Summer 2012 Examination

ST305

Actuarial Mathematics: Life

Suitable for All Candidates

Instructions to candidates

Time allowed: 3 hours

This paper contains 7 questions. Answer **All** questions.

You may use an electronic calculator (as prescribed by the examination regulations)

Taxes and expenses should be ignored unless otherwise indicated.

1. Consider two areas A and B with different population structures. The directly calculated standardised mortality rate for area A using the population of area B as standard is 0.1. The indirectly calculated standardised mortality ratio of area B using the mortality rates of area A as standard is 1.015. The population of area B is 64000. Is there evidence that the mortality experience of the two areas is different? What is the limitation of this calculation?

(3+2=5 *marks*)

2. Lives are subject to two causes of death; death due to accident that occurs with force $\mu^{(1)}_{x+t}$ and death due to natural causes that occurs with force $\mu^{(2)}_{x+t}$. Consider two lives aged x and y. Given that the second death of these two lives occured at time t, provide an expression for the conditional probability that the death was accidental using the forces and survival probabilities.

(6 *marks*)

3. Consider two independent lives aged x and y. Using the quantities ${}_5p_x$, ${}_5p_y$, $\bar{a}_x$, $\bar{a}_y$, $\bar{a}_{xy}$, $\bar{a}_{x+5}$, $\bar{a}_{y+5}$, $\bar{a}_{x+5:y+5}$ and the force of interest r only, provide expressions for the expected present value of the following annuity contracts

   (a) An annuity payable till 5 years after the second death of the two lives

   (6 *marks*)

   (b) An annuity payable for as long as one life is alive but guaranteed for 5 years.

   (6 *marks*)

4. Consider a health-sickness model with three states "*healthy*", "*sick*" and "*dead*". The sickness rate for a life aged $x+t$ is σ_{x+t}, the recovery rate is ρ_{x+t}, the mortality rate for a healthy life is μ_{x+t} and the mortality rate for a sick life is ν_{x+t}. An office is issuing a policy under which a continuous sickness benefit at a rate of b per annum payable while the life is at the "*sick*" state is provided. The policy is financed by a continuous premium payable at a rate π per annum payable while the life is at the "*healthy*' state. The policy duration is n years. After that period has elapsed or earlier death there are no more benefits or premiums. The force of interest at time t is r_t. In order to calculate

the premium the office uses a first order basis where the sickness rate is σ^*_{x+t}, the recovery rate is ρ^*_{x+t}, the mortality rate for a healthy life is μ^*_{x+t}, the mortality rate for a sick life is ν^*_{x+t} and the force of interest is r^*. Initially, the life is healthy.

(a) Show that surplus per active policy emerges at a rate

$$(r_t - r^*)\, V_t^H + \left(\mu_{x+t} - \mu^*_{x+t}\right) V_t^H - \left(\sigma_{x+t} - \sigma^*_{x+t}\right)\left(V_t^S - V_t^H\right)$$

when the life is in state "*healthy*" and

$$(r_t - r^*)\, V_t^S + \left(\nu_{x+t} - \nu^*_{x+t}\right) V_t^S + \left(\rho_{x+t} - \rho^*_{x+t}\right)\left(V_t^S - V_t^H\right)$$

when the life is in state "*sick*", where V_t^H and V_t^S are prospective reserves calculates using the first order basis.

(8 *marks*)

(b) Discuss how the office will choose a safe first order basis.

(3 *marks*)

(c) All surplus will be distributed as extra benefit to sick lives. Explain how you would calculate the extra benefit rate at time t.

(6 *marks*)

5. We are now at the 1st of January 2012. An employee with current salary 40000 per annum payable continuously became a member of a *defined contributions* pension scheme on the 1st of January 2010 then aged 30. The contributions are 20% of her salary payable continuously. During the period between the 1st of January 2010 and the 1st of January 2012, salary inflation occured at a force of 0.01 per annum and the force of interest was constant and equal to 0.02 per annum. On the 1st of January 2040, the employee will be entitled to the accumulation of the contributions. There is no benefit on earlier death. In order to predict that amount the office will use a deterministic salary inflation rate at a force of 0.01 per annum and a stochastic interest rate model with two states. When at state 1 the force of interest will be 0.02 and when at state 2 it will be 0.04. The transition rate from state 1 to state 2 is 0.5 and the transition rate from state 2 to state 1 is 1. Assume also a force of mortality μ_{30+t} at time t.

(a) Explain how you would calculate the accumulated value of past contributions on the 1st of January 2012.

(4 *marks*)

(b) Develop equations to calculate the expected value of the amount she will be entitled to on the 1st of January 2040.

(13 *marks*)

6. Police officers can be working either in uniform or in plain clothes. The rate of transition from working in uniform to working in plain clothes is 0.1 and the rate of transition from working in plain clothes to working in uniform is 0.05. Police officers aged x are also subject to a force of mortality μ_{x+t} at time t regardless of where they are working. Let $p_1(t)$ denote the probability that a police officer is working in uniform at time t and $p_2(t)$ denote the probability that he is working in plain clothes at time t. The following quantities are provided:

$$ {}_{30}p_{30} = 0.84 $$

At a force of interest of 0.05

$$ \bar{a}_{30:\overline{30|}} = 15 $$

At a force of interest of 0.10

$$ \bar{a}_{30:\overline{30|}} = 9.28 $$

At a force of interest of 0.15

$$ \bar{a}_{30:\overline{30|}} = 6.49 $$

At a force of interest of 0.20

$$ \bar{a}_{30:\overline{30|}} = 4.93 $$

(a) A police officer is currently (time 0) working in plain clothes and is aged 30. Write down the forward equations and show that

$$ p_2(t) = \left(\frac{1}{3}\exp(-0.15t) + \frac{2}{3}\right) {}_tp_{30} $$

and

$$ p_1(t) = \left(\frac{1}{3} - \frac{1}{3}\exp(-0.15t)\right) {}_tp_{30} $$

where

$$ {}_tp_x = \exp\left(-\int_0^t \mu_{x+s}ds\right). $$

(7 *marks*)

(b) The police force is offering him a death benefit of 100000 provided death occurs while working in uniform. The benefit will stop when the employee reaches the age of 60. Use a force of interest of 0.05 to calculate the expected present value of the benefit.

(9 *marks*)

(c) Suppose now that the benefit in part (b) applies to the first continuous period in uniform only. Moreover, on return to plain clothes after the first period in uniform there is a lump sum benefit of 5000. There are no more benefits thereafter and no benefits are payable after the age of 60. Calculate the expected present value of the benefits.

(10 *marks*)

7. The next two pages are the printout of a MAPLE worksheet with calculations about a certain life insurance product. The sum of 2451.58 represents the annual premium payable continuously.

(a) Identify the product by laying out its terms, the benefits, the ages of the life(s) involved, the interest rate and any expenses. Make sure you clearly state the circumstances under which all benefits are paid.

(6 *marks*)

(b) What does the amounts of 10951.05 and 34959.58 in (19) represent?

(2+2=4 *marks*)

(c) Ten years after the policy originated the life(s) involved are both alive but can not pay any more premiums and the policy is converted to a pure endowment with the same maturity date. There will be no more premiums payable, there is a charge for expenses of 800 and there will be a lump sum payable at the end provided all the life(s) involved are still alive. Use the information provided to calculate the lump sum.

(7 *marks*)

This is the force of mortality:

```
> m := t -> 0.0005+0.00007585775*10^(0.038*t) ;
```

$$m := t \rightarrow 0.0005 + 0.00007585775\ 10^{(0.038\ t)} \tag{1}$$

```
> evalf(0.00007585775*10^(0.038*30)/(0.038*ln(10)));
```

$$0.01196742383 \tag{2}$$

```
> p1 := t -> exp(-0.0005*t-.01196742383*(10^(0.038*t)-1));
```

$$p1 := t \rightarrow e^{\left(-0.0005\,t - 0.01196742383\,\left(10^{(0.038\,t)} - 1\right)\right)} \tag{3}$$

```
> evalf(0.00007585775*10^(0.038*40)/(0.038*ln(10)));
```

$$0.02870785022 \tag{4}$$

```
> p2 := t -> exp(-0.0005*t-.02870785022*(10^(0.038*t)-1));
```

$$p2 := t \rightarrow e^{\left(-0.0005\,t - 0.02870785022\,\left(10^{(0.038\,t)} - 1\right)\right)} \tag{5}$$

Some survival probabilities:

```
> evalf(p1(10));
```

$$0.9784941933 \tag{6}$$

```
> evalf(p1(20));
```

$$0.9352906781 \tag{7}$$

```
> evalf(p2(10));
```

$$0.9558469375 \tag{8}$$

```
> evalf(p2(20));
```

$$0.8637355917 \tag{9}$$

```
> evalf( Int( exp(-0.02*t)*p1(t)*p2(t)*m(30+t), t=0..20 ));
```

$$0.04774116749 \tag{10}$$

```
> evalf( Int( exp(-0.02*t)*p1(t)*p2(t), t=0..20 ));
```

$$15.34781696 \tag{11}$$

```
> evalf(p1(20)*exp(-0.02*20));
```

$$0.6269440904 \tag{12}$$

```
> evalf(100000*0.04774116749);
```

$$4774.116749 \tag{13}$$

```
> evalf(50000*0.6269440904);
```

$$31347.20452 \tag{14}$$

```
> evalf((4774.116749+31347.20452)/(15.34781696*.96));
```

$$2451.578386 \tag{15}$$

```
> dsys  := {diff(v1(t),t) =  (0.02+m(40+t)+m(30+t))*v1(t) +
  2451.578386 - 0.04*2451.578386 - m(30+t)*100000-m(40+t)*v2(t),
  diff(v2(t),t) =(0.02+m(30+t))*v2(t), v1(20)=50000, v2(20)=50000};
```

(16)

$$dsys := \left\{ v1(20) = 50000,\ v2(20) = 50000,\ \frac{d}{dt}\, v1(t) = \left(0.0210 + 0.00007585775\ 10^{(1.520 + 0.038\,t)} + 0.00007585775\ 10^{(1.140 + 0.038\,t)}\right) v1(t) + 2303.515251 - 7.585775000\ 10^{(1.140 + 0.038\,t)} - \left(0.0005 + 0.00007585775\ 10^{(1.520 + 0.038\,t)}\right) v2(t),\ \frac{d}{dt}\, v2(t) = \left(0.0205 + 0.00007585775\ 10^{(1.140 + 0.038\,t)}\right) v2(t) \right\} \quad (16)$$

```
> dsol := dsolve(dsys, numeric, range = 0..20);
```

$$dsol := \mathbf{proc}(x\_rkf45)\ \ldots\mathbf{end\ proc} \quad (17)$$

```
> dsol(0);
```

$$[t = 0.,\ v1(t) = -0.0156727747435070342,\ v2(t) = 31347.1900888073251] \quad (18)$$

```
> dsol(5);
```

$$[t = 5.,\ v1(t) = 10951.0521387238004,\ v2(t) = 34959.5801488387150] \quad (19)$$

```
> dsol(10);
```

$$[t = 10.,\ v1(t) = 22812.5193655145668,\ v2(t) = 39129.0425814939445] \quad (20)$$

```
> dsol(15);
```

$$[t = 15.,\ v1(t) = 35725.0956709153398,\ v2(t) = 44040.8208613628667] \quad (21)$$

```
> dsol(20);
```

$$[t = 20.,\ v1(t) = 50000.,\ v2(t) = 50000.] \quad (22)$$

```
>
```