2010 ST305 Solutions

1. We have

$$p_{x+1} = \frac{0.90}{0.96} = 0.9375$$

and so

$$p_{[x]} = \exp\left(- \int_0^1 0.75\mu_{x+s}\mathrm{ds} \right) = 0.96^{0.75} = 0.9698$$

and

$$p_{[x]+1} = \exp\left(- \int_0^1 0.9\mu_{x+1+s}\mathrm{ds} \right) = 0.9375^{0.9} = 0.9436$$

and

$$p_{[x]}p_{[x]+1} = 0.9151$$

Also

$$7.94 = \ddot{a}_x = 1 + \exp(-0.03)0.96 + 0.90\exp(-0.06)\ddot{a}_{x+2}$$

and so

$$\exp(-0.06)\ddot{a}_{x+2} = \frac{7.94 - 1 - \exp(-0.03)0.96}{0.90} = 6.676.$$

Hence

$$\ddot{a}_{[x]} = 1 + \exp(-0.03)0.9698 + 0.9151\exp(-0.06)\ddot{a}_{x+2} =$$

$$1 + \exp(-0.03)0.9698 + 0.9151 \times 6.676 = 8.0503.$$

2.

a)

$$\mathrm{CDR} = \frac{1+2+4+18}{400+1000+1500+2100} = \frac{25}{5000} = 0.005$$

The standardised mortality ratio is

$$\frac{1+2+4+18}{400 \times 0.0008 + 1000 \times 0.0010 + 1500 \times 0.0040 + 2100 \times 0.0110} = \frac{25}{30.42} = 0.822$$

b) The expected number of deaths is 30.2. The standard deviation is approximately

$$\sqrt{30.42} = 5.515$$

so testing the hypothesis there is no difference

$$\frac{25 - 30.42}{5.515} = -0.98$$

This is well below 1.96; there does not appear to be evidence they are different.

3.

a) By choosing $\alpha = 0$, the amount that goes into the cash fund is

$$\mu_{x+t}(b - U_t)_+,$$

so the cash fund does not grow at all as this is exactly enough to cover the death costs. Hence it stays at 0 all the time. At time n there might be a shortfall

$$(b - U_n)_+$$

which in general has a positive probability of being positive. The office would have made no provision to cover it.

b) The growth of the cash fund W_t is given by the equation

$$\frac{dW_t}{dt} = (r^* + \mu_{x+t})W_t + (1+\alpha)\mu_{x+t}(b - U_t)_+ - \mu_{x+t}(b - U_t)$$

with $W_0 = 0$. The growth of the unit fund is given by the equation

$$\frac{dU_t}{dt} = rU_t + \pi - (1+\alpha)\mu_{x+t}(b - U_t)_+$$

with $U_0 = 0$. If π is such that $\pi > (1+\alpha)\mu_{x+t}(b - U_t)_+$ the fund is increasing in value and since $U_n < b$, $U_t < b$ for all t and so

$$\frac{dU_t}{dt} = rU_t + \pi - (1+\alpha)\mu_{x+t}(b - U_t)$$

with $U_0 = 0$. Re write this as

$$\frac{dU_t}{dt} = (r + (1+\alpha)\mu_{x+t})\,U_t + \pi - (1+\alpha)\mu_{x+t}b$$

and we have

$$U_t = \int_0^t \exp\left(\int_s^t (r + (1+\alpha)\mu_{x+u})\mathrm{du}\right)(\pi - (1+\alpha)\mu_{x+s}b)\mathrm{ds}$$

The equation for W_t becomes

$$\frac{dW_t}{dt} = (r^* + \mu_{x+t})W_t + \alpha\mu_{x+t}(b - U_t)$$

and therefore

$$W_t = \int_0^t \exp\left(\int_s^t (r + \mu_{x+u})\mathrm{du}\right)\alpha\mu_{x+s}(b - U_s)\mathrm{ds}$$

where U_t is given above. We then set

$$\int_0^n \exp\left(\int_s^n (r + (1+\alpha)\mu_{x+u})\mathrm{du}\right)(\pi - (1+\alpha)\mu_{x+s}b)\mathrm{ds} +$$

$$\int_0^n \exp\left(\int_s^n (r + \mu_{x+u})\mathrm{du}\right)\alpha\mu_{x+s}(b - U_s)\mathrm{ds}$$

and solve for π.

4.

a)

$$p_H(t) = \exp\Big(- \int_0^t (\mu(s) + \sigma(s))\mathrm{ds}\Big)$$

b)

$$p_S(t) = \int_0^t \exp\Big(- \int_0^s (\mu(u) + \sigma(u))\mathrm{du}\Big)\sigma(s)\exp\Big(- \int_s^t (\nu(u) + \rho(u))\mathrm{du}\Big)\mathrm{ds}$$

c)

$$\int_q^n p_S(s-q)\exp\Big(- \int_{s-q}^s (\nu(u) + \rho(u))\mathrm{du}\Big)\exp(-rs)\mathrm{ds}$$

(In the integral, the life should be sick at time $s-q$ and remain sick till time s).

d) If

$$\int_{t+q-q_1}^n \exp\Big(- \int_t^s (\nu(u) + \rho(u))\mathrm{du}\Big)\exp(-r\,(s-t))\mathrm{ds}$$

when $q_1 < t \leq n - q + q_1$ and 0 when $t > n - q + q_1$.

e) The value of the **extra** benefit is

$$p_S(n-q)\int_n^\infty \exp\Big(- \int_{n-q}^s (\nu(u) + \rho(u))\mathrm{du}\Big)\exp(-rs)\mathrm{ds}$$

(life is sick at time $n-q$ and continues to be so at time $s > n$).

So the answer is

$$\int_q^n p_S(s-q)\exp\Big(- \int_{s-q}^s (\nu(u) + \rho(u))\mathrm{du}\Big)\exp(-rs)\mathrm{ds} +$$

$$p_S(n-q)\int_n^\infty \exp\Big(- \int_{n-q}^s (\nu(u) + \rho(u))\mathrm{du}\Big)\exp(-rs)\mathrm{ds}.$$

5.

a) Let V_t^* be the reserve based on the first order basis satisfies Thiele's differential equation

$$\frac{dV_t^*}{dt} = (r^* + \mu_{x+t}^*)V_t^* + \pi - \mu_{x+t}^* b$$

where π is such that $V_0^* = 0$ and $V_n^* = b$. (Students do not need to state the equation here but it will be used later).

The rate at which surplus emerges per survivor is given by

$$c_t = (r_t - r^*)V_t^* + (\mu_{x+t}^* - \mu_{x+t})(b - V_t^*).$$

b) The reserve (using a first order basis) W_t^* for a temporary assurance contract with amount payable b satisfies the equation

$$\frac{dW_t^*}{dt} = r^* W_t^* - \mu_{x+t}^*(b - W_t^*)$$

with $W_n^* = 0$. W_t^* is the single premium paid for such a contract if it is bought at time t. So the number of temporary assurance units purchased at time t is $q_t dt$ where

$$q_t = \frac{W_t^*}{c_t}$$

So the total number of additional temporary assurance units purchase up to time t is

$$Q_t = \int_0^t q_s \mathrm{ds}.$$

The total surplus generated at time t is then given by

$$c_t = (r_t - r^*)(V_t^* + Q_t W_t^*) + (\mu_{x+t}^* - \mu_{x+t})(b - V_t^* + Q_t(b - W_t^*)).$$

We therefore have the equations

$$\frac{dQ_t}{dt} = \frac{W_t^*}{(r_t - r^*)(V_t^* + Q_t W_t^*) + (\mu_{x+t}^* - \mu_{x+t})(b - V_t^* + Q_t(b - W_t^*))}$$

$$\frac{dW_t^*}{dt} = r^* W_t^* - \mu_{x+t}^*(b - W_t^*)$$

$$\frac{dV_t^*}{dt} = r^* V_t^* + \pi - \mu_{x+t}^*(b - V_t^*)$$

with side conditions $Q_0 = 0$, $W_n^* = 0$, $V_n^* = 0$, The second equation has a terminal condition attached to it, while the other two have initial conditions. To fix this we calculated W_0^* from

$$\frac{dW_t^*}{dt} = (r^* + \mu_{x+t}^*)W_t^* - \mu_{x+t}^*(b - W_t^*)$$

with $W_n^* = 0$ and the solve the system

$$\frac{dQ_t}{dt} = \frac{W_t^{(1)}}{(r_t - r^*)\left(V_t^* + Q_t W_t^{(1)}\right) + (\mu_{x+t}^* - \mu_{x+t})\left(b - V_t^* + Q_t\left(b - W_t^{(1)}\right)\right)}$$

$$\frac{dW_t^{(1)}}{dt} = r^* W_t^{(1)} - \mu_{x+t}^*\left(b - W_t^{(1)}\right)$$

$$\frac{dV_t^*}{dt} = r^* V_t^* + \pi - \mu_{x+t}^*(b - V_t^*)$$

with initial conditions $Q_0 = 0$, $W_0^{(1)} = W_0^*$, $V_n^* = 0$. The amount payable on death at time t is $b(1 + Q_t)$.

6.

a) For this part we consider the probability that the wife is in the scheme rather than the probability of survival. So by "alive" we mean in the scheme. For $t>10$, this is

$$\exp(-0.2)\,{}_tp_{30}$$

The present value of the benefit is

$$100000\int_{35}^{\infty}{}_tp_{30}\exp(-0.04t)\mathrm{dt}+$$

$$80000\int_{35}^{\infty}\exp(-0.2)\,{}_tp_{30}(1-{}_tp_{30})\exp(-0.04t)\mathrm{dt}+$$

$$80000\bar{a}_{\bar{5}|}\int_{35}^{\infty}\exp(-0.2)\,{}_tp_{30}(1-{}_tp_{30})\exp(-0.04t)\mu_{30+t}\mathrm{dt}=$$

$$100000{}_{35}p_{30}\exp(-0.04\times 35)\bar{a}_{65}+$$

$$80000{}_{35}p_{30}\exp(-0.04\times 35)\exp(-0.2)\left(\bar{a}_{65}-\bar{a}_{65:65}\right)+$$

$$80000{}_{35}p_{30}\exp(-0.04\times 35)\exp(-0.2)\bar{a}_{\bar{5}|}\left(\bar{A}_{65}-\frac{1}{2}\bar{A}_{65:65}\right).$$

We calculate

$$\bar{a}_{\bar{5}|}=\frac{1-\exp(-0.04\times 5)}{0.04}=4.5317,$$

$$\bar{A}_{65}=1-0.04\times 10.57=0.5772$$

and

$$\bar{A}_{65:65}=1-0.04\times 7.92=0.6832$$

Substituting we get that the present value of the benefits is 274022. Also

$$\bar{a}_{30:\overline{35}|}=19.93-0.77\times\exp(-0.04\times 35)\times 10.57=17.923$$

and the annual premium will be

$$\frac{274022}{17.923\times 0.95}=16094.$$

b) If only the husband is alive it is

$$100000\int_{0}^{\infty}{}_tp_{65}\exp(-0.04t)\mathrm{dt}=100000\bar{a}_{65}=1057000$$

If only the wife is alive it is

$$80000\left(\int_0^\infty {}_tp_{65}\exp(-0.04t)\mathrm{dt} + 4.5317 \times \int_0^\infty {}_tp_{65}\mu_{65+t}\exp(-0.04t)\mathrm{dt}\right) =$$

$$80000(10.57 + 4.5317 \times 0.5772) = 1054856.$$

If they are both alive it is

$$100000\int_0^\infty {}_tp_{65}\exp(-0.04t)\mathrm{dt} +$$

$$80000\int_0^\infty {}_tp_{65}(1 - {}_tp_{65})\exp(-0.04t)\mathrm{dt} +$$

$$80000\bar{a}_{\bar{5}|}\int_0^\infty {}_tp_{65}(1 - {}_tp_{65})\exp(-0.04t)\mu_{65+t}\mathrm{dt} =$$

$$1057000 + 80000\left(10.57 - 7.92 + 4.5317\left(0.5772 - \frac{0.6832}{2}\right)\right) = 1354414.$$

c)

$$\bar{a}_{\overline{45-t}|}$$

for $40 < t < 45$ and 0 for $t > 45$.

d) Let $V_0(t)$ be the reserve if they are both in the scheme, $V_H(t)$ be the reserve if the husband only is the scheme and $V_W(t)$ if the wife only is in the scheme. We can find reserves at time t by solving the equations

$$\frac{dV_0(t)}{dt} = 0.04V_0(t) + 16094.$$

$$\left(\mu_{30+t} + 0.02\mathbf{1}_{\{t<10\}}\right)(V_H(t) - V_0(t)) + \mu_{30+t}(V_W(t) - V_0(t))$$

$$\frac{dV_H(t)}{dt} = (0.04 + \mu_{30+t})V_H(t) + 16094$$

$$\frac{dV_W(t)}{dt} = (0.04 + \mu_{30+t})V_W(t)$$

subject to $V_0(35) = 1354414$, $V_H(35) = 1057000$ and $V_W(35) = 1054856$.

e) Yes, if the wife dies soon after time 0 (or they get divorced) the only benefit will be his pension (but will still be paying the same premium),; clearly the present value of the benefits will be much smaller than the present value of the premium.

f) The office could charge a different premium (or no premium) if his wife dies (or they get divorced).

SOLUTION OF Q2 OF THE RESIT PAPER

a)

$$\int_0^\infty {}_tp_{40}\, {}_tp_{50}(1-{}_tp_{60})\mu_{40+t}e^{-rt}\mathrm{dt} + \int_0^\infty {}_tp_{40}\, {}_tp_{50}(1-{}_tp_{60})\mu_{50+t}e^{-rt}\mathrm{dt} +$$

$$\int_0^\infty {}_tp_{40}\, {}_tp_{60}(1-{}_tp_{50})\mu_{40+t}e^{-rt}\mathrm{dt} + \int_0^\infty {}_tp_{40}\, {}_tp_{60}(1-{}_tp_{50})\mu_{60+t}e^{-rt}\mathrm{dt} +$$

$$\int_0^\infty {}_tp_{50}\, {}_tp_{60}(1-{}_tp_{40})\mu_{50+t}e^{-rt}\mathrm{dt} + \int_0^\infty {}_tp_{50}\, {}_tp_{60}(1-{}_tp_{40})\mu_{60+t}e^{-rt}\mathrm{dt}.$$

b) It is $\frac{A}{B}$, where

$$A = \int_0^\infty {}_tp_{40}\, {}_tp_{60}(1-{}_tp_{50})\mu_{40+t}\mathrm{dt} + \int_0^\infty {}_tp_{50}\, {}_tp_{60}(1-{}_tp_{40})\mu_{50+t}\mathrm{dt}$$

and

$$B = \int_0^\infty {}_tp_{40}\, {}_tp_{50}(1-{}_tp_{60})\mu_{40+t}\mathrm{dt} + \int_0^\infty {}_tp_{40}\, {}_tp_{50}(1-{}_tp_{60})\mu_{50+t}\mathrm{dt} +$$

$$\int_0^\infty {}_tp_{40}\, {}_tp_{60}(1-{}_tp_{50})\mu_{40+t}\mathrm{dt} + \int_0^\infty {}_tp_{40}\, {}_tp_{60}(1-{}_tp_{50})\mu_{60+t}\mathrm{dt} +$$

$$\int_0^\infty {}_tp_{50}\, {}_tp_{60}(1-{}_tp_{40})\mu_{50+t}\mathrm{dt} + \int_0^\infty {}_tp_{50}\, {}_tp_{60}(1-{}_tp_{40})\mu_{60+t}\mathrm{dt}.$$