ST305 2007 Solutions

1.

a) A continuous life annuity for a 65 year old guaranteed for 5 years.

b) A continuous annuity for a 65 year old payable till 5 years after the time of death.

c) A payment of 1 on the death of a life aged x provided this occurs no later than five years after the death of a life aged y

2. Let the unit fund accumulation be $U(t)$. We have that

$$\frac{dU(t)}{dt} = rU(t) + (1-\gamma_t)\pi$$

with $U(0)=0$. For the cash fund flow to be 0, we must have $\gamma_t\pi = \mu_{x+t}(g - U(t))^+$. Hence the differential equation can be rewritten as

$$\frac{dU(t)}{dt} = rU(t) + \pi - \mu_{x+t}(g - U(t))^+.$$

For those t that $U(t) \le g$,

$$\frac{dU(t)}{dt} = rU(t) + \pi - \mu_{x+t}(g - U(t))$$

and solving subject to $U(0)=0$ or arguing directly

$$U(t) = \int_0^t (\pi - g\mu_{x+s})\exp\Big(\int_s^t r + \mu_{x+u} du\Big).$$

We then have

$$g - U(t) = g - \int_0^t (\pi - g\mu_{x+s})\exp\Big(\int_s^n r + \mu_{x+u} du\Big) = \exp\Big(\int_0^n r + \mu_{x+u} du\Big)\big(g\,\bar{A}_{x:\overline{t}|} - \pi\bar{a}_{x:\overline{t}|}\big).$$

This is a decreasing function from $g\exp\big(\int_0^n r + \mu_{x+u}du\big)$ to a negative (or zero) value. Let m be the time such that

$$g = \pi\frac{\bar{a}_{x:\overline{m}|}}{\bar{A}_{x:\overline{m}|}}.$$

We then have

$$U(t) = \int_0^t (\pi - g\mu_{x+s})\exp\Big(\int_s^t r + \mu_{x+u} du\Big)$$

for $t \le m$ and

$$U(t) = \exp(r(t-m))U(m) + \int_m^t \pi\exp(r(s-m))ds =$$

$$\exp(r(t-m))g + \pi\frac{\exp(r(t-m)) - 1}{r}$$

for $m \le t \le n$. The allocation is such that

$$\gamma_t = \mu_{x+t}(g - U(t))$$

for $t \le m$ and $\gamma_t = 0$ for $m \le t \le n$ a the unit fund will have then grown enough so as not to need to finance the guarantee. The guarantee is adequately financed at all times and there is no need to set reserves. The final payoff is

$$U(n) = \exp(r(n-m))U(m) + \int_m^n \pi \exp(r(s-m))ds.$$

3.

a) The maximum salary is achieved at age 60. The benefit is

$$5S_0 \exp\left(\int_0^{25} a(s)ds\right).$$

The premium is

$$5S_0 \exp\left(\int_0^{25} a(s)ds\right) \exp\left(-\int_0^{30} (r(u) + \mu_{x+u})du\right) + c\int_0^{30} \exp\left(-\int_0^t (r(u) + \mu_{x+u})du\right)dt$$

b) We need to calculate

$$5S_0 \exp\left(\int_0^{25} a(s)ds\right) E\left(\exp\left(-\int_t^{30} (r(u) + \mu_{x+u})du\right)\middle|\mathcal{F}_t\right) = 5S_0 \exp\left(\int_0^{25} a(s)ds\right) {}_{30-t}p_{35+t} E\left(\exp\left(-\int_t^{30} r(u)du\right)\middle| r(t) = r^{(e)}\right).$$

Define

$$W_e(t) = E\left(\exp\left(-\int_t^{30} r(u)du\right)\middle| r(t) = r^{(e)}\right).$$

Using a backward argument

$$W_1(t-dt) = \left(1 - r^{(1)}dt\right)(1-\lambda dt)W_1(t) + \lambda dt W_2(t) + o(dt)$$

and hence

$$\frac{W_1(t-dt) - W_1(t)}{dt} = -\left(r^{(1)} + \lambda\right)W_1(t) + \lambda W_2(t) + \frac{o(dt)}{dt}.$$

Let $dt \rightharpoonup 0$ and we have

$$\frac{dW_1(t)}{dt} = \left(r^{(1)} + \lambda\right)W_1(t) - \lambda W_2(t).$$

Similarly

$$\frac{dW_2(t)}{dt} = \left(r^{(2)} + \lambda\right)W_2(t) - \lambda W_1(t).$$

The two equations should be solved subject to

$$W_1(30) = W_2(30) = 5S_0 \exp\left(\int_0^{25} a(s)ds\right).$$

The premium will be $W_1(0)\,{}_{30}p_{35}$.

4.

a) The forward equations are

$$\frac{dp_1(t)}{dt} = -(\mu_x + 0.1)p_1(t) + 0.1p_2(t)$$

and

$$\frac{dp_2(t)}{dt} = -(\mu_x + 0.1)p_2(t) + 0.1p_1(t)$$

with $p_1(0) = 1$ and $p_2(0) = 0$. We can now proceed either with direct substitution or observing that mortality is independent of other movements set $p_1(t) = {}_tp_{35}\bar{p}_1(t)$ and $p_2(t) = {}_tp_{35}\bar{p}_2(t)$. We then have $\bar{p}_2(t) = 1 - \bar{p}_1(t)$

$$\frac{d\,\bar{p}_1(t)}{dt} = -0.1\bar{p}_1(t) + 0.1(1 - \bar{p}_1(t))$$

with $\bar{p}_1(1) = 1$. Solving

$$\bar{p}_1(t) = \frac{1}{2} + \frac{1}{2}\exp(-0.2t)$$

and

$$\bar{p}_2(t) = \frac{1}{2} - \frac{1}{2}\exp(-0.2t).$$

b) The expected present value is

$$100000\int_0^{30} \exp(-0.05t)\left(\frac{1}{2} - \frac{1}{2}\exp(-0.2t)\right){}_tp_{35}\mu_{35+t}dt =$$

$$100000\left(\frac{1}{2}\int_0^{30} \exp(-0.05t)\,{}_tp_{35}\mu_{35+t}dt - \frac{1}{2}\int_0^{30} \exp(-0.25t)\,{}_tp_{35}\mu_{35+t}dt\right).$$

We also have that

$$\int_0^n \exp(-rt)\,{}_tp_x\mu_{x+t}dt = 1 - r\bar{a}_{x:\overline{n}|} - {}_np_x\exp(-rn).$$

So

$$\frac{1}{2}\int_0^{30} \exp(-0.05t)\,{}_tp_{35}\mu_{35+t}dt - \frac{1}{2}\int_0^{30} \exp(-0.25t)\,{}_tp_{35}\mu_{35+t}dt =$$

$100000 \times \frac{1}{2} \times (1 - 0.05 \times 14.75 - \exp(-30 \times 0.05) \times 0.78) - 100000 \times \frac{1}{2} \times (1 - 0.25 \times 3.95 - \exp(-30 \times 0.25) \times 0.78) = 4422.9 - 603.4 = 3819.5.$

c) The required probability is

${}_tp_{35}\int_0^t 0.1\exp(-0.1s)\exp(-0.1(t-s))ds =_t p_{35}0.1\exp(-0.1t)\int_0^t ds = {}_tp_{35}0.1t\exp(-0.1t).$

So the value of the benefit is

$$100000\int_0^t \exp(-0.05t)0.1t\exp(-0.1t)\,{}_tp_{35}\,\mu_{35+t}dt = 10000\times 0.233 = 2330$$

5.

a) We first calculate the expected present value of the benefit. To this effect we have to solve the system

$$\frac{dW_H(t)}{dt} = (r(t)+\mu(t))W_H(t) - \sigma(t)(W_S(t)-W_H(t))$$

$$\frac{dW_S(t)}{dt} = (r(t)+\nu(t))W_S(t) - b - \rho(t)(W_H(t)-W_S(t))$$

subject to the conditions $W_H(25) = W_S(25) = 0$. $W_H(0)$ would then represent the expected present value of the benefits. Then we also solve the system

$$\frac{dU_H(t)}{dt} = (r(t)+\mu(t))U_H(t) - 1 - \sigma(t)(U_S(t)-U_H(t))$$

$$\frac{dU_S(t)}{dt} = (r(t)+\nu(t))U_S(t) - \rho(t)(U_H(t)-U_S(t))$$

subject to the conditions $U_H(25) = U_S(25) = 0$ $U_H(0)$ would then represent the expected present value of a continuous annuity of 1 payable for 25 years while the life is healthy. The continuous premium is given by $\pi = \frac{W_H(0)}{U_H(0)}$. We can then calculate reserves represented by $V_H(t)$ and $V_S(t)$ by solving the system

$$\frac{dV_H(t)}{dt} = (r(t)+\mu(t))V_H(t) + \pi - \sigma(t)(V_S(t)-V_H(t))$$

$$\frac{dV_S(t)}{dt} = (r(t)+\nu(t))V_S(t) - b - \rho(t)(V_H(t)-V_S(t))$$

subject to the conditions $V_H(25) = V_S(25) = 0$ (or $V_H(0) = V_S(0) = 0$).

b) Towards the end of the policy the value of the benefits will be smaller than the value of the premiums if the life is healthy which will result to negative reserves and on incentive for the life to keep paying the premiums. One way to deal with this is to restrict the period that premiums are payable to the first 15 (say) years.

c) The expected present value of the benefits remains as before. We now modify the terminal conditions for the equations for $U_H(t)$ and $U_S(t)$, to $U_H(15) = U_S(15) = 0$. The premium is $\pi = \frac{W_H(0)}{U_H(0)}$, but with the new value of $U_H(0)$. We can then calculate reserves represented by $V_H(t)$ and $V_S(t)$ by solving the system

$$\frac{dV_H(t)}{dt} = (r(t)+\mu(t))V_H(t) + \pi\mathbf{1}_{(t<15)} - \sigma(t)(V_S(t)-V_H(t))$$

$$\frac{dV_S(t)}{dt} = (r(t)+\nu(t))V_S(t) - b - \rho(t)(V_H(t)-V_S(t))$$

subject to the conditions $V_H(25) = V_S(25) = 0$ (or $V_H(0) = V_S(0) = 0$).

6.

a) The policy pays out the sum of 80000 at time 20 or upon earlier death of a life now aged 35. The payment is conditional on another life aged 45 being alive at the time. The force of interest is 0.03 and expenses that are 4% of the continuous premium are assumed.

b) The prospective (or retrospective) reserve for the rest of the policy at time 5, provided both lives are alive at the time.

c) The reserve at that time is 30300, so 30300-300=30000 will be used as a single premium. The lump sum will be

$$\frac{30000}{{}_{10}p_{45}\,{}_{10}p_{55}}\exp(0.03 \times 10).$$

We already know ${}_{10}p_{45} = 0.935$ and we can calculate

$${}_{10}p_{55} = \frac{{}_{20}p_{45}}{{}_{10}p_{45}} = \frac{0.8014}{0.935} = 0.887.$$

So the lump sum will be

$$\frac{30000}{0.935 \times 0.887}\exp(0.03 \times 10) = 48829.$$