ST305 2008 Solutions

1. An expression is

$$\int_0^\infty \left[\, {}_tp_x(1-{}_tp_y)+{}_tp_y(1-{}_tp_x)\right]dt$$

where

$${}_tp_x=\exp\Big(-\int_0^t \mu_{x+s}ds\Big)$$

2. The expected present value of the benefits is

$$\frac{0.5S_0}{5}\int_{30}^{35}\int_{y-5}^{y}\exp\Big(\int_0^s a(u)du\Big)ds\int_y^\infty \exp\Big(-\int_0^s (r(u)+\mu_{x+u})du\Big)ds\frac{1}{5}dy$$

and of the contributions

$$\alpha S_0\int_{30}^{35}\int_0^y \exp\Big(-\int_0^s (-a(u)+r(u)+\mu_{x+u})du\Big)ds\frac{1}{5}dy.$$

Equating the two expressions yields the value of α.

3.

a) The terminal bonus is .

$$\int_0^{30}\exp\Big(\int_s^{30}(r(u)+\mu_{30+u})du\Big)(r(s)-0.05)V_sds$$

where $r(t)$ is the interest rate at any time and V_t is the prospective reserve (calculated at the safe basis) at any time t. This can be rewritten as

$$A(t)U(t)+C(t)$$

where

$$A(t)=\exp\Big(\int_t^{30}(r(u)+\mu_{30+u})du\Big),$$

$$U(t)=\int_0^t \exp\Big(\int_s^t (r(u)+\mu_{30+u})du\Big)(r(s)-0.05)V_sds$$

and

$$C(t)=\int_t^{30}\exp\Big(\int_t^{30}(r(u)+\mu_{30+u})du\Big)(r(s)-0.05)V_sds.$$

$U(t)$ has already been observed, but $A(t)$ and $C(t)$ need to be predicted. Define

$$W_1^{(1)}(t) = E(A(t)|r(t) = 0.04),$$

$$W_2^{(1)}(t) = E(A(t)|r(t) = 0.07),$$

$$W_1^{(2)}(t) = E(C(t)|r(t) = 0.04)$$

and

$$W_2^{(2)}(t) = E(A(t)|r(t) = 0.07).$$

We can then see that

$$W_1^{(1)}(t-dt) = W_1^{(1)}(t)(1 + 0.04dt + \mu_{30+t}dt)(1 - 0.8dt) + 0.8dtW_2^{(1)}(t) + o(dt)$$

leading to

$$\frac{dW_1^{(1)}(t)}{dt} = -(0.04 + \mu_{30+t})W_1^{(1)}(t) + 0.8\Big(W_1^{(1)}(t) - W_2^{(1)}(t)\Big).$$

Also

$$W_1^{(2)}(t-dt) = W_1^{(2)}(t)(1 - 0.8dt) + (0.04 - 0.05)W_1^{(1)}(t)V_t dt + 0.8dtW_2^{(2)}(t) + o(dt)$$

leading to

$$\frac{dW_1^{(2)}(t)}{dt} = -(0.04 - 0.05)W_1^{(1)}(t)V_t + 0.8\Big(W_1^{(2)}(t) - W_2^{(2)}(t)\Big).$$

So we have to solve the following system of equations

$$\frac{dV_t}{dt} = (0.05 + \mu_{30+t})V_t + 3340 - 200000\mu_{30+t}$$

$$\frac{dW_1^{(1)}(t)}{dt} = -(0.04 + \mu_{30+t})W_1^{(1)}(t) + 0.8\Big(W_1^{(1)}(t) - W_2^{(1)}(t)\Big)$$

$$\frac{dW_2^{(1)}(t)}{dt} = -(0.07 + \mu_{30+t})W_2^{(1)}(t) + 0.8\Big(W_2^{(1)}(t) - W_1^{(1)}(t)\Big)$$

$$\frac{dW_1^{(2)}(t)}{dt} = -(0.04 - 0.05)W_1^{(1)}(t)V_t + 0.8\Big(W_1^{(2)}(t) - W_2^{(2)}(t)\Big)$$

$$\frac{dW_2^{(2)}(t)}{dt} = -(0.07 - 0.05)W_2^{(1)}(t)V_t + 0.8\Big(W_2^{(2)}(t) - W_1^{(2)}(t)\Big)$$

subject to the terminal conditions: $V_{30} = 200000$, $W_1^{(1)}(30) = W_2^{(1)}(30) = 1$ and $W_2^{(1)}(30) = W_2^{(2)}(30) = 0$. The prediction value is $W_2^{(2)}(0)$.

b) The prediction value is $W_1^{(1)}(5)U(5) + W_1^{(2)}(5)$. This means we have to calculate $U(5)$. This can be done by solving

$$\frac{dU(t)}{dt} = \big(\mathbf{1}_{\{t<1\}}0.07 + \mathbf{1}_{\{t>1\}}0.04 + \mu_{30+t}\big)U(t) + 3340 - 200000\mu_{30+t}$$

subject to the initial condition $U(0) = 0$. $W_1^{(1)}(5)$ and $W_1^{(2)}(5)$ are calculated from the system of equations in (a).

4.

a) Let $p_1(t)$ be the probability that the life has no symptoms, $p_2(t)$ be the probability that the life has mild symptoms and $p_3(t)$ be the probability that the life has a full version of the disease. We then have

$$p_1(t) = \exp(-0.05t)_t p_x$$

and

$$p_2(t) = \int_0^t p_1(s)\exp(-0.1(t-s))\,_{t-s}p_{x+s}ds =$$

$$\int_0^t \exp(-0.05s)\,_s p_x \exp(-0.1(t-s))\,_{t-s}p_{x+s}ds =$$

$$\exp(-0.1t)_t p_x \int_0^t \exp(0.05s)ds = (\exp(-0.05t) - \exp(-0.1t))\,_t p_x\,.$$

Also

$$p_3(t) = \int_0^t p_2(s)\exp(-0.3(t-s))\,_{t-s}p_{x+s}ds =$$

$$p_3(t) = \int_0^t (\exp(-0.05s) - \exp(-0.1s))\,_s p_x \exp(-0.3(t-s))\,_{t-s}p_{x+s}ds =$$

$$\exp(-0.3t)_t p_x \int_0^t (\exp(0.25s) - \exp(-0.2s))\mathrm{ds} =$$

$$(0.4\exp(-0.05t) - 0.5\exp(-0.1t) + 0.1\exp(-0.3t))\,_t p_x.$$

b) This means that the life will die while healthy; the probability is (actuarial functions at 0.05)

$$\int_0^\infty \exp(-0.05t)_t p_x \mu_{x+t}dt = \bar{A}_{30} = 1 - 0.05\bar{a}_{30} = 1 - 0.05 \times 17.101 = 0.145.$$

c) The value of the first benefit is

$$100000\int_0^\infty \exp(-0.05t)p_2(t)\mu_{x+t}dt =$$

$$100000\int_0^\infty \exp(-0.05t)(\exp(-0.05t) - \exp(-0.1t))\,_t p_x \mu_{x+t}dt =$$

$$100000\int_0^\infty (\exp(-0.1t) - \exp(-0.15t))\,_t p_x \mu_{x+t}dt = 10000\left(\bar{A}_x^{(0.1)} - \bar{A}_x^{(0.15)}\right)$$

For $x = 30$, this is

$$100000((1 - 0.1 \times 9.605) - (1 - 0.15 \times 6.544)) = 2110$$

The value of the second benefit is

$$100000\int_0^\infty \exp(-0.05t)p_3(t)dt =$$

$$10000\int_0^\infty \exp(-0.05t)(0.4\exp(-0.05t)-0.5\exp(-0.1t)+0.1\exp(-0.3t))\,{}_tp_x dt$$

$$= 10000\Big(0.4\bar{a}_x^{(0.1)} - 0.5\bar{a}_x^{(0.15)} + 0.1\bar{a}_x^{(0.35)}\Big).$$

For $x = 30$, this is

$$10000(0.4\times 9.605 - 0.5\times 6.544 + 0.1\times 2.842) = 8542$$

The answer is $8542 + 2110 = 10652$.

d) First note that for a force of interest of 0.1

$$\bar{a}_{60} = \bar{a}_{30} - {}_{30}p_{30}\exp(-3)\,\bar{a}_{30:\overline{30}} = 9.605 - 0.845\times\exp(-3)\times 9.283 = 9.214.$$

The value of the reserve at $t = 30$ is the present value for a 60 year old. That is

$$10000\Big(\bar{A}_{60}^{(0.1)} - \bar{A}_{60}^{(0.15)}\Big) + 10000\Big(0.4\bar{a}_{60}^{(0.1)} - 0.5\bar{a}_{60}^{(0.15)} + 0.1\bar{a}_{60}^{(0.35)}\Big) =$$

$$10000((1-0.1\times 9.214) - (1-0.15\times 6.211)) +$$

$$10000(0.4\times 9.214 - 0.5\times 6.211 + 0.1\times 2.708) = 1025 + 8509 = 9534$$

So the value of the product is

$$10652 - \exp(-3)\times 0.845\times 9534 = 10251$$

5.

a) The two lives are exposed while they are both alive to mortality rate $m(30+t)$ and $m(40+t)$ independently of each other. On top of this they are exposed to a force $m2(t) = \frac{1}{400-t}$ that causes them to die together. When the 30 year old alone is alive (s)he is exposed to a force of mortality of $m(30+t) + m2(t)$ and when the 40 year old alone is alive (s)he is exposed to a force of mortality of $m(40+t) + m2(t)$.

b) 100000 payable on the second death provided this happens within 20 years. The premium is payable while they are both alive and for no more than 20 years.

c) The first one is the expected present value of the benefits at the outset. The second one is the expected present value of an annuity of 1 per annum payable for as long as both lives are alive. The third is the reserve the office should hold at time 10 if at the time only the 30 year old is alive.

d)

$$\frac{3900.12\times 1.01}{12.63\times 0.98} = 308.80\times\frac{1.01}{0.98} = 318.25$$

e) This means also getting a new 10 yer old policy at that time. The premium for this will be

$$\frac{2425.06}{7.6885} = 315.41.$$

The total premium from then on will be 308.80+315.41=624.21.

f) As we notice in (18) the reserve is negative at $t = 15$ and is probably negative till the end of the policy. If the policy lapses during that period the office will make a loss.

g) Let $V_{\mathrm{xy}}(t)$ be the reserve at time t while they are both alive, $V_x(t)$ be the reserve at time t while x only is alive and $V_y(t)$ be the reserve at time t while y only is alive. Let μ_{x+t} be the mortality force of x at time t and μ_{y+t} be the mortality force of y at time t. Let also r be the force of interest and π be the premium. $V_{\mathrm{xy}}(t)$ satisfies

$$\frac{dV_{\mathrm{xy}}(t)}{dt} = rV_{\mathrm{xy}}(t) - \mu_{y+t}(V_x(t) - V_{\mathrm{xy}}(t)) + \mu_{y+t}(V_x(t) - V_{\mathrm{xy}}(t)) + \pi.$$

Note also that $V_{\mathrm{xy}}(n-) = V_x(n-) = V_y(n-) = 0$ which then implies that

$$\lim_{t \to n} \frac{dV_{\mathrm{xy}}(t)}{dt} = \pi > 0.$$

This means that $V_{\mathrm{xy}}(t)$ is an increasing function in an interval $(t_1,\ n)$. Since $V_{\mathrm{xy}}(n-) = 0$ that means that $V_{\mathrm{xy}}(t)$ will be negative in this interval.