpha L_{1,\tau_1^S}^S - \beta L_{2,\tau_1^S}^S} \middle| A_k^1\right) \\ = \; & E\left(e^{-\alpha\left(\sum_{n=1}^{k-1} U_{1,n}^S + d_1\right) - \beta \sum_{n=1}^{k-1} U_{2,n}^S} \middle| U_{1,1}^S < d_1, \cdots, U_{1,k-1}^S < d_1, U_{2,1}^S < d_2, \cdots, U_{2,k-1}^S < d_2\right) \\ = \; & e^{-\alpha d_1} \left\{\int_0^{d_1} e^{-\alpha u} \frac{p_{12}(u)}{P_{12}(d_1)} \mathrm{d}u\right\}^{k-1} \left\{\int_0^{d_2} e^{-\beta u} \frac{p_{21}(u)}{P_{21}(d_2)} \mathrm{d}u\right\}^{k-1}. \end{aligned}$$

Also

$$P(A_k^1) = P_{12}(d_1)^{k-1} P_{21}(d_2)^{k-1} \bar{P}_{12}(d_1).$$

We have therefore

$$\begin{aligned} & E\left(e^{-\alpha L_{1,\tau^S}^S - \beta L_{2,\tau^S}^S} \mathbf{1}_{\{\tau_1^S < \tau_2^S\}}\right) \\ = \; & \sum_{k=1}^{\infty} e^{-\alpha d_1} \left\{\int_0^{d_1} e^{-\alpha u} \frac{p_{12}(u)}{P_{12}(d_1)} \mathrm{d}u\right\}^{k-1} \left\{\int_0^{d_2} e^{-\beta u} \frac{p_{21}(u)}{P_{21}(d_2)} \mathrm{d}u\right\}^{k-1} P_{12}(d_1)^{k-1} P_{21}(d_2)^{k-1} \bar{P}_{12}(d_1) \\ = \; & \frac{e^{-\alpha d_1} \bar{P}_{12}(d_1)}{1 - \int_0^{d_1} e^{-\alpha s} p_{12}(s) \mathrm{d}s \int_0^{d_2} e^{-\beta s} p_{21}(s) \mathrm{d}s}. \end{aligned}$$

Similarly, we have for the case when $\tau_1^S > \tau_2^S$

$$
\begin{aligned}
& E\left(e^{-\alpha L^S_{1,\tau^S}-\beta L^S_{2,\tau^S}}\mathbf{1}_{\{\tau_1^S>\tau_2^S\}}\right) \\
= & \sum_{k=1}^{\infty} E\left(e^{-\alpha L^S_{1,\tau^S}-\beta L^S_{2,\tau^S}}\Big| A_k^2\right) P\left(A_k^2\right) \\
= & \sum_{k=1}^{\infty} E\left(e^{-\alpha L^S_{1,\tau_2^S}-\beta L^S_{2,\tau_2^S}}\Bigg| A_k^2\right) P\left(A_k^2\right) \\
= & \sum_{k=1}^{\infty} e^{-\beta d_2}\left\{\int_0^{d_1} e^{-\alpha u}\frac{p_{12}(u)}{P_{12}(d_1)}\mathrm{d}u\right\}^k\left\{\int_0^{d_2} e^{-\beta u}\frac{p_{21}(u)}{P_{21}(d_2)}\mathrm{d}u\right\}^{k-1} P_{12}(d_1)^{k-1}P_{21}(d_2)^{k-1}\bar{P}_{21}(d_2) \\
= & \frac{e^{-\beta d_2}\bar{P}_{21}(d_2)\int_0^{d_1} e^{-\alpha u}p_{12}(u)\mathrm{d}u}{1-\int_0^{d_1} e^{-\alpha s}p_{12}(s)\mathrm{d}s\int_0^{d_2} e^{-\beta s}p_{21}(s)\mathrm{d}s}.
\end{aligned}
$$

Therefore

$$
E\left(e^{-\alpha L^S_{1,\tau^S}-\beta L^S_{2,\tau^S}}\right) = \frac{e^{-\alpha d_1}\bar{P}_{12}(d_1)+e^{-\beta d_2}\bar{P}_{21}(d_2)\int_0^{d_1} e^{-\beta u}p_{12}(u)\mathrm{d}u}{1-\int_0^{d_1} e^{-\alpha s}p_{12}(s)\mathrm{d}s\int_0^{d_2} e^{-\beta s}p_{21}(s)\mathrm{d}s}.
$$

□

From lemma 1 we can derive several interesting results. We show some of them here.

Firstly, noticed that $L^X_{1,\tau^X}+L^X_{2,\tau^X}=\tau^X$, we can easily obtain the Laplace transform of τ^X:

$$
E\left(e^{-\beta\tau^X}\right) = \frac{e^{-\beta d_1}\bar{P}_{12}(d_1)+e^{-\beta d_2}\bar{P}_{21}(d_2)\int_0^{d_1} e^{-\beta s}p_{12}(s)\mathrm{d}s}{1-\int_0^{d_1} e^{-\beta s}p_{12}(s)\mathrm{d}s\int_0^{d_2} e^{-\beta s}p_{21}(s)\mathrm{d}s}. \tag{12}
$$

If we are only interested in the excursion at one side of the barrier, we can set the required length of excursion at another side to be $+\infty$. For example, when $d_1\to+\infty$, $\tau_1^S\to+\infty$ according to its definition. We have therefore

$$
\tau^S=\tau_1^S\wedge\tau_2^S\to\tau_2^S,
$$

and hence

$$
L^S_{i,\tau^S}\to L^S_{i,\tau_2^S},\quad i=1,2.
$$

We also know that

$$
\left|e^{-\alpha L^S_{1,\tau^S}-\beta L^S_{2,\tau^S}}\right|<1,\quad \text{for any } \alpha\geq0,\beta\geq0.
$$

By the dominated convergence theorem, we have

$$
E\left(e^{-\alpha L^S_{1,\tau_1^S}-\beta L^S_{2,\tau_1^S}}\right)=E\left(\lim_{d_1\to\infty} e^{-\alpha L^S_{1,\tau^S}-\beta L^S_{2,\tau^S}}\right)=\lim_{d_1\to\infty} E\left(e^{-\alpha L^S_{1,\tau^S}-\beta L^S_{2,\tau^S}}\right).
$$

As a result, we can deduce that

$$E\left(e^{-\alpha L^X_{1,\tau_2^X}-\beta L^X_{2,\tau_2^X}}\right)=\frac{e^{-\beta d_2}\bar{P}_{21}(d_2)\int_0^{+\infty}e^{-\alpha s}p_{12}(s)\mathrm{d}s}{1-\int_0^{+\infty}e^{-\alpha s}p_{12}(s)\mathrm{d}s\int_0^{d_2}e^{-\beta s}p_{21}(s)\mathrm{d}s}. \tag{13}$$

Also, by using $L^X_{1,\tau_2^X}+L^X_{2,\tau_2^X}=\tau_2^X$, we get

$$E\left(e^{-\beta\tau_2^X}\right)=\frac{e^{-\beta d_2}\bar{P}_{21}(d_2)\int_0^{+\infty}e^{-\beta s}p_{12}(s)\mathrm{d}s}{1-\int_0^{+\infty}e^{-\beta s}p_{12}(s)\mathrm{d}s\int_0^{d_2}e^{-\beta s}p_{21}(s)\mathrm{d}s}. \tag{14}$$

We can also derive the results for the other side by setting $d_2\to+\infty$.

4 Examples

In this section, we are going to apply the results to the Brownian Motion $W_t^\mu=\mu t+W_t$, with $W_0^\mu=0$ and $\mu\geq 0$.

In §**??** we have stated that the main difficulty with the Brownian Motion is that its origin point is regular, i.e. the probability that W_t^μ will return to the origin at arbitrarily small time is 1. We have therefore introduced the new processes $(X_t^{(\epsilon)},Z_t^X)$, with transition densities for Z_t^X

$$p_{12}(t)=\frac{\epsilon}{\sqrt{2\pi t^3}}\exp\left\{-\frac{(\epsilon+\mu t)^2}{2t}\right\},\quad p_{21}(t)=\frac{\epsilon}{\sqrt{2\pi t^3}}\exp\left\{-\frac{(\epsilon-\mu t)^2}{2t}\right\}.$$

In order to simplify expressions, we define

$$\Psi(x)=2\sqrt{\pi x}\mathscr{N}\left(\sqrt{2x}\right)-\sqrt{\pi x}+e^{-x},$$

where $\mathscr{N}(.)$ is the cumulative distribution function for a standard Normal Distribution.

From lemma 1 we have

$$E\left(\exp\left\{-\alpha L^X_{1,\tau^X}-\beta L^X_{2,\tau^X}\right\}\right)=\frac{e^{-\alpha d_1}\bar{P}_{12}(d_1)+e^{-\beta d_2}\bar{P}_{21}(d_2)\int_0^{d_1}e^{-\beta u}p_{12}(u)\mathrm{d}u}{1-\int_0^{d_1}e^{-\alpha s}p_{12}(s)\mathrm{d}s\int_0^{d_2}e^{-\beta s}p_{21}(s)\mathrm{d}s}.$$

Since $X_t^{(\epsilon)}\to W_t^\mu$ a.s., when we calculate the limit as $\epsilon\to 0$,

$$E\left(\exp\left\{-\alpha L^X_{1,\tau^X}-\beta L^X_{2,\tau^X}\right\}\right)\to E\left(e^{-\alpha L^{W^\mu}_{1,\tau^{W^\mu}}-\beta L^{W^\mu}_{2,\tau^{W^\mu}}}\right)$$

(see the appendix). As a result, for a Brownian Motion with drift we have

$$\begin{aligned}&E\left(e^{-\alpha L^{W^\mu}_{1,\tau^{W^\mu}}-\beta L^{W^\mu}_{2,\tau^{W^\mu}}}\right) \qquad (15)\\ =\;&\frac{e^{-\alpha d_1}\left\{\sqrt{d_2}\Psi\left(\frac{\mu^2 d_1}{2}\right)+\mu\sqrt{\frac{d_1 d_2\pi}{2}}\right\}+e^{-\beta d_2}\left\{\sqrt{d_1}\Psi\left(\frac{\mu^2 d_2}{2}\right)-\mu\sqrt{\frac{d_1 d_2\pi}{2}}\right\}}{\sqrt{d_2}\Psi\left\{\frac{(2\alpha+\mu^2)d_1}{2}\right\}+\sqrt{d_1}\Psi\left\{\frac{(2\beta+\mu^2)d_2}{2}\right\}}.\end{aligned}$$

If we set $\mu = 0$, we can get the result for the standard Brownian motion

$$E\left(e^{-\alpha L^W_{1,\tau^W} - \beta L^W_{2,\tau^W}}\right) = \frac{\sqrt{d_2}e^{-\alpha d_1} + \sqrt{d_1}e^{-\beta d_2}}{\sqrt{d_2}\Psi(\alpha d_1) + \sqrt{d_1}\Psi(\beta d_2)}. \tag{16}$$

By setting $\alpha = \beta$ in (15) and (16), we have our first important result.

Theorem 1 *For a Brownian Motion W^μ_t with $W^\mu_0 = 0$, $\tau_1^{W^\mu}$, $\tau_2^{W^\mu}$ and τ^{W^μ} have been defined in §**??** (2), (3) and (4) with $S_t = W^\mu_t$. We then have*

$$E\left(e^{-\beta\tau^{W^\mu}} \mathbf{1}_{\{\tau_1^{W^\mu} < \tau_2^{W^\mu}\}}\right) = \frac{e^{-\beta d_1}\left\{\sqrt{d_2}\Psi\left(\frac{\mu^2 d_1}{2}\right) + \mu\sqrt{\frac{d_1 d_2 \pi}{2}}\right\}}{\sqrt{d_2}\Psi\left\{\frac{(2\beta+\mu^2)d_1}{2}\right\} + \sqrt{d_1}\Psi\left\{\frac{(2\beta+\mu^2)d_2}{2}\right\}}, \tag{17}$$

$$E\left(e^{-\beta\tau^{W^\mu}} \mathbf{1}_{\{\tau_1^{W^\mu} > \tau_2^{W^\mu}\}}\right) = \frac{e^{-\beta d_2}\left\{\sqrt{d_1}\Psi\left(\frac{\mu^2 d_2}{2}\right) - \mu\sqrt{\frac{d_1 d_2 \pi}{2}}\right\}}{\sqrt{d_2}\Psi\left\{\frac{(2\beta+\mu^2)d_1}{2}\right\} + \sqrt{d_1}\Psi\left\{\frac{(2\beta+\mu^2)d_2}{2}\right\}}. \tag{18}$$

$$E\left(e^{-\beta\tau^{W^\mu}}\right) = \frac{e^{-\beta d_1}\left\{\sqrt{d_2}\Psi\left(\frac{\mu^2 d_1}{2}\right) + \mu\sqrt{\frac{d_1 d_2 \pi}{2}}\right\} + e^{-\beta d_2}\left\{\sqrt{d_1}\Psi\left(\frac{\mu^2 d_2}{2}\right) - \mu\sqrt{\frac{d_1 d_2 \pi}{2}}\right\}}{\sqrt{d_2}\Psi\left\{\frac{(2\beta+\mu^2)d_1}{2}\right\} + \sqrt{d_1}\Psi\left\{\frac{(2\beta+\mu^2)d_2}{2}\right\}}. \tag{19}$$

For a standard Brownian Motion, the special case when $\mu = 0$, we have

$$E\left(e^{-\beta\tau^{W}} \mathbf{1}_{\{\tau_1^{W} < \tau_2^{W}\}}\right) = \frac{\sqrt{d_2}e^{-\beta d_1}}{\sqrt{d_2}\Psi(\beta d_1) + \sqrt{d_1}\Psi(\beta d_2)}, \tag{20}$$

$$E\left(e^{-\beta\tau^{W}} \mathbf{1}_{\{\tau_1^{W} > \tau_2^{W}\}}\right) = \frac{\sqrt{d_1}e^{-\beta d_2}}{\sqrt{d_2}\Psi(\beta d_1) + \sqrt{d_1}\Psi(\beta d_2)}, \tag{21}$$

$$E\left(e^{-\beta\tau^{W}}\right) = \frac{\sqrt{d_2}e^{-\beta d_1} + \sqrt{d_1}e^{-\beta d_2}}{\sqrt{d_2}\Psi(\beta d_1) + \sqrt{d_1}\Psi(\beta d_2)}. \tag{22}$$

According to the definition, τ^{W^μ} is the first time of either the length of the excursion above 0 reaches d_1 or the length of the excursion below 0 reaches d_2. The results in theorem 1 are what we need to price our newly-defined *single-barrier two-sided Parisian options.*

Letting $\beta \to 0$, we have the following remarkable results.

Corollary 1.1 *The probability that W^μ_t achieves an excursion above 0 with the length as least d_1 before it achieves an excursion below 0 with the length at least d_2 is*

$$P\left(\tau_1^{W^\mu} < \tau_2^{W^\mu}\right) = \frac{\sqrt{d_2}\Psi\left(\frac{\mu^2 d_1}{2}\right) + \mu\sqrt{\frac{d_1 d_2 \pi}{2}}}{\sqrt{d_2}\Psi\left(\frac{\mu^2 d_1}{2}\right) + \sqrt{d_1}\Psi\left(\frac{\mu^2 d_2}{2}\right)}; \tag{23}$$

And the probability of its opposite event is given by

$$P\left(\tau_1^{W^\mu} > \tau_2^{W^\mu}\right) = \frac{\sqrt{d_1}\Psi\left(\frac{\mu^2 d_2}{2}\right) - \mu\sqrt{\frac{d_1 d_2 \pi}{2}}}{\sqrt{d_2}\Psi\left(\frac{\mu^2 d_1}{2}\right) + \sqrt{d_1}\Psi\left(\frac{\mu^2 d_2}{2}\right)}. \tag{24}$$

Similarly, for a standard Brownian motion

$$P\left(\tau_1^W < \tau_2^W\right) = \frac{\sqrt{d_2}}{\sqrt{d_1} + \sqrt{d_2}}, \tag{25}$$

$$P\left(\tau_1^W > \tau_2^W\right) = \frac{\sqrt{d_1}}{\sqrt{d_1} + \sqrt{d_2}}. \tag{26}$$

Remark 1: If we set $d_1 = d_2 = d$ in (23), (24), (25) and (26), we have for a standard Brownian motion

$$P\left(\tau_1^W < \tau_2^W\right) = P\left(\tau_1^W > \tau_2^W\right) = \frac{1}{2},$$

which can be well explained by the symmetry of the standard Brownian motion;

Remark 2: For a Brownian motion with positive drift,

$$P\left(\tau_1^{W^\mu} < \tau_2^{W^\mu}\right) = \frac{1}{2} + \frac{\mu\sqrt{\frac{d\pi}{2}}}{\Psi\left(\frac{\mu^2 d}{2}\right)} > \frac{1}{2}, \quad P\left(\tau_1^{W^\mu} > \tau_2^{W^\mu}\right) = \frac{1}{2} - \frac{\mu\sqrt{\frac{d\pi}{2}}}{\Psi\left(\frac{\mu^2 d}{2}\right)} < \frac{1}{2},$$

because it has a tendency to move upwards.

Moreover, by setting $d_1 = d_2 = d$, we can get the result for a reflected Brownian motion $R_t^\mu = |W_t^\mu|$.

Theorem 2 *For a reflected Brownian Motion $R_t^\mu = |W_t^\mu|$ with $R_0^\mu = 0$, We have*

$$E\left(e^{-\beta\tau^{R^\mu}}\right) = \frac{e^{-\beta d}\left\{\Psi\left(\frac{\mu^2 d}{2}\right) + \mu\sqrt{\frac{d\pi}{2}}\right\}}{\Psi\left\{\frac{(2\beta+\mu^2)d}{2}\right\} - \mu\sqrt{\frac{\pi d}{2}}}. \tag{27}$$

When $\mu = 0$, we have

$$E\left(e^{-\beta\tau^R}\right) = \frac{e^{-\beta d}}{\Psi(\beta d)}. \tag{28}$$

Now we are going to concentrate on the excursion below the barrier. The results for the excursion above the barrier can be easily obtained by the same methods.

In §?? we have proved that, in order to get the results regarding to the excursion below the barrier, we just need to set $d_1 \to +\infty$. From (13), we have

$$\begin{aligned} & E\left(\exp\left\{-\alpha L^{W^\mu}_{1,\tau_2^{W^\mu}} - \beta L^{W^\mu}_{2,\tau_2^{W^\mu}}\right\}\right) \\ = & \lim_{\epsilon\to 0} E\left(\exp\left\{-\alpha L^{X^{(\epsilon)}}_{1,\tau_2^{X^{(\epsilon)}}} - \beta L^{X^{(\epsilon)}}_{2,\tau_2^{X^{(\epsilon)}}}\right\}\right) \\ = & \lim_{\epsilon\to 0} \frac{e^{-\beta d_2}\bar{P}_{21}(d_2)\int_0^{+\infty} e^{-\alpha s}p_{12}(s)\mathrm{d}s}{1-\int_0^{+\infty} e^{-\alpha s}p_{12}(s)\mathrm{d}s\int_0^{d_2} e^{-\beta s}p_{21}(s)\mathrm{d}s} \\ = & \frac{e^{-\beta d_2}\left\{\Psi\left(\frac{\mu^2 d_2}{2} - \sqrt{\frac{\pi\mu^2 d_2}{2}}\right)\right\}}{\Psi\left\{\frac{(2\beta+\mu^2)d_2}{2}\right\} + \sqrt{\frac{\pi(2\alpha+\mu^2)d_2}{2}}}. \end{aligned} \tag{29}$$

When $\mu = 0$, we have

$$E\left(\exp\left\{-\alpha L^{W}_{1,\tau_2^{W}} - \beta L^{W}_{2,\tau_2^{W}}\right\}\right) = \frac{e^{-\beta d_2}}{\Psi(\beta d_2)+\sqrt{\pi\alpha d_2}}. \tag{30}$$

These can also be verified by setting $d_1 \to +\infty$ in (15) and (16). Followed by (29) and (30) we can get the result applied in pricing Parisian options by setting $\alpha = \beta$ and using the relation $\tau_2^W = L^W_{1,\tau_2^W} + L^W_{2,\tau_2^W}$.

Theorem 3 *For a Brownian Motion W_t^μ with $W_0^\mu = 0$, $\tau_2^{W^\mu}$ has been defined in §??, (3) with $S_t = W_t^\mu$. We then have*

$$E\left(e^{-\beta\tau_2^{W^\mu}}\right) = \frac{e^{-\beta d_2}\left\{\Psi\left(\frac{\mu^2 d_2}{2}\right) - \mu\sqrt{\frac{d_2\pi}{2}}\right\}}{\Psi\left\{\frac{(2\beta+\mu^2)d_2}{2}\right\} + \sqrt{\frac{(2\beta+\mu^2)d_2}{2}}}. \tag{31}$$

When $\mu = 0$, we have

$$E\left(e^{-\beta\tau_2^{W}}\right) = \frac{e^{-\beta d_2}}{\Psi(\beta d_2)+\sqrt{\pi\beta d_2}}. \tag{32}$$

The result presented by (32) has been obtained in [6] for Parisian option pricing by a very different technique which can only be applied for a standard Brownian motion.

By setting $\beta \to 0$ in (31), we can calculate the Parisian type ruin probability:

Corollary 3.1 *For a Brownian motion W^μ, $\mu \geq 0$ and $W_0^\mu = 0$, the probability that the length of the excursion below 0 ever reaches d_2 is*

$$P\left(\tau_2^{W^\mu} < \infty\right) = 1 - \frac{\mu}{\frac{1}{\sqrt{2\pi d_2}}e^{-\frac{\mu^2 d_2}{2}} + \mu\mathscr{N}\left(\mu\sqrt{d_2}\right)}. \tag{33}$$

As a result, for a Brownian motion with positive drift, with probability

$$\frac{\mu}{\frac{1}{\sqrt{2\pi d_2}}e^{-\frac{\mu^2 d_2}{2}} + \mu\mathcal{N}(\mu\sqrt{d_2})}$$

the process will never stay in state 2, i.e. below the barrier, for longer than d_2, while for a standard Brownian motion ($\mu = 0$), this event will happen with probability 0.

We hereby introduce a new type of ruin probability, the *Parisian type ruin probabilities*. As what we have briefly mentioned in §1, if $T_0 = \inf\{t > 0 \,|\, S_t < 0\}$, i.e. the first time the process hits 0, we have, in risk theory, $P(T_0 < \infty)$ as the ruin probability. Here we extend this concept and define the *Parisian type ruin probabilities* to be

$$P(\tau_2^S < \infty),$$

i.e the probability that the event that the process falls below 0 and stays below 0 constantly for at least d_2 ever happens. Therefore, the Parisian type ruin probability for a Brownian motion with positive drift is given by (33). From an operational point of view, this is a more realistic model of insolvency as it gives the company an opportunity to put its finances back in order.

5 Appendix

We will now show that we can take limits of Laplace transforms when $\epsilon \to 0$ as we did earlier. We have studied the following variables: τ_1^S,τ_2^S, τ^S, L_{1,τ^S}^S, $L_{1,\tau_1^S}^S$, $L_{1,\tau_2^S}^S$, L_{2,τ^S}^S, $L_{2,\tau_1^S}^S$ and $L_{2,\tau_2^S}^S$. In order to simplify the notations, we define $R^S = (R_1^S, R_2^S, \cdots, R_9^S)$, where R_i^S, $i = 1, 2, \cdots, 9$, stand for each of the above variables.

Firstly, according to the definition of $X^{(\epsilon)}$, we know that

$$X_t^{(\epsilon)} \xrightarrow{a.s} W_t^\mu, \quad \text{for all } t.$$

Therefore,

$$R_i^{X^{(\epsilon)}} \xrightarrow{a.s} R_i^{W^\mu}, \qquad \text{for } i = 1, 2, \cdots, 9.$$

So for given non-negative constants β_i, $i = 1, 2, \cdots, 9$,

$$\exp\left\{-\sum_{i=1}^{9} \beta_i R_i^{X^{(\epsilon)}}\right\} \xrightarrow{a.s} \exp\left\{-\sum_{i=1}^{9} \beta_i R_i^{W}\right\}.$$

Since $R_i^{X^{(\epsilon)}} \geq 0$, we also have,

$$\left|\exp\left\{-\sum_{i=1}^{9} \beta_i R_i^{X^{(\epsilon)}}\right\}\right| < 1.$$

By the Dominated Convergence Theorem,

$$E\left(\exp\left\{-\sum_{i=1}^{9}\beta_i R_i^W\right\}\right) = E\left(\lim_{\epsilon\to 0}\exp\left\{-\sum_{i=1}^{9}\beta_i R_i^{X^{(\epsilon)}}\right\}\right) = \lim_{\epsilon\to 0} E\left(\exp\left\{-\sum_{i=1}^{9}\beta_i R_i^{X^{(\epsilon)}}\right\}\right).$$

When $\mu = 0$, we can get the same conclusion for the standard Brownian motion by the above argument.

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