Summer 2008 Examination

ST305

Actuarial Mathematics: Life
Suitable for all students

Instructions to candidates

Time allowed: 3 hours

This paper contains 5 questions. Answer **All** questions.

You may use an electronic calculator (as prescribed by the examination regulations)

Taxes and expenses should be ignored unless otherwise indicated.

1. Two independent lives are aged x and y and at time t they are subject to mortality forces μ_{x+t} and μ_{y+t}. Provide a single integral expression involving the forces and the corresponding survival probabilities only for the expected value of the time interval between their deaths.

(4 *marks*)

2. A pension plan provides a pension starting at any time between the ages of 60 and 65 of half the average salary over the last 5 service years. The member who is entitled to the benefit is now aged 30. Retirement is equally likely to occur at any time between the ages of 60 and 65. The force of interest is assumed to be deterministic and given by $r(t)$. The salary is also calculated as a deterministic function, where the salary of the member at time t is given by $S_t = S_0 \exp\left(\int_0^t a(s)\,\mathrm{ds}\right)$, where S_0 is the current salary. Assume a force of mortality μ_{30+t} at time t and expenses at a constant rate c per annum. The plan is financed by a continuous premium that is equal to αS_t. The premium is payable until the time of retirement. Write down integral expressions and use them to calculate α.

(8 *marks*)

3. An office issued with profits 30 year endowment assurance policies to 30 year old lives with a sum assured of €200000. The premium is €3340 per annum payable continuously and was calculated using a force of mortality μ_x, where x a life's age, a force of interest of 0.05 per annum and no expenses. The real force of interest is stochastic and can take the values 0.04 and 0.07. The transition rates between the two values are 0.8. The real mortality force is also μ_x at age x. The force of interest is 0.07 initially. The policyholders will receive all surplus in the form of a terminal bonus.

   (a) Set out a system of equations and explain how you would use it to predict the terminal bonus. *(Note that although you are not required to derive any of the equations, some sort of derivation will give you partial credit if things go wrong. You should in any case define the functions involved in the equations).* (16 *marks*)

(b) Suppose that we are at time t=5 and the force of interest has been 0.07 for the first year and 0.04 for the next four years and it still is 0.04. Explain how you would predict the terminal bonus *(Hint: You have to calculate the accumulated surplus at $t = 5$)*

(9 *marks*)

(Total: 25 *marks*)

4. The following calculations might be useful to you for this question. They all refer to a force of mortality μ_x at age x.

$$ {}_{30}p_{30} = 0.845 $$

At a force of interest 0.05

$$ \bar{a}_{30} = 17.101 \qquad \bar{a}_{60} = 11.183 $$

At a force of interest 0.1

$$ \bar{a}_{30} = 9.605 \qquad \bar{a}_{30:\overline{30|}} = 9.283 $$

At a force of interest 0.15

$$ \bar{a}_{30} = 6.544 \qquad \bar{a}_{60} = 6.211 $$

At a force of interest 0.3

$$ \bar{a}_{30} = 3.312 \qquad \bar{a}_{60} = 3.123 $$

At a force of interest 0.35

$$ \bar{a}_{30} = 2.842 \qquad \bar{a}_{60} = 2.708 $$

An office is offering special policies to lives that are prone to a relatively rare disease. The lives are originally healthy and at some stage they might develop mild symptoms. This happens with rate 0.05 per annum. Lives with mild symptoms might develop the full version of the disease with rate 0.1 per annum. The progress of the disease is irreversible. Lives aged x that are healthy or have mild symptoms only have a force of mortality μ_x. Lives with the full version of the disease have a force of mortality $0.3 + \mu_x$.

(a) Show that the probability that a healthy life initially aged x has mild symptoms at time t is

$$(\exp(-0.05t) - \exp(-0.1t))\, {}_tp_x$$

and the probability that the life has the full version of the disease is

$$(0.4\exp(-0.05t) - 0.5\exp(-0.1t) + 0.1\exp(-0.3t))\, {}_tp_x$$

where

$${}_tp_x = \exp\left(-\int_0^t \mu_{x+s} ds\right).$$

(8 *marks*)

(b) Suppose now that $x = 30$. Find the probability that the life will never develop any symptoms. (5 *marks*)

(c) A healthy life initially aged 30 is entitled to a death benefit of €100,000 if death occurs while exhibiting mild symptoms only and a continuous benefit of €10,000 per annum while having the full version of the disease. The policy is financed by a single premium up front and it is valid for life. Calculate the premium using a force of interest of 0.05 per annum. (10 *marks*)

(d) Suppose now that the policy is only valid for 30 years unless the life has developed symptoms or has the full disease at $t = 30$ in which case the policy continues for life. Calculate the premium in this case using a force of interest of 0.05 per annum. (7 *marks*)

(Total: 30 *marks*)

5. The last two pages of this examination paper are the printout of a MAPLE worksheet with calculations about a certain life insurance product. The product is on two lives aged 30 and 40 that are not independent of each other. The sum of €308.80 represents the annual premium, the force of interest is 0.04 per annum and there are no expenses.

(a) Describe the mortality model for the two lives. (3 *marks*)

(b) What are the benefits of the policy? When is the premium being paid? (3 *marks*)

(c) What do the amounts of €3900.12 in (7), €12.63 in (13) and €5579.29 in (19) represent? (Equation numbers in brackets refer to lines of MAPLE code) (9 *marks*)

(d) Suppose now that there are expenses 2% of each annual premium and 1% of the sum assured. What would the premium be in this case?

(4 *marks*)

(e) Now ignore expenses again. At time $t = 10$ both lives are alive and want to double the benefits by increasing the annual premium which will still be payable continuously. What would the new premium be? (6 *marks*)

(f) Identify a problem that this policy has. (2 *marks*)

(g) Show that for two independent lives, this product will always have a similar problem as in in (f) for any benefit and premium.

(6 *marks*)

(Total: 33 *marks*)

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> m := t -> 0.0005+0.00007585775*10^(0.038*t);
```

$$m := t \rightarrow 0.0005 + 0.00007585775\ 10^{0.038\,t} \tag{1}$$

```
> m2 := t -> 1/(400-t);
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$$m2 := t \rightarrow \frac{1}{400 - t} \tag{2}$$

```
> dsys1 := {diff(w0(t),t)=0.04*w0(t)-m(30+t)*(w2(t)-w0(t))-m(40+t)
  *(w1(t)-w0(t))-m2(t)*(100000-w0(t)), diff(w1(t),t)=0.04*w1(t)-(m
  (30+t)+m2(t))*(100000-w1(t)),  diff(w2(t),t)=0.04*w2(t)-(m(40+t)
  +m2(t))*(100000-w2(t)), w0(20)=0, w1(20)=0, w2(20)=0};
>
```

$$\begin{aligned} dsys1 := \Big\{ & w2(20) = 0, w1(20) = 0, w0(20) = 0, \frac{d}{dt}\, w1(t) = 0.04\, w1(t) - \Big(0.0005 \\ & + 0.00007585775\ 10^{1.140 + 0.038\,t} + \frac{1}{400 - t}\Big)\,(100000 - w1(t)), \frac{d}{dt}\, w2(t) = 0.04\, w2(t) \\ & - \Big(0.0005 + 0.00007585775\ 10^{1.520 + 0.038\,t} + \frac{1}{400 - t}\Big)\,(100000 - w2(t)), \frac{d}{dt}\, w0(t) \\ & = 0.04\, w0(t) - \big(0.0005 + 0.00007585775\ 10^{1.140 + 0.038\,t}\big)\,(w2(t) - w0(t)) - \big(0.0005 \\ & + 0.00007585775\ 10^{1.520 + 0.038\,t}\big)\,(w1(t) - w0(t)) - \frac{100000 - w0(t)}{400 - t} \Big\} \end{aligned} \tag{3}$$

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> dsol1 := dsolve(dsys1, numeric, range = 0..20);
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$$dsol1 := \mathbf{proc}(x\_rkf45)\ \ldots\ \mathbf{end\ proc} \tag{4}$$

```
> dsol1(20);
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$$[t = 20., w0(t) = 0., w1(t) = 0., w2(t) = 0.] \tag{5}$$

```
> dsol1(10);
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$$\begin{aligned} [& t = 10., w0(t) = 2425.06200266858014, w1(t) = 5579.28975170987452, w2(t) \\ & = 9678.27063770329188] \end{aligned} \tag{6}$$

```
> dsol1(0);
```

$$\begin{aligned} [& t = 0., w0(t) = 3900.12413785380886, w1(t) = 7322.37212416162402, w2(t) \\ & = 11574.6744356303916] \end{aligned} \tag{7}$$

```
> deq1  := diff(u(t),t) =(0.04+m(40+t)+m(30+t)+m2(t))*u(t) -1 ;
```

$$\begin{aligned} deq1 := \frac{d}{dt}\, u(t) = & \Big(0.0410 + 0.00007585775\ 10^{1.520 + 0.038\,t} + 0.00007585775\ 10^{1.140 + 0.038\,t} \\ & + \frac{1}{400 - t}\Big)\, u(t) - 1 \end{aligned} \tag{8}$$

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> tc1   := u(20) = 0;
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$$tc1 := u(20) = 0 \tag{9}$$

```
> dsol2 := dsolve({deq1,tc1}, numeric, range=0..20);
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$$\tag{10}$$

$$dsol2 := \mathbf{proc}(x\_rkf45)\ \ldots\ \mathbf{end\ proc} \tag{10}$$

```
> dsol2(20);
```

$$[t = 20.,\ u(t) = 0.] \tag{11}$$

```
> dsol2(10);
```

$$[t = 10.,\ u(t) = 7.68853575734880934] \tag{12}$$

```
> dsol2(0);
```

$$[t = 0.,\ u(t) = 12.6298135520422008] \tag{13}$$

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> evalf(3900.12413785380886/12.6298135520422008);
```

$$308.8029861 \tag{14}$$

```
> dsys2 := {diff(v0(t),t)=0.04*v0(t)-m(30+t)*(v2(t)-v0(t))-m(40+t)
  *(v1(t)-v0(t))-m2(t)*(100000-v0(t))+308.8029861, diff(v1(t),t)=
  0.04*v1(t)-(m(30+t)+m2(t))*(100000-v1(t)),  diff(v2(t),t)=0.04*
  v2(t)-(m(40+t)+m2(t))*(100000-v2(t)), v0(20)=0, v1(20)=0, v2(20)
  =0};
```

$$\begin{aligned} dsys2 := \Big\{ & v0(20) = 0,\ v1(20) = 0,\ v2(20) = 0,\ \frac{d}{dt}\ v1(t) = 0.04\ v1(t) - \Big(0.0005 \\ & + 0.00007585775\ 10^{1.140 + 0.038\,t} + \frac{1}{400 - t}\Big)\ (100000 - v1(t)),\ \frac{d}{dt}\ v2(t) = 0.04\ v2(t) \\ & - \Big(0.0005 + 0.00007585775\ 10^{1.520 + 0.038\,t} + \frac{1}{400 - t}\Big)\ (100000 - v2(t)),\ \frac{d}{dt}\ v0(t) \\ & = 0.04\ v0(t) - \big(0.0005 + 0.00007585775\ 10^{1.140 + 0.038\,t}\big)\ (v2(t) - v0(t)) - \big(0.0005 \\ & + 0.00007585775\ 10^{1.520 + 0.038\,t}\big)\ (v1(t) - v0(t)) - \frac{100000 - v0(t)}{400 - t} + 308.8029861 \Big\} \end{aligned} \tag{15}$$

```
> dsol3 := dsolve(dsys2, numeric, range = 0..20);
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$$dsol3 := \mathbf{proc}(x\_rkf45)\ \ldots\ \mathbf{end\ proc} \tag{16}$$

```
> dsol3(20);
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$$[t = 20.,\ v0(t) = 0.,\ v1(t) = 0.,\ v2(t) = 0.] \tag{17}$$

```
> dsol3(15);
```

$$[t = 15.,\ v0(t) = -25.1228573275004230,\ v1(t) = 3537.96684236209649,\ v2(t) = 6444.56880737622396] \tag{18}$$

```
> dsol3(10);
```

$$[t = 10.,\ v0(t) = 50.8196762898640842,\ v1(t) = 5579.28991200066866,\ v2(t) = 9678.27113344908504] \tag{19}$$

```
> dsol3(0);
```

$$[t = 0.,\ v0(t) = 0.0000325881121128190898,\ v1(t) = 7322.37178726971251,\ v2(t) = 11574.6737363741014] \tag{20}$$

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>
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