Summer 2009 Examination

ST305

Actuarial Mathematics: Life
Suitable for all students

Instructions to candidates

Time allowed: 3 hours

This paper contains 5 questions. Answer **All** questions.

You may use an electronic calculator (as prescribed by the examination regulations)

Taxes and expenses should be ignored unless otherwise indicated.

1. Two independent lives are aged x and y and at time t they are subject to mortality forces μ_{x+t} and μ_{y+t}. The force of interest is r. Provide an integral expression involving the forces and the corresponding survival probabilities only, for the expected value of a benefit of 1 payable on the second death provided this happens within 10 years of the first death.

(7 *marks*)

2. The following calculations might be useful to you for this question. They all refer to a force of mortality μ_x.
At a force of interest 0.02

$\bar{a}_{30:40} = 19.17$ $\qquad\qquad$ $\bar{a}_{30:30:40:40} = 23.07$

At a force of interest 0.03

$\bar{a}_{30:40} = 16.88$ $\qquad\qquad$ $\bar{a}_{30:30:40:40} = 19.82$

(a) Show that

$$\bar{A}_{xy\overset{1}{x}y} + \bar{A}_{xyx\overset{1}{y}} = \frac{1}{2}\bar{A}_{xyxy}$$

(2 *marks*)

(b) Two sisters aged 30 and 40 are subject to the mortality forces μ_{30+t} and μ_{40+t} at time t. They are taking care of a disabled relative who is subject to a mortality force $0.01 + \mu_{30+t} + \mu_{40+t}$ at time t. The deaths of the three lives are independent events. An insurance contact provides a benefit of 200000 when one of the two sisters dies while the disabled relative is still alive. If the second sister dies as well while the disabled relative remains alive, then the relative will receive a life annuity of 50000 per annum payable continuously. Calculate the single premium to be paid at the outset for these benefits using a force of interest of 0.02 per annum. (9 *marks*)

(Total: 11 *marks*)

3. The flow-chart in Figure 1 shows the state space and the transition intensities as functions of policy duration t for a life insurance policy with surrender and lapse options issued to an x-year old. Suppose that the benefit is an n-year term insurance with sum assured b payable

immediately upon death and that the premiums are an n-year life annuity payable continuously at level rate c. If the insured surrenders the policy at time t, then he receives the one-off surrender value $w(t)$. If he lapses at time t, then the sum asssured is reduced to $q(t)\,b$, where $q(t) \in [0,1]$. Assume that the interest rate r is constant.

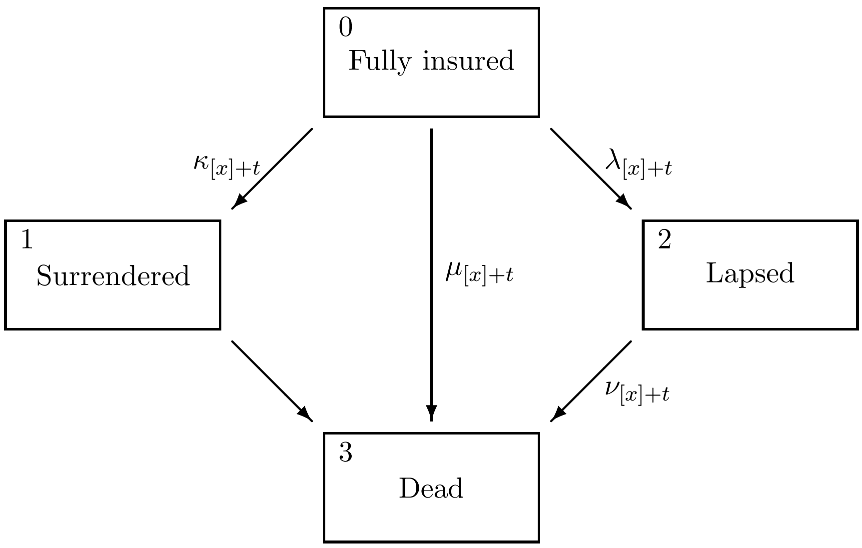

Figure 1: A model for a single-life insurance with surrender and lapse options.

In state 1 the policy has expired and no reserve is to be provided; $V_1(t) = 0$ for all t. In state 2, at policy duration t and state duration s, the reserve is

$$V_2(s,t) = q(t-s)\,\bar{V}_2(t)\,, \tag{1}$$

where $\bar{V}_2(t)$ is the premium reserve in respect of the full benefit specified in the contract:

$$\bar{V}_2(t) \;=\; b \int_t^n e^{-\int_t^\tau (r+\nu_{[x]+u})\,du}\,\nu_{[x]+\tau}\,d\tau\,. \tag{2}$$

(a) Write down without proof the general partial Thiele's differential equations for the state-wise reserves when payments may depend on state duration. Explain why the reserve V_0 in state 0 in the present situation is a function of policy duration only and is given by the ordinary differential equation

$$\begin{aligned}\frac{d}{dt}V_0(t) &= V_0(t)\,r + c - \mu_{[x]+t}\,(b - V_0(t)) \\ &\quad - \kappa_{[x]+t}\,(w(t) - V_0(t)) - \lambda_{[x]+t}\,(q(t)\bar{V}_2(t) - V_0(t)),\end{aligned}$$

subject to the terminal condition $V_0(n-) = 0$.

(6 *marks*)

(b) The functions w and q, which are not specified in the contract, are to be determined. Motivate the standard choice, which is $w(t) = V_0(t)$ and $q(t) = V_0(t)/\bar{V}_2(t)$.

(5 *marks*)

(c) Observe that, with standard w and q as in (b), the reserve $V_0(t)$ becomes the same as for a standard policy without options. Give an intuitive explanation to this.

(5 *marks*)

(d) Using the general partial Thiele's differential equations in (a), write down the partial differential equation for $V_2(s,t)$. Verify that the solution is indeed the function given by (1) and (2).

(6 *marks*)

(Total: 22 *marks*)

4. Under an occupational pension scheme the contributions are a fixed proportion π of the salary and the benefit is an endowment payable upon retirement at age 65. At any calendar time t the scheme's investments yield interest at rate $r(t)$ and its y-year old members have mortality intensity $\mu(t,y)$. Consider a member who enters the scheme at age x at time 0 (say), earns salary at rate $S(t) = e^{\int_0^t a(u)\,du}$ at any time t between 0 and $m := 65 - x$, and will receive an endowment of K at time m. For given outcomes of interest, mortality, and salary, the expected present value at time 0 of the contributions is

$$C := \pi \int_0^m e^{\int_0^\tau (a(u) - r(u) - \mu(u, x+u))\,du}\,d\tau\,,$$

and the expected present value at time 0 of the benefit is

$$B := e^{-\int_0^m (r(u) + \mu(u,x+u))\,du}\,K\,.$$

(a) If the scheme is of "defined contribution" type, then the endowment K is given by the equivalence relation $B = C$, that is,

$$K = \pi \int_0^m e^{\int_0^\tau a(u)\,du + \int_\tau^m (r(u)+\mu(u,x+u))\,du}\, d\tau\,.$$

Since equivalence is attained for any outcome of interest, mortality, and salary, there is no need to model these factors for the purpose of assessing the insolvency risk. However, for the purpose of forecasting future pension benefits, the actuary of the pension fund uses the following model. The economic and demographic environment is governed by a continuous time Markov chain $Y(t)$, $t \geq 0$, with state space $\mathcal{Y} = \{1, \ldots, J\}$ and constant intensities of transition λ_{ef}, $e \neq f$. If $Y(t) = e$, then $r(t) = r_e$, $a(t) = a_e$, $\mu(t, y) = m_e\,\mu_y$, where (r_e, a_e, m_e), $e = 1, \ldots, J$, and μ_y, $y > 0$, are known. Explain in detail how to calculate the conditional expected value of K at time t, given the past history of Y. (You need to derive differential equations.)

(14 *marks*)

(b) If the scheme is of "defined benefits" type and based on final salary, then the endowment K is

$$K := k\,S(m) := k\,e^{\int_0^m a(u)\,du}\,, \tag{3}$$

where k is fixed at the outset. Equivalence is not guaranteed in general for all outcomes of interest, mortality, and salary. Outline how the insolvency risk could be managed by letting k fulfil the equivalence requirement for prudently chosen first order elements a^* (constant), r^* (constant), and μ^*_y, $y > 0$, and by adding bonus to the contractual endowment at time m.

(8 *marks*)

(Total: 22 *marks*)

5. The last two pages are the printout of a MAPLE worksheet with calculations concerning a 35 year health insurance product on a 30 year old life. Sickness has been split into two types A and B . The benefits are a continuous payment of 20000 per annum during any period of type A sickness and a continuous payment of 30000 per annum during any period of type B sickness. The force of interest is 0.05 per annum and there are no expenses.

(a) Describe the Markov Chain model underpinning the contract identifying the appropriate transition forces. You may draw a diagram if you like. (5 *marks*)

(b) What is unusual about one of the transition forces? (2 *marks*)

(c) A 50 year old life has just fallen sick. What is the probability this is type B sickness? (4 *marks*)

(d) What is the premium and when is it paid? (3 *marks*)

(e) An alternative would be a continuous premium payable throughout the duration of the policy. What would be the problem with this? (3 *marks*)

(f) What do the amounts of 16668.87 in (10), 39055.62 in (12), 12.75 in (16) and 6462.40 in (24) represent? (12 *marks*)

(g) Suppose now that there are expenses 2% of the premium and 3% of the benefits. What would the premium be in this case? (4 *marks*)

(h) Now ignore expenses again. At time $t = 10$ the policyholder while healthy wants to double the benefits for the rest of the duration of the policy. The premium will still be payable as before but will be increased. What would the new premium be? (5 *marks*)

(Total: 38 *marks*)

This is the force of mortality for healthy lives

```
> m := t -> 0.0005+0.00007585775*10^(0.038*t);
```

$$m := t \rightarrow 0.0005 + 0.00007585775\ 10^{(0.038\ t)} \tag{1}$$

We now define the mortality force for sick lives (both kinds)

```
> ms:= t -> 0.0002+m(t+5);
```

$$ms := t \rightarrow 0.0002 + m(t+5) \tag{2}$$

This is the force of transition from the healthy state to the type A sick state.

```
> sa := t -> 0.02*(1+t/50);
```

$$sa := t \rightarrow 0.02\left(1 + \frac{1}{50}\,t\right) \tag{3}$$

This is the force of transition from the healthy state to the type B sick state.

```
> sb := t -> 0.03*(1-t/200);
```

$$sb := t \rightarrow 0.03\left(1 - \frac{1}{200}\,t\right) \tag{4}$$

This is the force of transition from the type A sick state to the healthy state.

```
> ra := t -> 3*(1+t/40)^(-1);
```

$$ra := t \rightarrow \frac{3}{1 + \frac{1}{40}\,t} \tag{5}$$

This is the force of transition from the type B sick state to the healthy state.

```
> rb := t -> 2*(1+t/50)^(-1);
```

$$rb := t \rightarrow \frac{2}{1 + \frac{1}{50}\,t} \tag{6}$$

Consider the following system of equations

```
> dsys1 := {diff(wh(t),t)=(m(30+t)+sa(30+t)+sb(30+t)+0.05)*wh(t)-sa
  (30+t)*wa(t)-sb(30+t)*wb(t), diff(wa(t),t)=(ms(30+t)+ra(30+t)
  +0.05)*wa(t)-ra(30+t)*wh(t)-20000,diff(wb(t),t)=(ms(30+t)+rb(30+t)
  +0.05)*wb(t)-rb(30+t)*wh(t)-30000, wh(35)=0, wa(35)=0, wb(35)=0};
```

$$dsys1 := \Bigg\{ wh(35)=0,\ wa(35)=0,\ wb(35)=0,\ \frac{\mathrm{d}}{\mathrm{d}t}\,wh(t) = \big(0.1080000000 + 0.00007585775\ 10^{(1.140\,+\,0.038\,t)} + 0.0002500000000\,t\big)\,wh(t) - (0.03200000000 + 0.0004000000000\,t)\,wa(t) - (0.02550000000 - 0.0001500000000\,t)\,wb(t),\ \frac{\mathrm{d}}{\mathrm{d}t}\,wa(t) = \left(0.0507 + 0.00007585775\ 10^{(1.330\,+\,0.038\,t)} + \frac{3}{\frac{7}{4} + \frac{1}{40}\,t}\right)wa(t) - \frac{3\,wh(t)}{\frac{7}{4} + \frac{1}{40}\,t} - 20000,\ \frac{\mathrm{d}}{\mathrm{d}t}\,wb(t) = \left(0.0507 + 0.00007585775\ 10^{(1.330\,+\,0.038\,t)} + \frac{2}{\frac{8}{5} + \frac{1}{50}\,t}\right)wb(t) - \frac{2\,wh(t)}{\frac{8}{5} + \frac{1}{50}\,t} - 30000 \Bigg\} \tag{7}$$

```
> dsol1 := dsolve(dsys1, numeric, range = 0..35);
```

$$dsol1 := \mathbf{proc}(x\_rkf45)\ \ldots\mathbf{end\ proc} \tag{8}$$

```
> dsol1(35);
```

$$[t=35.,\ wa(t)=0.,\ wb(t)=0.,\ wh(t)=0.] \tag{9}$$

```
> dsol1(0);
```

$$[t=0.,\ wa(t)=27541.2928805420488,\ wb(t)=39184.1285203428051,\ wh(t)=16668.8662092926098] \tag{10}$$

```
>  dsol1(10);
```

$$[t=10.,\ wa(t)=27509.8876665882782,\ wb(t)=40274.0039858502642,\ wh(t)=15243.1450327642961] \tag{11}$$

```
> dsol1(20);
```

$$[t=20.,\ wa(t)=25260.4576817229609,\ wb(t)=39055.6241655602789,\ wh(t)=11677.5567780068122] \tag{12}$$

Consider also the system

```
> dsys2 := {diff(vh(t),t)=(m(30+t)+sa(30+t)+sb(30+t)+0.05)*vh(t)-sa
  (30+t)*va(t)-sb(30+t)*vb(t)-1, diff(va(t),t)=(ms(30+t)+ra(30+t)
  +0.05)*va(t)-ra(30+t)*vh(t),diff(vb(t),t)=(ms(30+t)+rb(30+t)+0.05)
  *vb(t)-rb(30+t)*vh(t), vh(25)=0, va(25)=0, vb(25)=0};
```

$$dsys2 := \left\{ vh(25)=0,\ va(25)=0,\ vb(25)=0,\ \frac{d}{dt}\,vh(t)=\left(0.1080000000 + 0.00007585775\,10^{(1.140\,+\,0.038\,t)} + 0.0002500000000\,t\right) vh(t) - (0.03200000000 + 0.0004000000000\,t)\,va(t) - (0.02550000000 - 0.0001500000000\,t)\,vb(t) - 1,\ \frac{d}{dt}\,va(t) = \left(0.0507 + 0.00007585775\,10^{(1.330\,+\,0.038\,t)} + \frac{3}{\frac{7}{4}+\frac{1}{40}\,t}\right) va(t) - \frac{3\,vh(t)}{\frac{7}{4}+\frac{1}{40}\,t},\ \frac{d}{dt}\,vb(t) = \left(0.0507 + 0.00007585775\,10^{(1.330\,+\,0.038\,t)} + \frac{2}{\frac{8}{5}+\frac{1}{50}\,t}\right) vb(t) - \frac{2\,vh(t)}{\frac{8}{5}+\frac{1}{50}\,t} \right\} \tag{13}$$

```
> dsol2 := dsolve(dsys2, numeric, range = 0..25);
```

$$dsol2 := \mathbf{proc}(x\_rkf45)\ \ldots\mathbf{end\ proc} \tag{14}$$

```
> dsol2(25);
```

$$[t=25.,\ va(t)=0.,\ vb(t)=0.,\ vh(t)=0.] \tag{15}$$

```
> dsol2(0);
```

$$[t=0.,\ va(t)=12.7473174367779798,\ vb(t)=12.5483860031442784,\ vh(t)=13.3025545626635200] \tag{16}$$

```
> dsol2(10);
```

$$[t=10.,\ va(t)=9.16140722565402044,\ vb(t)=8.95014113326149286,\ vh(t)=9.79175234958992212] \tag{17}$$

```
> dsol2(20);
```

$$[t=20.,\ va(t)=3.46511954679237588,\ vb(t)=3.24954808505827320,\ vh(t)=4.16191844674697010] \quad (18)$$

```
> evalf(16668.8662092926098/13.3025545626635200);
```

$$1253.057534 \quad (19)$$

```
>
```

Finally, consider the system

```
> dsys3 := {diff(uh(t),t)=(m(30+t)+sa(30+t)+sb(30+t)+0.05)*uh(t)-sa(30+t)*ua(t)-sb(30+t)*ub(t)+1253.057534*0.5*(signum(25-t)+1), diff(ua(t),t)=(ms(30+t)+ra(30+t)+0.05)*ua(t)-ra(30+t)*uh(t)-20000,diff(ub(t),t)=(ms(30+t)+rb(30+t)+0.05)*ub(t)-rb(30+t)*uh(t)-30000, uh(35)=0, ua(35)=0, ub(35)=0};
>
```

$$dsys3 := \left\{ uh(35)=0,\ ua(35)=0,\ ub(35)=0,\ \frac{d}{dt}\,ua(t) = \left(0.0507 + 0.00007585775\ 10^{(1.330\ +\ 0.038\ t)} + \frac{3}{\frac{7}{4}+\frac{1}{40}\,t}\right) ua(t) - \frac{3\ uh(t)}{\frac{7}{4}+\frac{1}{40}\,t} - 20000,\ \frac{d}{dt}\,uh(t) = \left(0.1080000000 + 0.00007585775\ 10^{(1.140\ +\ 0.038\ t)} + 0.0002500000000\ t\right) uh(t) - (0.03200000000 + 0.0004000000000\ t)\ ua(t) - (0.02550000000 - 0.0001500000000\ t)\ ub(t) - 626.5287670\ \mathrm{signum}(t-25) + 626.5287670,\ \frac{d}{dt}\,ub(t) = \left(0.0507 + 0.00007585775\ 10^{(1.330\ +\ 0.038\ t)} + \frac{2}{\frac{8}{5}+\frac{1}{50}\,t}\right) ub(t) - \frac{2\ uh(t)}{\frac{8}{5}+\frac{1}{50}\,t} - 30000 \right\} \quad (20)$$

```
> dsol3 := dsolve(dsys3, numeric, range = 0..35);
```

$$dsol3 := \mathbf{proc}(x\_rkf45)\ \ldots\mathbf{end\ proc} \quad (21)$$

```
> dsol3(0);
```

$$[t=0.,\ ua(t)=11568.1588935817964,\ ub(t)=23460.2664675624364,\ uh(t)=-0.0126199712412358168] \quad (22)$$

```
> dsol3(35);
```

$$[t=35.,\ ua(t)=0.,\ ub(t)=0.,\ uh(t)=0.] \quad (23)$$

```
> dsol3(20);
```

$$[t=20.,\ ua(t)=20918.4243948118746,\ ub(t)=34983.7146781015034,\ uh(t)=6462.39698463377500] \quad (24)$$

```
> dsol3(10);
```

$$[t=10.,\ ua(t)=16030.0905199441695,\ ub(t)=29058.9372411689183,\ uh(t)=2973.49495855669012] \quad (25)$$

```
>
```