ST305 2012 Solutions

1. The expected number of deaths for area B will then be

$$64000 \times 0.1 = 6400$$

and the actual deaths $6400 \times 1.015 = 6496$. Calculating

$$\Phi\left(\frac{6496 - 6400}{\sqrt{6400}}\right) = \Phi(1.2) = 0.885$$

leading to a p-value of 0.23. So there isn't any significant evidence of a difference. Of course it could be the case there is different mortality experience for different age groups. Single figure indices can not capture that.

2.

$$\frac{{}_tp_x(1 - {}_tp_y)\mu_{x+t}^{(1)} + {}_tp_y(1 - {}_tp_x)\mu_{y+t}^{(1)}}{{}_tp_x(1 - {}_tp_y)\mu_{x+t}^{(1)} + {}_tp_y(1 - {}_tp_x)\mu_{y+t}^{(1)} + {}_tp_x(1 - {}_tp_y)\mu_{x+t}^{(2)} + {}_tp_y(1 - {}_tp_x)\mu_{y+t}^{(2)}}$$

3. We use the convention ${}_tp_x = 1$ for $t \leq 0$. We also observe that the probability at least one life is alive at time t is

$$1-(1-{}_tp_x)(1-{}_tp_y) = {}_tp_x + {}_tp_y - {}_tp_x\,{}_tp_y$$

a) This will be

$$\int_0^\infty \exp(-rt)({}_{t-5}p_x + {}_{t-5}p_y - {}_{t-5}p_x\,{}_{t-5}p_y)dt =$$

$$\int_0^5 \exp(-rt)dt + \int_5^\infty \exp(-rt)({}_{t-5}p_x + {}_{t-5}p_y - {}_{t-5}p_x\,{}_{t-5}p_y)dt =$$

$$\frac{1-\exp(-5r)}{r} + \exp(-5r)\int_5^\infty \exp(-r(t-5))({}_{t-5}p_x + {}_{t-5}p_y - {}_{t-5}p_x\,{}_{t-5}p_y)dt =$$

$$\frac{1-\exp(-5r)}{r} + \exp(-5r)\int_0^\infty \exp(-rt)({}_tp_x + {}_tp_y - {}_tp_x\,{}_tp_y)dt =$$

$$\frac{1-\exp(-5r)}{r} + \exp(-5r)(\bar{a}_x + \bar{a}_y - \bar{a}_{xy}).$$

b) This will be

$$\frac{1-\exp(-5r)}{r} + \int_5^\infty \exp(-rt)({}_tp_x + {}_tp_y - {}_tp_x\,{}_tp_y)dt =$$

$$\frac{1-\exp(-5r)}{r} + \exp(-5r)\,{}_5p_x\int_5^\infty \exp(-r(t-5))\,{}_{t-5}p_{x+5}dt +$$

$$\exp(-5r)\,{}_5p_y\int_5^\infty \exp(-r(t-5))\,{}_{t-5}p_{y+5}dt -$$

$$\exp(-5r)\,{}_5p_x\,{}_5p_y\int_5^\infty \exp(-r(t-5))\,{}_{t-5}p_{x+5}\,{}_{t-5}p_{y+5}dt =$$

$$\frac{1-\exp(-5r)}{r} + \exp(-5r)\,{}_5p_x\int_0^\infty \exp(-rt)\,{}_tp_{x+5}dt +$$

$$\exp(-5r)\,{}_5p_y\int_0^\infty \exp(-rt)\,{}_tp_{y+5}dt -$$

$$\exp(-5r)\,{}_5p_x\,{}_5p_y\int_0^\infty \exp(-rt)\,{}_tp_{x+5}\,{}_tp_{y+5}dt =$$

$$\frac{1-\exp(-5r)}{r} + \exp(-5r)({}_5p_x\bar{a}_{x+5} + {}_5p_y\bar{a}_{y+5} - {}_5p_x\,{}_5p_y\bar{a}_{x+5:y+5}).$$

4.

a) The discounted surplus up to time t is

$$S_t = \int_0^t \exp\left(-\int_0^s r_u du\right)\pi \mathbf{1}_{\{X(s)=H\}} ds - \int_0^t \exp\left(-\int_0^s r_u du\right) b \mathbf{1}_{\{X(s)=S\}} ds -$$

$$-\exp\left(-\int_0^t r_u du\right) V_t^H \mathbf{1}_{\{X(t)=H\}} - \exp\left(-\int_0^t r_u du\right) V_t^S \mathbf{1}_{\{X(t)=S\}}.$$

We then have that when $X(t)=H$

$$dS_t = \pi \exp\left(-\int_0^t r_u du\right) dt + r_t \exp\left(-\int_0^t r_u du\right) V_t^H dt - \exp\left(-\int_0^t r_u du\right) dV_t^H -$$

$$\exp\left(-\int_0^t r_u du\right) \sigma_{x+t}(V_t^S - V_t^H) dt - \exp\left(-\int_0^t r_u du\right) \mu_{x+t}(0 - V_t^H) dt$$

and since

$$dV_t^H = (r^* + \mu^*_{x+t}) V_t^H dt + \pi dt - \sigma^*_{x+t}(V_t^S - V_t^H) dt$$

we have

$$dS_t = \exp\left(-\int_0^t r_u du\right) dt ((r_t - r^* + \mu_{x+t} - \mu^*_{x+t}) V_t^H - (\sigma_{x+t} - \sigma^*_{x+t})(V_t^S - V_t^H))$$

so the emerging surplus rate is

$$c_t^H = (r_t - r^* + \mu_{x+t} - \mu^*_{x+t}) V_t^H - (\sigma_{x+t} - \sigma^*_{x+t})(V_t^S - V_t^H).$$

Similarly when $X(t)=S$,

$$dS_t = -b \exp\left(-\int_0^t r_u du\right) dt + r_t \exp\left(-\int_0^t r_u du\right) V_t^S dt - \exp\left(-\int_0^t r_u du\right) dV_t^S -$$

$$\exp\left(-\int_0^t r_u du\right) \rho_{x+t}(V_t^H - V_t^S) dt - \exp\left(-\int_0^t r_u du\right) \nu_{x+t}(0 - V_t^S) dt$$

and since

$$dV_t^S = (r^* + \nu^*_{x+t}) V_t^S dt - b dt - \rho^*_{x+t}(V_t^H - V_t^S) dt$$

we have

$$dS_t = \exp\left(-\int_0^t r_u du\right) dt((r_t - r^* + \nu_{x+t} - \nu^*_{x+t})V_t^S + (\rho_{x+t} - \rho^*_{x+t})(V_t^S - V_t^H))$$

so the emerging surplus rate is

$$c_t^S = (r_t - r^* + \nu_{x+t} - \nu^*_{x+t})V_t^S + (\rho_{x+t} - \rho^*_{x+t})(V_t^S - V_t^H).$$

b) Note that $V_t^S > V_t^H$ and so the safe basis should be such that $r^* < r_t$, $\mu^*_{x+t} < \mu_{x+t}$, $\nu^*_{x+t} < \nu_{x+t}$, $\sigma^*_{x+t} > \sigma_{x+t}$ and $\rho^*_{x+t} < \rho_{x+t}$.

c) The rate will be

$$\frac{p^H(t)c_t^H + p^S(t)c_t^S}{p^S(t)},$$

where the probabilities that a life is healthy or sick $p^H(t)$ and $p^S(t)$ can be calculated by solving the forward equations

$$\frac{dp^H(t)}{dt} = \rho_{x+t}p^S(t) - (\mu_{x+t} + \sigma_{x+t})\, p^H(t)$$

$$\frac{dp^S(t)}{dt} = \sigma_{x+t}p^H(t) - (\nu_{x+t} + \rho_{x+t})\, p^S(t)$$

subject to $p^H(0) = 1$ and $p^S(0) = 0$.

5.

a) The value is

$$8000\exp(-0.02)\int_0^2 e^{0.01s}e^{0.02(2-s)}\exp\left(\int_s^2 \mu_{30+u} du\right) ds.$$

We can either calculate the integral directly or solve the equation

$$\frac{dU(t)}{dt} = (0.02 + \mu_{30+t})U(t) + 40000e^{0.01t}$$

subject to $U(0)=0$ and take $U(2)$.

b) The accumulated amount will be

$$8000\exp(-0.02)\int_0^{30} e^{0.01s}\exp\Big(\int_s^{30}(r_u+\mu_{30+u})du\Big)ds=$$

$$8000\exp(-0.02)\int_0^{t} e^{0.01s}\exp\Big(\int_s^{t}(r_u+\mu_{30+u})du\Big)ds\,\exp\Big(\int_t^{30}(r_u+\mu_{30+u})du\Big)+$$

$$8000\exp(-0.02)\int_t^{30} e^{0.01s}\exp\Big(\int_s^{30}(r_u+\mu_{30+u})du\Big)ds=$$

$$U(t)\exp\Big(\int_t^{30}(r_u+\mu_{30+u})du\Big)+8000\exp(-0.02)\int_t^{30} e^{0.01s}\exp\Big(\int_s^{30}(r_u+\mu_{30+u})du\Big)ds.$$

Define now

$$V_1(t)=E\Big(\exp\Big(\int_t^{30}(r_u+\mu_{30+u})du\Big)|\,X(t)=1\Big),$$

$$V_2(t)=E\Big(\exp\Big(\int_t^{30}(r_u+\mu_{30+u})du\Big)|\,X(t)=2\Big),$$

$$W_1(t)=E\Big(\int_t^{30} e^{0.01s}\exp\Big(\int_s^{30}(r_u+\mu_{30+u})du\Big)ds|\,X(t)=1\Big)$$

and

$$W_2(t)=E\Big(\int_t^{30} e^{0.01s}\exp\Big(\int_s^{30}(r_u+\mu_{30+u})du\Big)ds|\,X(t)=2\Big).$$

We have

$$V_1(t-dt)=(1+(0.02+\mu_{30+t})dt)V_1(t)(1-0.5dt)+0.5dtV_2(t)+o(dt)$$

leading to

$$\frac{dV_1(t)}{dt}=-(0.02+\mu_{30+t})V_1(t)+0.5(V_1(t)-V_2(t)).$$

Similarly

$$V_2(t-dt)=(1+(0.04+\mu_{30+t})dt)V_2(t)(1-dt)+dtV_1(t)+o(dt)$$

leading to

$$\frac{dV_2(t)}{dt}=-(0.04+\mu_{30+t})V_2(t)+(V_2(t)-V_1(t)).$$

Moreover

$$W_1(t-dt)=(W_1(t)+8000\exp(-0.02)e^{0.01t}V_1(t))(1-0.5dt)+0.5dtW_2(t)+o(dt)$$

leading to

$$\frac{dW_1(t)}{dt}=-8000\exp(-0.02)e^{0.01t}V_1(t)+0.5(W_1(t)-W_2(t))$$

and

$$W_2(t-dt) = (W_2(t) + 8000\exp(-0.02)e^{0.01t}V_2(t))(1-dt) + dt\,W_1(t) + o(dt)$$

leading to

$$\frac{dW_2(t)}{dt} = -8000\exp(-0.02)e^{0.01t}V_2(t) + (W_2(t) - W_1(t)).$$

We then solve the four differential equations simultaneously subject to the terminal conditions $V_1(30)=1,\ V_2(30)=1,\ W_1(30)=0$ and $W_2(30)=0$. The prediction will be $U(2)V_1(2)+W_1(2)$.

6.

a) The forward equations are

$$\frac{dp_1(t)}{dt} = -(\mu_x + 0.1)p_1(t) + 0.05p_2(t)$$

and

$$\frac{dp_2(t)}{dt} = -(\mu_x + 0.05)p_2(t) + 0.1p_1(t)$$

with $p_1(0)=0$ and $p_2(0)=1$. We can now proceed either with direct substitution or observing that mortality is independent of other movements set $p_1(t) = {}_tp_{30}\bar{p}_1(t)$ and $p_2(t) = {}_tp_{30}\bar{p}_2(t)$. We then have $\bar{p}_1(t) = 1 - \bar{p}_2(t)$

$$\frac{d\,\bar{p}_2(t)}{dt} = -0.05\bar{p}_2(t) + 0.1(1 - \bar{p}_2(t)\,)$$

with $\bar{p}_2(0)=1$. Solving

$$\bar{p}_2(t) = \frac{2}{3} + \frac{1}{3}\exp(-0.15t)$$

and

$$\bar{p}_1(t) = \frac{1}{3} - \frac{1}{3}\exp(-0.15t).$$

b) The expected present value is

$$100000\int_0^{30} \exp\left(-0.05t\right)\left(\frac{1}{3} - \frac{1}{3}\exp(-0.15t)\right){}_tp_{30}\mu_{30+t}dt =$$

$$100000\left(\frac{1}{3}\int_0^{30} \exp\left(-0.05t\right){}_tp_{30}\mu_{30+t}dt - \frac{1}{3}\int_0^{30} \exp\left(-0.2t\right){}_tp_{30}\mu_{30+t}dt\right).$$

We also have that

$$\int_0^{n} \exp\left(-rt\right){}_tp_x\mu_{x+t}dt = 1 - r\bar{a}_{x:\overline{n}|} - {}_np_x\exp(-rn).$$

So

$$\frac{1}{3}\int_0^{30} \exp\left(-0.05t\right) {}_tp_{30}\mu_{30+t}dt - \frac{1}{3}\int_0^{30} \exp\left(-0.20t\right) {}_tp_{30}\mu_{30+t}dt =$$

$100000 \times \frac{1}{3} \times (1 - 0.05 \times 15 - \exp\left(-30 \times 0.05\right) \times 0.84) - 100000 \times \frac{1}{3} \times (1 - 0.2 \times 4.93 - \exp\left(-30 \times 0.2\right) \times 0.84) = \frac{10000}{3}(0.06257 - 0.01192) = 1688.43$

c) The probability that he is in uniform for the first time at time t is

$${}_tp_{30}\int_0^t 0.05\exp(-0.05s)\exp(-0.1(t-s))ds =$$

$${}_tp_{30}\exp(-0.1t)\int_0^t 0.05\exp(0.05s)\exp ds =$$

$${}_tp_{30}(\exp\left(-0.05t\right) - \exp\left(-0.1t\right))$$

So the value of the benefit is

$$100000\int_0^{30} {}_tp_{30}\exp(-0.05t)(\exp\left(-0.05t\right) - \exp\left(-0.1t\right))\mu_{30+t}dt +$$

$$5000\int_0^{30} {}_tp_{30}\exp(-0.05t)(\exp\left(-0.05t\right) - \exp\left(-0.1t\right))0.1dt =$$

$$100000 \times (1 - 0.1 \times 9.28 - \exp\left(-30 \times 0.1\right) \times 0.84) -$$

$$100000 \times (1 - 0.15 \times 6.49 - \exp\left(-30 \times 0.15\right) \times 0.84) +$$

$$5000 \times 0.1 \times (9.28 - 6.49) =$$

$$100000(0.0302 - 0.01717) + 500 \times 2.79 = 1301 + 1395 = 2696$$

7.

a) The policy pays out the sum of 100000 upon death of a life now aged 30 provided another life aged 40 is alive at the time. Moreover, there is a survival benefit for the 30 year old of 50000 at time 20. This is not conditional on the 40 year old being alive at the time. The force of interest is 0.02 and expenses that are 4% of the continuous premium are assumed.

b) The prospective (or retrospective) reserve for the rest of the policy at time 5; the first figure is if both lives are alive and the second figure if the 30 year old only is alive.

c) The reserve at that time is 22812.52, so 22812.52-800=22012.52 will be used as a single premium. The lump sum will be

$$\frac{22012.52}{{}_{10}p_{40}\,{}_{10}p_{50}}\exp(0.02 \times 10).$$

We already know ${}_{10}p_{40} = 0.956$ and we can calculate

$${}_{10}p_{50} = \frac{{}_{20}p_{40}}{{}_{10}p_{40}} = \frac{0.8637}{0.956} = 0.9036$$

So the lump sum will be

$$\frac{22012.52}{0.956 \times 0.9036}\exp(0.02 \times 10) = 31127.$$