ST305 2009 Solutions

1. An expression is

$$\int_0^{10}\left[{}_tp_x\left(1-{}_tp_y\right)\mu_{x+t}+{}_tp_y\left(1-{}_tp_x\right)\mu_{y+t}\right]dt+$$

$$\int_{10}^{\infty}\left[{}_tp_x\left({}_{t-10}p_y-{}_tp_y\right)\mu_{x+t}+{}_tp_y\left({}_{t-10}p_x-{}_tp_x\right)\mu_{y+t}\right]dt$$

where

$${}_tp_x=\exp\left(-\int_0^t\mu_{x+s}ds\right)$$

and

$${}_tp_y=\exp\left(-\int_0^t\mu_{y+s}ds\right)$$

2. a) We have

$$\bar{A}_{xyxy}=\int_0^{\infty}\left(\mu_{x+t}+\mu_{y+t}+\mu_{x+t}+\mu_{y+t}\right){}_tp_x{}_tp_x{}_tp_y{}_tp_ydt$$

and

$$\bar{A}_{xy\overset{1}{x}y}+\bar{A}_{xyx\overset{1}{y}}=\int_0^{\infty}\left(\mu_{x+t}+\mu_{y+t}\right){}_tp_x{}_tp_x{}_tp_y{}_tp_ydt.$$

Hence

$$\bar{A}_{xy\overset{1}{x}y}+\bar{A}_{xyx\overset{1}{y}}=\frac{1}{2}\bar{A}_{xyxy}.$$

b) The survival probability function of the disabled relative is

$$\exp\left(-\int_0^t\left(0.01+\mu_{30+s}+\mu_{40+s}\right)ds\right)=e^{-0.01t}{}_tp_{30}{}_tp_{40}$$

Therefore, the value of the first death benefit is

$$200000\int_0^{\infty}e^{-0.02t}e^{-0.01t}{}_tp_{30}{}_tp_{40}{}_tp_{30}{}_tp_{40}\left(\mu_{30+s}+\mu_{40+s}\right)ds=$$

$$200000\int_0^{\infty}e^{-0.03t}{}_tp_{30}{}_tp_{40}{}_tp_{30}{}_tp_{40}\left(\mu_{30+s}+\mu_{40+s}\right)ds=100000\bar{A}_{30:30:40:40}$$

calculated at a force of interest 0.03 so it is equal to

$$100000\left(1-0.03\times 19.82\right)=40540.$$

The value of the other death benefit is

$$50000\int_0^{\infty}e^{-0.02t}e^{-0.01t}{}_tp_{30}{}_tp_{40}\left(1-{}_tp_{30}\right)\left(1-{}_tp_{40}\right)dt=$$

$$50000\left(\bar{a}_{30:40}+\bar{a}_{30:40:30:40}-\bar{a}_{30:30:40}-\bar{a}_{30:40:40}\right)$$

(all calculated at 0.03). Unfortunately the values of the last two annuities are not given.

3. a) The general partial Thiele equations are

$$
\begin{aligned}
\frac{\partial}{\partial t}V_j(s,t) &= r(t)V_j(s,t) - \frac{\partial}{\partial s}V_j(s,t) - b_j(s,t) \\
&\quad - \sum_{k;k\neq j} \mu_{jk}(s,t)\,(b_{jk}(s,t) + V_k(0,t) - V_j(s,t))\,.
\end{aligned}
$$

In state 0 the state duration coincides with policy (it is the initial state with no return) duration and is therefore redundant. Inserting the particulars of the situation in the general Thiele equation, one obtains the claimed result.

b) The reserve can be seen as the insured's outstanding debt from the insurer. With the standard choice the insured gets this debt back upon alteration of the contract: upon surrender he gets the debt back in cash, and upon lapse the outstanding debt is used as a single premium for a benefit of the same form as specified in the contract.

c) The differential equation becomes

$$
\frac{d}{dt}V_0(t) = V_0(t)\,r + c - \mu_{[x]+t}\,(b - V_0(t))\,,
$$

which is just the one for the contract without surrender and lapse options. The intuitive explanation is that upon alteration of the contract the insured gets his outstanding debt back, so nothing is bequeathed to those who stay in the scheme on normal terms and conditions. In particular, the premium rate c, determined by the equivalence requirement $V_0(0) = 0$, is not affected by the options.

d) The partial differential equation for $V_2(s,t)$ is

$$
\frac{\partial}{\partial t}V_2(s,t) = V_2(s,t)\,r - \frac{\partial}{\partial s}V_2(s,t) - \nu_{[x]+t}\,(b - V_2(s,t))
$$

We need to check that the function in (**??**) satisfies the PDE. For this $V_2(s,t)$ we have

$$
V_2(s,t) = q(t-s)\,\bar{V}_2(t)\,,
$$

$$
\frac{\partial}{\partial t}V_2(s,t) = q'(t-s)\,\bar{V}_2(t) + q(t-s)\,\left(\bar{V}_2(t)\,r - \nu_{[x]+t}\,b\right)
$$

and

$$
\frac{\partial}{\partial s}V_2(s,t) = -q'(t-s)\,\bar{V}_2(t)\,.
$$

which fits into the differential equation. The terminal condition is clearly satisfied.

4. a) Start from the conditional expectation of K given $\mathcal{F}_t^Y$:

$$E\left[\int_0^m e^{\int_0^\tau a(u)\,du + \int_\tau^m (r(u)+\mu(u,x+u))\,du}\,d\tau \,\middle|\, \mathcal{F}_t\right]$$

$$= \int_0^t e^{\int_0^\tau a(u)\,du + \int_\tau^t (r(u)+\mu(u,x+u))\,du}\,d\tau \; E\left[e^{\int_t^m (r(u)+\mu(u,x+u))\,du} \,\middle|\, \mathcal{F}_t\right]$$

$$+ e^{\int_0^t a(u)\,du}\, E\left[\int_t^m e^{\int_t^\tau a(u)\,du + \int_\tau^m (r(u)+\mu(u,x+u))\,du}\,d\tau \,\middle|\, \mathcal{F}_t\right].$$

We need to determine the functions

$$W_e^{(1)}(t) \;=\; E\left[e^{\int_t^m (r(u)+\mu(u,x+u))\,du} \,\middle|\, Y(t)=e\right],$$

$$W_e^{(2)}(t) \;=\; E\left[\int_t^m e^{\int_t^\tau a(u)\,du + \int_\tau^m (r(u)+\mu(u,x+u))\,du}\,d\tau \,\middle|\, Y(t)=e\right].$$

Direct backward construction for $W_e^{(1)}(t)$:

$$\begin{aligned}
W_e^{(1)}(t) \;&=\; (1-\lambda_{e\cdot}\,dt)\,e^{(r_e+m_e\mu_{x+u})\,dt}\,W_e^{(1)}(t+dt) \;+\; \sum_{f;\,f\neq e}\lambda_{ef}\,dt\,W_f^{(1)}(t) \;+\; o(dt)\\
&=\; (1-\lambda_{e\cdot}\,dt)\;(1+(r_e+m_e\mu_{x+u})\,dt)\left(W_e^{(1)}(t)+\frac{d}{dt}W_e^{(1)}(t)\,dt\right)\\
&\qquad + \sum_{f;\,f\neq e}\lambda_{ef}\,dt\,W_f^{(1)}(t) \;+\; o(dt)\\
&=\; W_e^{(1)}(t) \;+\; (-\lambda_{e\cdot}+r_e+m_e\mu_{x+u})\;dt\,W_e^{(1)}(t)+\frac{d}{dt}W_e^{(1)}(t)\,dt\\
&\qquad + \sum_{f;\,f\neq e}\lambda_{ef}\,dt\,W_f^{(1)}(t) \;+\; o(dt)
\end{aligned}$$

gives the differential equation

$$\frac{d}{dt}W_e^{(1)}(t) \;=\; (\lambda_{e\cdot}-r_e-m_e\mu_{x+u})\;W_e^{(1)}(t) \;-\; \sum_{f;\,f\neq e}\lambda_{ef}\,W_f^{(1)}(t)$$

with side conditions

$$W_e^{(1)}(m-) \;=\; 1\,.$$

Direct backward construction for $W_e^{(2)}(t)$:

$$\begin{aligned}
W_e^{(2)}(t) \;&=\; (1-\lambda_{e\cdot}\,dt)\left[W_e^{(1)}(t)\,dt \;+\; e^{a_e\,dt}\,W_e^{(2)}(t+dt)\right] \;+\; \sum_{f;\,f\neq e}\lambda_{ef}\,dt\,W_f^{(2)}(t) \;+\; o(dt)\\
&=\; (1-\lambda_{e\cdot}\,dt)\left[W_e^{(1)}(t)\,dt \;+\; (1\;+\;a_e\,dt)\left(W_e^{(2)}(t)\;+\;\frac{d}{dt}W_e^{(2)}(t)\,dt\right)\right]\\
&\qquad + \sum_{f;\,f\neq e}\lambda_{ef}\,dt\,W_f^{(2)}(t) \;+\; o(dt)\\
&=\; W_e^{(1)}(t)\,dt \;+\; (1+(a_e-\lambda_{e\cdot})\,dt)\;W_e^{(2)}(t) \;+\; \frac{d}{dt}W_e^{(2)}(t)\,dt \;+\; \sum_{f;\,f\neq e}\lambda_{ef}\,dt\,W_f^{(2)}(t) \;+
\end{aligned}$$

gives the differential equation

$$\frac{d}{dt}W_e^{(2)}(t) \;=\; (\lambda_{e\cdot} - a_e)\, W_e^{(2)}(t) - W_e^{(1)}(t) - \sum_{f;\, f \neq e} \lambda_{ef}\, dt\, W_f^{(2)}(t) + o(dt)$$

with side conditions

$$W_e^{(2)}(m-) \;=\; 0\,.$$

b) k is determined by

$$\pi \int_0^m e^{\int_0^\tau \left(a^* - r^* - \mu^*_{x+u}\right) du}\, d\tau \;=\; k\, e^{\int_0^m \left(a^* - r^* - \mu^*_{x+u}\right) du}\,,$$

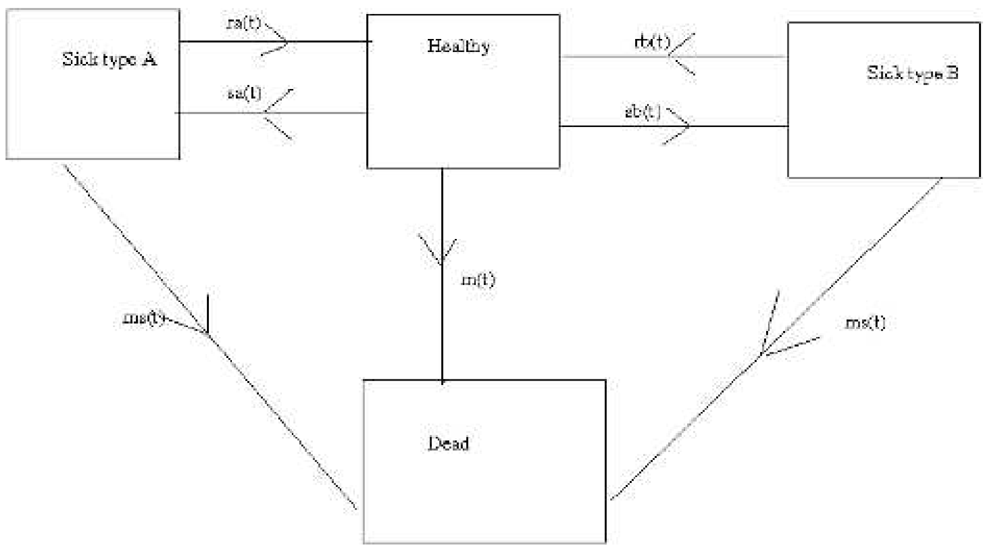
or

$$k \;=\; \pi \int_0^m e^{\int_\tau^m \left(-a^* + r^* + \mu^*_{x+u}\right) du}\, d\tau\,.$$

The endowment is

$$\begin{aligned} k\, S(m) \;&=\; \pi \int_0^m e^{\int_\tau^m \left(-a^* + r^* + \mu^*_{x+u}\right) du}\, d\tau\, e^{\int_0^m a(u)\, du} \\ &=\; \pi \int_0^m e^{\int_\tau^m (a(u) - a^*)\, du + \int_0^\tau a(u)\, du + \int_\tau^m \left(r^* + \mu^*_{x+u}\right) du}\, d\tau\,. \end{aligned}$$

If one chooses $a^* \geq a(u)$, $r *^{\leq} r(u)$, $\mu^*_{x+u} \leq \mu(u, x+u)$ for all u, then this endowment is less than or equal to the one in (a) which gives equivalence. One simply pays out the difference as bonus at time m.



5. a) See diagram
   b) The force for the type B sickness is decreasing with time.
   c)

$$\frac{0.03\left(1 - \frac{50}{200}\right)}{0.02\left(1 + \frac{50}{50}\right) + 0.03\left(1 - \frac{50}{200}\right)} = \frac{9}{25} = 0.36$$

d) The premium is 1253.06 per annum payable continuously while the policy holder is healthy. It is payable during the first 25 years of the policy only.

e) If the premium was payable till the end of the 35 years, there would be negative reserves at the end as there would not be enough time for the benefit to be received.

f) 16668.87 in (10): The single premium payable up front if the policy were to be financed this way.
39055.62 in (12): The value at time 20 of another 15 years of benefits assuming the policyholder is suffering from type B sickness at the time.
12.747 in (16): The value of an annuity of 1 payable for 25 years during periods of health by a 30 year old currently suffering form a type A sickness.
6462.40 in (24): The reserves the office should set aside for at time 20 if the policyholder is healthy at that time.

g)
$$\frac{16668.87 \times 1.03}{13.30255 \times 0.98} = 1253.06 \times \frac{1.03}{0.98} = 1316.99$$

h) This means also getting a new 25 year policy at that time. The premium for this will be payable for 15 years and it will be

$$\frac{15243.145}{9.79175} = 1556.73$$

The total premium from then on will be 1253.06+1556.73=2809.79.