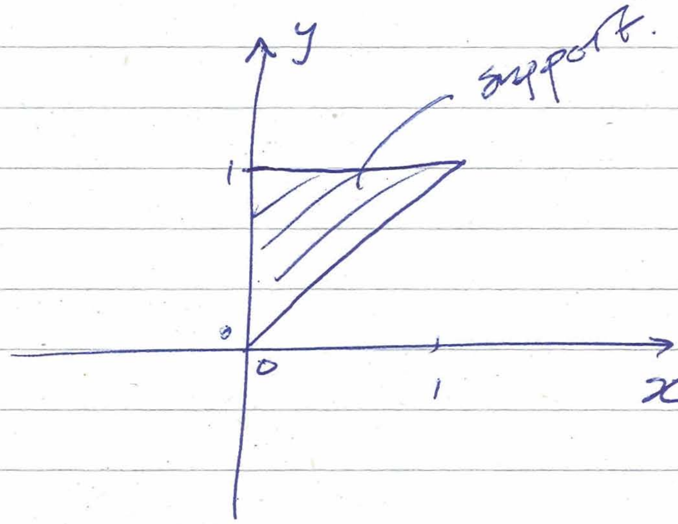
ST402 2007 Solutions

1. (a)



(b) for $0 \le x \le 1$

$$f_X(x) = \int_x^1 8xy \, dy = 8x \frac{y^2}{2} \Big|_x^1$$

$$= -4x^3 + 4x = 4x(1-x^2)$$

so $f_X(x) = \begin{cases} 4x(1-x^2) & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$

for $0 \le y \le 1$,

$$f_Y(y) = \int_0^y 8xy \, dx = 8\frac{x^2}{2} \cdot y \Big|_0^y = 4y^3$$

so $f_Y(y) = \begin{cases} 4y^3 & 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{8xy}{4y^3} = \frac{2x}{y^2}$$

for $0 \le y \le 1$,

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{2x}{y^2} & 0 \le x \le y \\ 0 & \text{elsewhere} \end{cases}$$

1 (c) Not independent. $f_{X|Y}$ depends on y,

$f_{X,Y}(x,y) \neq f_X(x) f_Y(y)$,

(or more accurate similar statements of reasons)

(d) $E[Y^r] = \int_0^1 y^r \, 4y^3 \, dy = \left. \frac{y^{r+4}}{r+4} \right|_0^1 = \frac{1}{r+4}$

$E[Y] = \frac{1}{5}$, $E[Y^2] = \frac{1}{6}$.

(e) $E[X|Y=y] = \int_0^y x \cdot \frac{2x}{y^2} \, dx$

$= \left. \frac{2x^3}{3y^2} \right|_0^y = \frac{2y}{3}$. (for $0 \le y \le 1$)

(f)

$E[XY] = E[Y \, E[X|Y]]$

$= E[\frac{2Y^2}{3}] = \frac{2}{3} \cdot \frac{1}{6} = \frac{1}{9}$

$E[X] = E[E[X|Y]] = E[\frac{2Y}{3}] = \frac{2}{3} \times \frac{1}{5} = \frac{2}{15}$.

$\mathrm{Cov}(X,Y) = \frac{1}{9} - \frac{1}{5} \cdot \frac{2}{15} = \frac{25-6}{5^2 \times 3^2} = \frac{19}{225}$

(g) $E[e^{Vt}] = E[e^{(-4 \ln XY)t}] = E[(XY)^{-4t}]$

$= \int_0^1 \int_0^y \frac{1}{(xy)^{4t}} \, 8xy \, dx \, dy$

$= \int_0^1 8y^{1-4t} \left. \frac{x^{2-4t}}{2-4t} \right|_0^y dy = \int_0^1 \frac{8}{2-4t} \, y^{3-8t} \, dy$

$= \frac{8}{2-4t} \cdot \frac{1}{4-8t} = \frac{1}{(1-2t)^2} = \frac{1}{(1-2t)^{4/2}}$.

$\Rightarrow V$ is $\chi^2_{(4)}$.

2. (a) $E[e^{Yt} \mid N=n]$

$$= \begin{cases} E\left[e^{\sum_1^n X_i t}\right] & n>0 \\ 1 & n=0 \end{cases}$$

$$= \begin{cases} \prod_{i=1}^{n} E[e^{X_i t}] & n>0 \\ 1 & n=0 \end{cases} \qquad \text{(independence)}$$

$$= \begin{cases} [M_X(t)]^n & n>0 \\ 1 & n=0 \end{cases}$$

$$= M_X(t)^n.$$

$$M_Y(t) = E[e^{Yt}] = E[E[e^{Yt} \mid N]] = E[M_X(t)^N]$$

$$= E[e^{N \ln M_X(t)}]$$

$$= M_N(\ln M_X(t)).$$

2(b) If N has a Poisson (λ) then

$$M_N(t) = e^{-\lambda(1-e^t)}.$$

$$M_X(t) = (1-p)e^{0t} + pe^t.$$

$$M_Y(t) = e^{-\lambda(1-(1-p)+pe^t)}$$

$$= e^{-\lambda p(1-e^t)}$$

$\Rightarrow$ Y is Poisson mean λp.

If X takes values $0, 1, 2$

$$M_X(t) = (1-p_1-p_2) + p_1 e^t + p_2 e^{2t}$$

$$M_Y(t) = \exp\{-\lambda(1-(1-p_1-p_2) - p_1 e^t - p_2 e^{2t})\}$$

$$= \exp\{-\lambda p_1(1-e^t) - \lambda p_2(1-e^{2t})\}.$$

$$= \exp\{-\lambda p_1(1-e^t)\}\exp\{-\lambda p_2(1-e^{2t})\}.$$

However,

$$M_{u_1+2u_2}(t) = M_{u_1}(t)\, M_{2u_2}(t)$$

$$= e^{-\lambda p_1(1-e^t)} \cdot e^{-\lambda p_2(1-e^{2t})}$$

So have result.

3.

(a) Joint probability is

$$f(\underline{x};\lambda) = \prod_{i=1}^{n} e^{-\lambda} \frac{\lambda^{x_i}}{x_i!}$$

$$= e^{-n\lambda} \frac{\lambda^{\Sigma x_i}}{\Pi x_i!}$$

$$\ln L = -n\lambda + \Sigma x_i \ln \lambda - \ln \Pi x_i!$$

Score function is $\frac{d\ln L}{d\lambda} = -n + \frac{\Sigma x_i}{\lambda}$

ie

$$s(\lambda) = -\frac{n}{\lambda}(\lambda - \bar{x}).$$

(b) $$J = Var(s(\lambda)) = \frac{n^2}{\lambda^2} Var(\bar{x})$$

$$= \frac{n^2}{\lambda^2} \cdot \frac{\lambda}{n} = \frac{n}{\lambda}.$$

(c) If $g(\lambda) = E[T]$, then C.R. bound is

$$Var(T) \geq \frac{[\dot{g}(\lambda)]^2}{J}$$

so for $g(\lambda) = \lambda^2$,

$$Var(T) \geq \frac{4\lambda^2}{\frac{n}{\lambda}} = \frac{4\lambda^3}{n}.$$

(d) $E[\bar{X}^2] = \text{var}(\bar{X}) + (E[\bar{X}])^2$

$= \frac{\lambda}{n} + \lambda^2$

So bias is $\frac{\lambda}{n}$.

(e) Draw bootstrap samples from Poisson $(\bar{x})$
So get bootstrap estimates

$\bar{x}_1^{2*}, \dots, \bar{x}_m^{2*}$

Estimate bias of $\bar{X}^2$ by

$$\frac{1}{m} \sum_1^m \bar{x}_j^{2} - \bar{x}^2$$

(f) Bias is, for $m \to \infty$ $E[\bar{u}^2] - \bar{x}^2$ where

$u \sim$ Poisson $(\bar{x})$

$E[\bar{u}^2] = \text{var}(\bar{u}) + [E(\bar{u})]^2$

$= \frac{\bar{x}}{n} + \bar{x}^2$

So Bootstrap bias estimate is $\frac{\bar{x}}{n}$.

Bias corrected estimator is

$$\bar{X}^2 - \frac{\bar{X}}{n} = \frac{1}{n} \bar{X} (n - \bar{X})$$

4 (a)

$$f_{\underset{\sim}{X}}(\underset{\sim}{x};\theta) = \prod_{i=1}^{n} \theta e^{-\theta x_i}$$

$$= \begin{cases} \theta^n e^{-\theta n \bar{x}} & x_i \ \forall > 0 \\ & \text{otherwise} \end{cases}$$

$\Rightarrow \bar{X}$ is a sufficient statistic for θ.

(b) Loglikelihood is

$$l(\theta) = n \ln \theta - n\bar{x}\theta$$

$$\frac{dl}{d\theta} = \frac{n}{\theta} - n\bar{x} = \frac{n}{\theta}(1 - \theta \bar{x})$$

$$\frac{d^2 l}{d\theta^2} = -\frac{n}{\theta^2}$$

So,

Maximum at $\underline{\hat{\theta} = \frac{1}{\bar{x}}}$ from $\frac{dl}{d\theta} = 0$.

(c) $$f_{\underset{\sim}{Y}}(\underset{\sim}{y};\theta) = \frac{1}{\theta^n} \prod_{i=1}^{m} I_{(0,\theta)}(y_i)$$

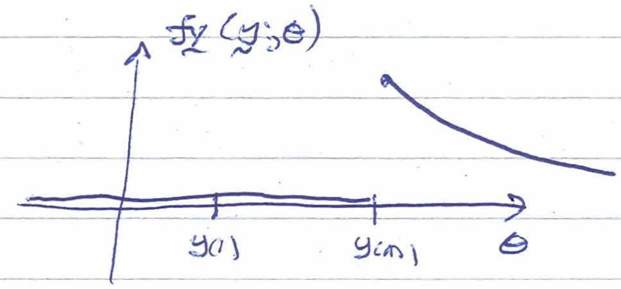
$$= \frac{1}{\theta^m} I_{(0,\infty)}(y_{(1)}) \, I_{(-\infty,\theta)}(y_{(m)}) \qquad I_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

where $y_{(i)}$ are ordered y_i.

$$\Rightarrow f_{\underset{\sim}{Y}}(\underset{\sim}{y};\theta) = \frac{1}{\theta^m} I_{(-\infty,\theta)}(y_{(m)}) \cdot I_{(0,\infty)}(y_{(1)})$$

So by factorisation criterion, $Y_{(m)}$ is sufficient for θ.

(d)



So, MLE of θ is $\underline{y_{(m)}}$.

4 (e) Joint density is

$$f_{\underset{\sim}{x},\underset{\sim}{y}}(\underset{\sim}{x},\underset{\sim}{y};\theta) = \theta^{n} e^{-n\bar{x}\theta} \frac{1}{\theta^{m}} I_{(0,\theta)}(y_{(m)})\, I_{(0,\infty)}(y_{(1)})$$

$$= \theta^{n-m} e^{-n\bar{x}\theta} I_{(0,\theta)}(y_{(m)})\, I_{(0,\infty)}(y_{(1)})$$

so by factorization criterion,

$(\bar{x}, y_{(m)})$ is sufficient for θ.

The likelihood is 0 for $\theta < y_{(m)}$, so we need consider only the behaviour of

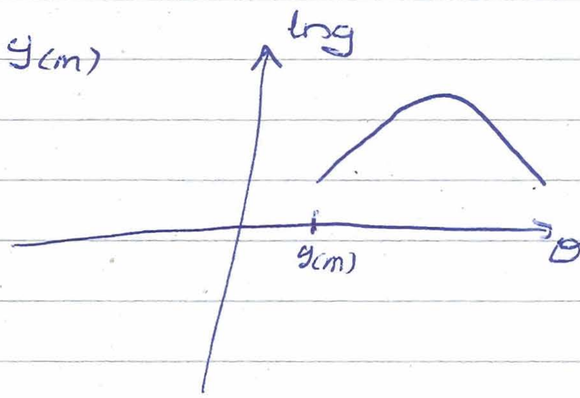
$$g(\theta) = \theta^{n-m} e^{-n\bar{x}\theta} \quad \text{for } \theta > y_{(m)}$$

$$\ln g(\theta) = (n-m)\ln\theta - n\bar{x}\theta$$

$$\frac{d\ln g(\theta)}{d\theta} = \frac{n-m}{\theta} - n\bar{x}$$

$$\frac{d^2\ln g(\theta)}{d\theta^2} = -\frac{(n-m)}{\theta^2}$$

So, for $n > m$, $\frac{n-m}{n\bar{x}} > y_{(m)}$



$$\underline{\hat{\theta} = \frac{n-m}{n\bar{x}}}$$

for $n > m$, $\frac{n-m}{n\bar{x}} < y_{(m)}$

$$\underline{\hat{\theta} = y_{(m)}}.$$

For $n < m$, $\underline{\hat{\theta} = y_{(m)}}$