ST305 2011 SOLUTIONS

1. We prove that the probability that y will die first is larger than the probability that x will die first. So we need to show that

$$\int_0^\infty {}_tp_x\, {}_tp_y \mu_{y+t} \mathrm{dt} > \int_0^\infty {}_tp_x\, {}_tp_y \mu_{x+t} \mathrm{dt}$$

but since μ_x is an increasing function of x $\mu_{y+t} > \mu_{x+t}$ for all t and the LHS is larger than the RHS.

2. If area A has an much older population than the population as a whole it will have a larger CDR. The SM Ratio will be less than 1 for area A as there will be fewer deaths than expected for all age groups. As an example supose that the one year mortality mortality are for young lives is 0.002 while for the county as a whole is 0.003. Also supose the equivalent rates for old lives are 0.01 and 0.012.. Area has 50% young lives and 50% old lives, while the contry as a whole has 70% young lives and 30% old lives. The CDRs are

$$0.5 \times 0.002 + 0.5 \times 0.01 = 0.006$$

and

$$0.6 \times 0.003 + 0.4 \times 0.011 = 0.0057.$$

However, the SM Ratio for A is

$$\frac{0.5 \times 0.002 + 0.5 \times 0.01}{0.5 \times 0.003 + 0.5 \times 0.012} = 0.8.$$

The SMR is always less than 1.

3. The expected present value of the benefits is

$$\frac{1}{80}\Big(15S_A + S\int_0^{25}\int_0^y \exp\Big(\int_0^s a(u)du\Big)ds\Big)\int_{25}^\infty \exp\Big(-\int_0^s (r(u)+\mu_{x+u})du\Big)ds.$$

The expected value of the expenses is

$$\int_0^{25} c\exp\Big(-\int_0^s (r(u)+\mu_{x+u})du\Big)ds$$

The expected value of the contributions plus the transfer value of his previous pension is

$$B + \alpha S_0\int_0^{25} \exp\Big(-\int_0^s (-a(u)+r(u)+\mu_{x+u})du\Big)ds.$$

Equating the sum of the first two expressions to the third gives us α the portion of salary that has to go towards pension contributions.

4.

   a)

$$2\int_0^\infty (1 - {}_tp_x)\,{}_tp_x\, dt$$

b)

$$\frac{\left({}_{t-5}p_x - {}_tp_x\right)^2}{\left(1 - {}_tp_x\right)^2}$$

c) It is

$$2\bar{A}_x - \bar{A}_{xx} = 2(1 - r\,\bar{a}_x) - (1 - r\,\bar{a}_{xx}).$$

d) It is an annuity commencing on the first death and ending on the second plus a 5 year annuity certain commencing on the second death

$$\int_0^\infty 2{}_tp_x\left(1 - {}_tp_x\right)e^{-rt}\mathrm{dt} + (2(1 - r\,\bar{a}_x) - (1 - r\,\bar{a}_{xx}))\bar{a}_{\bar{5}|} =$$

$$2(\bar{a}_x - \bar{a}_{xx}) + (2(1 - r\,\bar{a}_x) - (1 - r\,\bar{a}_{xx}))\frac{1 - e^{-5r}}{r}.$$

An alternative way to derive the second term is

$$\int_0^\infty (1 - {}_tp_x)^2 - \left(1 - {}_{t-5}p_x\right)^2 e^{-rt}\mathrm{dt}$$

leading to the same expression.

5. Let V_t be the reserve calculated according to the safe basis. As the emerging surplus per survivor is $(r(t) - 0.02)V_t$, the expected cost of the guarantee is

$$E\bigg(\int_0^{30} \exp\bigg(-\int_0^s r(u)\bigg)(0.02 - r(s))_+ V_s\ {}_sp_{30}\mathrm{ds}\bigg).$$

Define

$$W_1(t) = E\bigg(\int_t^{30} \exp\bigg(-\int_t^s r(u)\bigg)(0.02 - r(s))_+ V_s\ {}_sp_{30}\mathrm{ds}\,\bigg|\,r(t) = 0.01\bigg)$$

and

$$W_2(t) = E\bigg(\int_t^{30} \exp\bigg(-\int_t^s r(u)\bigg)(0.02 - r(s))_+ V_s\ {}_sp_{30}\mathrm{ds}\,\bigg|\,r(t) = 0.04\bigg).$$

We then have

$$W_1(t - \mathrm{dt}) = \left(-(0.02 - 0.01)_+ V_t\ {}_tp_{30}\mathrm{dt} + 0.01W_1(t)\right)(1 - 0.5\mathrm{dt}) + 0.5W_2(t)\mathrm{dt} + o(\mathrm{dt})$$

and letting dt$\downarrow$0 we have

$$\frac{dW_1(t)}{\mathrm{dt}} = 0.01V_t\ {}_tp_{30}\mathrm{dt} - 0.01W_1(t) + 0.5(W_1(t) - W_2(t)).$$

Similarly,

$$\frac{dW_2(t)}{\mathrm{dt}} = -0.04W_2(t) + 0.5(W_2(t) - W_1(t)).$$

The two equations should be solved together with

$$\frac{dV_t}{\mathrm{dt}} = (0.02 + \mu_{30+t})V_t + \pi - 10000\mu_{30+t}$$

and subject to the terminal conditions $W_1(30) = 0$, $W_2(30) = 0$ and $V_{30} = 10000$. If 0.01 was used for the safe basis then the interest rate can not move below the safe rate so there will never be a negative surplus emerging and the total cost of the guarantee is 0. There is an alternative solution where the expected cost of the guarantee is

$$E\Big(\int_0^{30} \exp\Big(-\int_0^s (r(u) + \mu_{30+u})\Big)(0.02 - r(s))_+ V_s \,\mathrm{ds}\Big),$$

$$W_1(t) = E\Big(\int_t^{30} \exp\Big(-\int_t^s (r(u) + \mu_{30+u})\Big)(0.02 - r(s))_+ V_s \,\mathrm{ds}\Big| r(t) = 0.01\Big)$$

$$W_1(t) = E\Big(\int_t^{30} \exp\Big(-\int_t^s (r(u) + \mu_{30+u})\Big)(0.02 - r(s))_+ V_s \,\mathrm{ds}\Big| r(t) = 0.04\Big)$$

and the two equations are

$$\frac{dW_1(t)}{\mathrm{dt}} = 0.01 V_t \,\mathrm{dt} - (0.01 + \mu_{30+t})W_1(t) + 0.5(W_1(t) - W_2(t))$$

and

$$\frac{dW_2(t)}{\mathrm{dt}} = -(0.04 + \mu_{30+t})W_2(t) + 0.5(W_2(t) - W_1(t)).$$

6.

a) We first calculate the expected present value of the benefit. To this effect we have to solve the system (HS is the state "healthy but been sick before")

$$\frac{dW_H(t)}{dt} = (r + \mu_{35+t})W_H(t) - \sigma_{35+t}(W_S(t) - W_H(t))$$

$$\frac{dW_S(t)}{dt} = (r + \nu_{35+t})W_S(t) - b - \rho_{35+t}(W_{\mathrm{HS}}(t) - W_S(t))$$

$$\frac{dW_{\mathrm{HS}}(t)}{dt} = (r + \tilde{\mu}_{35+t})W_{\mathrm{HS}}(t) - \tilde{\sigma}_{35+t}(W_S(t) - W_{\mathrm{HS}}(t))$$

subject to the conditions $W_H(25) = W_S(25) = W_{\mathrm{HS}}(25) = 0$. $W_H(0)$ would then represent the expected present value of the benefits. Then we also solve the system

$$\frac{dU_H(t)}{dt} = (r + \mu_{35+t})U_H(t) - 1 - \sigma_{35+t}(U_S(t) - U_H(t))$$

$$\frac{dU_S(t)}{dt} = (r + \nu_{35+t})U_S(t) - \rho_{35+t}(U_{\mathrm{HS}}(t) - U_S(t))$$

$$\frac{dU_{\mathrm{HS}}(t)}{dt} = (r + \tilde{\mu}_{35+t})U_{\mathrm{HS}}(t) - 1 - \tilde{\sigma}_{35+t}(U_S(t) - U_{\mathrm{HS}}(t))$$

subject to the conditions $U_H(25) = U_S(25) = U_{\mathrm{SH}}(25) = 0$ $U_H(0)$ would then represent the expected present value of a continuous annuity of 1 payable for 25 years while the life is healthy. The continuous premium is given by $\pi = \frac{W_H(0)}{U_H(0)}$.

b) We can then calculate reserves represented by $V_H(t)$ and $V_S(t)$ by solving the system

$$\frac{dV_H(t)}{dt} = (r + \mu_{35+t})V_H(t) + \pi - \sigma_{35+t}(V_S(t) - V_H(t))$$

$$\frac{dV_S(t)}{dt} = (r + \nu_{35+t})V_S(t) - b - \rho_{35+t}(V_{\mathrm{HS}}(t) - V_S(t))$$

$$\frac{dV_{\mathrm{HS}}(t)}{dt} = (r + \tilde{\mu}_{35+t})V_{\mathrm{HS}}(t) + \pi - \tilde{\sigma}_{35+t}(V_S(t) - V_{\mathrm{HS}}(t))$$

subject to the conditions $V_H(25) = V_S(25) = V_{\mathrm{HS}}(25) = 0$ (or $V_H(0) = V_S(0) = V_{\mathrm{HS}}(0) = 0$).

c) There is an extra benefit commencing at time 25 provided the life is then sick. The value of this benefit at time 25 is

$$A = \int_0^\infty \exp\left(-(r + \nu_{60+t} + \rho_{60+t})\right)\mathrm{dt}.$$

A can also be calculated by solving

$$\frac{dZ_S(t)}{dt} = (r + \nu_{60+t} + \rho_{60+t})Z_S(t) - b$$

subject to $\lim_{t\to\infty} Z(t) = 0$ and taking $A = Z(0)$.
We can then proced exactly as in (a) solving

$$\frac{dW_H(t)}{dt} = (r + \mu_{35+t})W_H(t) - \sigma_{35+t}(W_S(t) - W_H(t))$$

$$\frac{dW_S(t)}{dt} = (r + \nu_{35+t})W_S(t) - b - \rho_{35+t}(W_{\mathrm{HS}}(t) - W_S(t))$$

$$\frac{dW_{\mathrm{HS}}(t)}{dt} = (r + \tilde{\mu}_{35+t})W_{\mathrm{HS}}(t) - \tilde{\sigma}_{35+t}(W_S(t) - W_{\mathrm{HS}}(t))$$

but with the terminal conditions $W_H(25) = W_{\mathrm{HS}}(25) = 0$ and $W_S(25) = A$. The premium will be $\frac{W_H(0)}{U_H(0)}$, where $U_H(0)$ is exactly the same as in (a).

d) Without the modification in (c) there will be a negative reserve for states H and HS just before time 25. If the life falls ill just before time 25 it will not receive benefit for the duration of the sickness.

7.

a) A 20 year temporary annuity for a 40 year old and a 10 year temporary annuity for a 50 year old both calculated at a force of interest 0.03.

b)

$$ {}_5p_{45}\exp(-0.04\times 5)\bar{a}^{\text{at }0.03}_{50:10|} = \frac{0.9558}{0.9819}\times 0.8187\times 8.4811 = 6.7592 $$

c) Let π be the premium. The reserve for active policies is given by the equation

$$ \frac{dV_t}{\mathrm{dt}} = (0.03+\mu_{40+t})V_t + \pi + 0.02\times 0.5V_t\mathbf{1}_{\{t<5\}} $$

with $V_{20} = 100000$. The equation can be rewritten as

$$ \frac{dV_t}{\mathrm{dt}} = (a(t)+\mu_{40+t})V_t + \pi $$

where $a(t) = 0.04\mathbf{1}_{\{t<5\}} + 0.03\mathbf{1}_{\{t\geq 5\}}$. Solving and observing that $V_0 = 0$, we have that

$$ \pi\int_0^{20}\exp\Big(-\int_0^s (a(u)+\mu_{40+u})\mathrm{du}\Big) = 100000\exp\Big(-\int_0^{20}(a(u)+\mu_{40+u})\mathrm{du}\Big) $$

and therefore

$$ \pi\Big(\bar{a}^{\text{at }0.04}_{40:5|} + {}_5p_{40}\exp(-.04\times 5)\bar{a}^{\text{at }0.03}_{45:15|}\Big) = 100000\exp(-.04\times 5 - .03\times 15\Big)\,{}_{20}p_{40} $$

and hence

$$ \pi\big(\bar{a}^{\text{at }0.04}_{40:20|} - {}_5p_{40}\exp(-.04\times 5)\bar{a}^{\text{at }0.04}_{45:15|} + {}_5p_{40}\exp(-.04\times 5)\bar{a}^{\text{at }0.03}_{45:15|}\big) = $$

$$ 100000\exp(-.04\times 5 - .03\times 15)\,{}_{20}p_{40} $$

From the worksheet we have ${}_5p_{40} = 0.9819$, ${}_{20}p_{40} = 0.8637$, $\bar{a}^{\text{at }0.03}_{45:15|} = 11.824$, $\bar{a}^{\text{at }0.04}_{45:15|} = 11.049$ and $\bar{a}^{\text{at }0.04}_{40:20|} = 13.4775$. So

$$ \pi(13.4775 + 0.9819\times 0.8187(11.824-11.049)) = 100000\times 0.522\times 0.8637 $$

$$ \pi = \frac{45085.14}{14.1} = 3197.5 $$

d) The reserve at that time is

$$ 100000\,{}_{10}p_{50}\exp(-0.03\times 10) - 3197.5\bar{a}^{\text{at }0.03}_{50:10|} $$

Note that

$$ {}_{10}p_{50} = \frac{{}_{20}p_{40}}{{}_{10}p_{40}} = \frac{0.8637}{0.9558} = 0.9036 $$

and therefore the reserve is

$$ 100000\times 0.9036\times 0.7408 - 3197.5\times 8.481 = 39820.7 $$

Let b be the sum assured. We then have

$$ b\,\bar{A}_{50:\overline{10|}} = 39820.7 $$

so

$$b = \frac{39820.7}{1 - 0.03 \times 8.481} = \frac{39820.7}{0.74557} = 53409.73$$

e) One problem is that the lapse rate is constant. One might expect it to depend on time. Amore important issue is that lives that let their policies lapse might have higher mortality than other lives.