Solutions to selected ST305 exercises 2006/2007:

Exercise 1

(a)

$$\begin{aligned}\mathbb{P}\left[\bigcap_{i=0}^{r+1} Z(t_i)=j_i\right] &= \mathbb{P}\left[\bigcap_{i=0}^{r} Z(t_i)=j_i\right]\mathbb{P}\left[Z(t_{r+1})=j_{r+1}\,\middle|\,\bigcap_{i=0}^{r} Z(t_i)=j_i\right]\\ &= \mathbb{P}\left[\bigcap_{i=0}^{r} Z(t_i)=j_i\right] p_{j_r,j_{r+1}}(t_r,t_{r+1})\,.\end{aligned}$$

We have here used the Markov property of the process. Repeating this, we obtain

$$\mathbb{P}\left[\bigcap_{i=0}^{r+1} Z(t_i)=j_i\right] = p_{1j_0}(0,t_0)\prod_{i=1}^{r+1} p_{j_{i-1},j_i}(t_{i-1},t_i)\,. \tag{1}$$

Next, using (1),

$$\begin{aligned}\mathbb{P}\left[\bigcap_{i=1}^{r} Z(t_i)=j_i\,\middle|\,Z(t_0)=j_0,\,Z(t_{r+1})=j_{r+1}\right] &= \frac{\mathbb{P}\left[\bigcap_{i=0}^{r+1} Z(t_i)=j_i\right]}{\mathbb{P}\left[Z(t_0)=j_0\,,\,Z(t_{r+1})=j_{r+1}\right]}\\ &= \frac{p_{1j_0}(0,t_0)\prod_{i=1}^{r+1} p_{j_{i-1},j_i}(t_{i-1},t_i)}{p_{1j_0}(0,t_0)\,p_{j_0,j_{r+1}}(t_0,t_{r+1})}\\ &= \frac{\prod_{i=1}^{r+1} p_{j_{i-1},j_i}(t_{i-1},t_i)}{p_{j_0,j_{r+1}}(t_0,t_{r+1})}\,. \qquad (2)\end{aligned}$$

(b) For $s=t_0<t_1<\cdots<t_r<t_{r+1}=t$:

$$\begin{aligned}&\mathbb{P}\left[Z(t_r)=j_r\,\middle|\,\bigcap_{i=1}^{r-1} Z(t_i)=j_i\,,Z(s)=i,\,Z(t)=j\right]\\ =\;&\mathbb{P}\left[Z(t_r)=j_r\,\middle|\,\bigcap_{i=1}^{r-1} Z(t_i)=j_i\,,Z(t_0)=j_0,\,Z(t_{r+1})=j_{r+1}\right]\\ =\;&\frac{\mathbb{P}\left[\bigcap_{i=0}^{r+1} Z(t_i)=j_i\right]}{\mathbb{P}\left[\bigcap_{i=0,\ldots,r-1,r+1} Z(t_i)=j_i\right]}\\ =\;&\frac{p_{1j_0}(0,t_0)\prod_{i=1}^{r+1} p_{j_{i-1},j_i}(t_{i-1},t_i)}{p_{1j_0}(0,t_0)\prod_{i=1,\ldots,r-1} p_{j_{i-1},j_i}(t_{i-1},t_i)\,p_{j_{r-1},j_{r+1}}(t_{r-1},t_{r+1})}\\ =\;&\frac{p_{j_{r-1},j_r}(t_{r-1},t_r)\,p_{j_r,j}(t_r,t)}{p_{j_{r-1},j}(t_{r-1},t)}\,. \qquad (3)\end{aligned}$$

For given $t_r=t$, $j_{r+1}=j$ (and $t_0=s$, $j_0=i$) this is just a function $\tilde{p}_{j_{r-1},j_r}(t_{r-1},t_r)$ (say) of j_{r-1}, j_r, t_{r-1}, and t_r, showing that the conditional Markov chain is itself Markov.

The intensities of the conditional Markov chain are

$$\begin{aligned} \tilde{\mu}_{gh}(\tau) &= \lim_{d\tau \searrow 0} \frac{\tilde{p}_{gh}(\tau, \tau + d\tau)}{d\tau} \\ &= \lim_{d\tau \searrow 0} \frac{p_{gh}(\tau, \tau + d\tau)\, p_{hj}(\tau + d\tau, t)}{d\tau\, p_{gj}(\tau + d\tau, t)} \\ &= \mu_{gh}(\tau)\, \frac{p_{hj}(\tau, t)}{p_{gj}(\tau, t)}\,. \end{aligned} \tag{4}$$

$$\lim_{\tau \nearrow t} \tilde{\mu}_{gh}(\tau) = \begin{cases} \mu_{gh}(t)\, \frac{\mu_{hj}(t)}{\mu_{gj}(t)}\,, & g \neq j,\, h \neq j\,, \\ \infty\,, & g \neq j,\, h = j\,, \\ 0\,, & g = j,\, h \neq j\,. \end{cases}$$

The expression in the case $g \neq j$, $h \neq j$ is obtained upon writing

$$\frac{p_{hj}(\tau, t)}{p_{gj}(\tau, t)} = \frac{p_{hj}(\tau, t)/(t - \tau)}{p_{gj}(\tau, t)/(t - \tau)}$$

before taking the limit.

Think about these results - they are quite natural.

(e)

$$\tilde{\sigma}(\tau) = \sigma(\tau) \frac{p_{ii}(\tau, t)}{p_{ai}(\tau, t)}\,,$$

$$\tilde{\mu}(\tau) = \mu(\tau) \frac{p_{di}(\tau, t)}{p_{ai}(\tau, t)} = 0\,.$$

Constant intensities and no recovery:

$$\begin{aligned} \tilde{\sigma}(\tau) &= \sigma \frac{e^{-\nu\,(t-\tau)}}{\int_\tau^t e^{-(\mu+\sigma)(s-\tau)}\, \sigma\, e^{-\nu\,(t-s)}\, ds} = \frac{1}{\int_\tau^t e^{(\nu-\mu-\sigma)(s-\tau)}\, ds} \\ &= \begin{cases} \frac{\nu-\mu-\sigma}{e^{(\nu-\mu-\sigma)(t-\tau)}-1}\,, & \nu - \mu - \sigma \neq 0\,, \\ \frac{1}{t-\tau}\,, & \nu - \mu - \sigma = 0\,. \end{cases} \end{aligned}$$

Exercise 2

$$\begin{aligned} dS(t) &= e^{\alpha t + \beta N(t)} \alpha dt + dN(t) \left(e^{\alpha t + \beta (N(t-)+1)} - e^{\alpha t + \beta N(t-)} \right) \\ &= e^{\alpha t + \beta N(t-)} \alpha dt + dN(t) e^{\alpha t + \beta N(t-)} \left(e^\beta - 1 \right) \\ &= S(t-) \left(\alpha\, dt + \left(e^\beta - 1 \right) dN(t) \right) \\ &= S(t-) \left(\alpha + \lambda \left(e^\beta - 1 \right) \right)\, dt + S(t-) \left(e^\beta - 1 \right) dM(t)\,, \end{aligned}$$

where $M(t) = N(t) - \lambda t$, a so-called martingale (a process with conditionally zero mean and uncorrelated increments, here actually independent increments).

The rest is straightforward .

Exercise 3
First brute force:

$$ {}_t p_{\overline{x_1 \ldots x_r}} = 1 - \prod_j (1 - {}_t p_{x_j}) \,. $$

Using the rule for differentiating a product (special case of Itŏ),

$$ \frac{d}{dt} {}_t p_{\overline{x_1 \ldots x_r}} = - \sum_k - \frac{d}{dt} {}_t p_{x_k} \prod_{j;\, j \neq k} (1 - {}_t p_{x_j}) = - \sum_k {}_t p_{x_k} \, \mu_{x_k + t} \prod_{j;\, j \neq k} (1 - {}_t p_{x_j}) \,. $$

It follows that

$$ \mu_{\overline{x_1 \ldots x_r}}(t) = \frac{\sum_k {}_t p_{x_k} \, \mu_{x_k + t} \prod_{j;\, j \neq k} (1 - {}_t p_{x_j})}{1 - \prod_j (1 - {}_t p_{x_j})} \,. $$

Second, direct reasoning: The conditional probability that the last survivor dies in $(t, t + dt)$, given that there are survivors at time t, is dt times the expression above.

Exercise 6

(a) Expected PV at time 0 of benefits:

$$ \int_0^n e^{-r\tau} (1 - {}_{\tau/2} p_x) {}_\tau p_y \mu_{y+\tau} \, d\tau \,. $$

Expected PV at time 0 of premiums is π times

$$ \int_0^{n/2} e^{-r\tau} {}_\tau p_x {}_\tau p_y \, d\tau \,. $$

Equivalence premium π is the ratio between these expressions.

(b) A straightforward method is to define

$$ \begin{aligned} v_1(t) &= 1 - {}_{t/2} p_x , \\ v_2(t) &= {}_t p_y , \\ v_3(t) &= \int_0^t e^{-r\tau} (1 - {}_{\tau/2} p_x) {}_\tau p_y \mu_{y+\tau} \, d\tau \,, \end{aligned} $$

and solve numerically the system of differential equations

$$ \begin{aligned} v_1'(t) &= \frac{1}{2} \mu_{x+t/2} \, (1 - v_1(t)) \,, \\ v_2'(t) &= -\mu_{y+t} \, v_2(t) \,, \\ v_3'(t) &= e^{-rt} v_1(t) \, v_2(t) \, \mu_{y+t} \,, \end{aligned} $$

by a forward difference scheme starting from the conditions $v_1(0) = 0$, $v_2(0) = 1$, $v_3(0) = 0$.

A more sophisticated method is hinted at in the problem: Observe that

$$ {}_{t/2}p_x = \exp\left(-\int_0^{t/2} \mu_{x+s}\, ds\right) = \exp\left(-\int_0^{t} \frac{1}{2}\mu_{x+s/2}\, ds\right) , $$

formally a survival function with intensity $\tilde{\mu}(t) = \mu_{x+t/2}/2$. Then the single premium is the difference between the single premiums of two well-known simple products, which may be computed by solving their Thiele differential equations numerically. Or compute by e.g. the program 'prores1' the expected discounted value of an assurance of 1 payable upon transition from state 1 to state 3 in a four states Markov model on $\{0, 1, 2, 3\}$, starting from state 0, with transition intensities $\mu_{01}(t) = \mu_{23}(t) = \mu_{x+t/2}/2$, $\mu_{02}(t) = \mu_{13}(t) = \mu_{y+t}$, and all other intensities 0.

(c) Reserve V_t at time $t \in [0, n]$ depends on what is currently known about (x) and (y):
$T_y \leq t$: $V_t = 0$.
$T_y > t$, $t \geq n/2$, $T_x > n/2$: $V_t = 0$.
$T_y > t$, $T_x \leq n/2 \wedge t$: $V_t = \int_{2T_x \vee t}^{n} e^{-r(\tau - t)}\, {}_{\tau-t}p_{y+t}\mu_{y+\tau}\, d\tau$.
$T_y > t$, $t < n/2$, $T_x > t$: $V_t = \int_{2t}^{n} e^{-r(\tau-t)}(1 - {}_{\tau/2-t}p_{x+t})\, {}_{\tau-t}p_{y+t}\mu_{y+\tau}\, d\tau - \pi \int_t^{n/2} e^{-r(\tau-t)}\, {}_{\tau-t}p_{x+t}\, {}_{\tau-t}p_{y+t}\, d\tau$.

(d) In general, for a unit due at some random time, the non-central 2nd moment of present value is the same as the expected value, only with $2r$ instead of r.

Exercise 7

(a) Apology: Problems of this kind are one of the favorite sports of classical actuaries, not because they are so common in practice (market share in the per mille range), but rather because they can entertain and stimulate the brains of actuaries. The proposed product is not totally inconceivable, however: it might be useful for a couple that needs to secure economically the last survivor and also their children after the possible early death of the last survivor. It is also of some theoretical interest beyond that of mere parlor games as it is an example of a product where payments are dependent on the past history of the driving process. This is seen clearly if the problem is formulated in the set-up of the Markov chain models for two lives. Now to work:

As is almost always the case, the best method is to find the expected value of the discounted payment in each small time interval $(\tau, \tau + d\tau)$ and then sum over all times. For $\tau \leq 20$ the benefit is running if (and only if) $T_{xy} < \tau$. For $\tau > 20$ the benefit is running if $\tau - 20 < T_{xy} < \tau$ or if $T_{xy} < \tau - 20$ and $\tau - 10 < T_{\overline{xy}} < \tau$. We gather the following expected value of future discounted payments at time 0:

$$\int_0^{20} e^{-r\tau}\,(1 - {}_\tau p_x\, {}_\tau p_y)\, d\tau$$
$$+ \int_{20}^{\infty} e^{-r\tau}\,[{}_{\tau-20}p_x\, {}_{\tau-20}p_y - {}_\tau p_x\, {}_\tau p_y]\, d\tau$$
$$+ \int_{20}^{\infty} e^{-r\tau}[(1 - {}_{\tau-20}p_x)\,({}_{\tau-10}p_y - {}_\tau p_y) + (1 - {}_{\tau-20}p_y)\,({}_{\tau-10}p_x - {}_\tau p_x)]\, d\tau\,.$$

(b) The premium rate π is the ratio between the expected present value in item (a) and the expected present value

$$\int_0^{\infty} e^{-r\tau}\, {}_\tau p_x\, {}_\tau p_y\, d\tau\,.$$

The reserve is a long and tedious story. One must, at each time of consideration t, distinguish between all possible past histories of the two lives along. For instance, if $T_{xy} > t$, then the reserve is simply the first expression above minus π times the second expression above, with x and y replaced by $x+t$ and $y+t$.

Exercise 8

$$\int_0^{20} e^{-r\tau}\, {}_\tau p_x\, \mu_{x+\tau}\, {}_\tau p_z\, d\tau + \int_{20}^{\infty} e^{-r\tau}\, {}_\tau p_x\, \mu_{x+\tau}\, {}_{\tau-20}p_y\, {}_\tau p_z\, d\tau\,.$$

A benefit of 1 payable immediately upon the death of (x) if (z) is then still alive and (y) was alive 20 years ago.

Exercise 9

(a)

$$p_{ai}^{(1)}(0, t+dt) = p_{ai}^{(1)}(0,t)(1 - (\nu(t) + \rho(t))\, dt) + p_{\overline{aa}}(0,t)\,\sigma(t)\, dt$$

leads to

$$\frac{d}{dt} p_{ai}^{(1)}(0,t) = -p_{ai}^{(1)}(0,t)(\nu(t) + \rho(t)) + p_{\overline{aa}}(0,t)\,\sigma(t)\,,$$

with side condition

$$p_{ai}^{(1)}(0,0) = 0\,.$$

Integrating gives the following integral expression, which could be put up by direct reasoning:

$$p_{ai}^{(1)}(0,t) = \int_0^t \exp(-\int_0^s (\mu + \sigma))\sigma(s)\, ds\, \exp(-\int_s^t (\nu + \rho))\,.$$

Next,

$$p_{aa}^{(1)}(0, t+dt) = p_{aa}^{(1)}(0,t)(1 - (\mu(t) + \sigma(t))\, dt) + p_{ai}^{(1)}(0,t)\,\rho(t)\, dt$$

leads to

$$\frac{d}{dt}p_{aa}^{(1)}(0,t) = -p_{aa}^{(1)}(0,t)(\mu(t)+\sigma(t)) \,+\, p_{ai}^{(1)}(0,t)\,\rho(t)\,,$$

with side condition

$$p_{aa}^{(1)}(0,0) = 0,.$$

Integral expression, which could be put up by direct reasoning:

$$p_{aa}^{(1)}(0,t) = \int_0^t p_{ai}^{(1)}(0,s)\,\rho(s)\,ds\,\exp(-\int_s^t(\mu+\sigma))\,.$$

As an exercise, repeat the argument for $k = 2, 3, ...$ to find differential equations for the probability $p_{ai}^{(k)}(0,t)$ of being disabled for the k-th time and the probability $p_{aa}^{(k)}(0,t)$ of being active after having been disabled k times.

One could attack this problem by redefining the state-space of the process as indicated in Figure 1, where the notation speaks for itself:

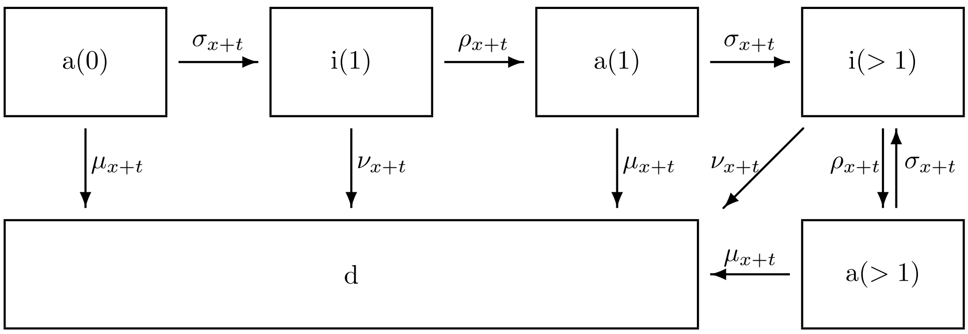


Figure 1: Problem 61

(b)

$$p_{ai}^{(1)}(0,t-q)p_{\overline{ii}}(t-q,t) = \int_0^{t-q} p_{\overline{aa}}(0,\tau)\,\sigma_{x+\tau}\,p_{\overline{ii}}(\tau,t)\,d\tau\,.$$

You should be able to interpret the integral expression as a sum of probabilities of mutually exclusive favourable events.

(c)

$$\pi = \frac{\int_q^n e^{-r\tau}p_{ai}^{(1)}(0,\tau-q)p_{\overline{ii}}(\tau-q,\tau)\,d\tau}{\int_0^n e^{-r\tau}p_{\overline{aa}}(0,\tau)\,d\tau}\,.$$

Comment: The premium plan is unacceptable in practice since it will produce a negative reserve if the insured is in premium paying state after time $n-q$ (then, for sure, no benefits will be received, but premium will still be paid). Therefore,

the premiums should be paid only over a shorter period and certainly not after time $n-q$.
Reserve: $\int_t^{t+q} e^{-r(\tau-t)} p_{\overline{ii}}(t,\tau)\, d\tau \;+\; \int_{t+q}^{n} e^{-r(\tau-t)} p_{ii}(t,\tau-q) p_{\overline{ii}}(\tau-q,\tau)\, d\tau\,.$

Exercise 10
See files Exercise10a.txt and Exercise10b.txt on public folder.

Exercise 12
Items (a) - (d) are rather theoretical and not typical exam questions in ST305. Anyway, we offer something for students who accept only statements that have been firmly proved.

(a) RTI$(T|T)$ is easy to prove:

$$\mathbb{P}[T > s \,|\, T > t] \;=\; \frac{\mathbb{P}[T > \max(s,t)]}{\mathbb{P}[T > t]} \;=\; \begin{cases} \frac{\mathbb{P}[T>s]}{\mathbb{P}[T>t]} & , \quad t < s\,, \\ 1 & , \quad t \geq s\,. \end{cases}$$

This is obviously an increasing (non-decreasing) function of t for fixed s. PQD(T,T) and AS(T,T) then follow.

We could also prove PQD(T,T) directly:

$$\mathbb{P}[T > s\,,\, T > t] = \mathbb{P}[T > \max(s,t)] \geq \mathbb{P}[T > s]\,\mathbb{P}[T > t]\,.$$

A direct proof of AS(T,T) goes as follows: Let $g(s,t)$ and $h(s,t)$ be increasing functions in both arguments. Then $\tilde{g}(t) = g(t,t)$ and $\tilde{h}(t) = h(t,t)$ are increasing functions in t. Thus, marginal association is enough, see notes 'depend-l.pdf'.

(b) PQD$(-S|T)$ means

$$\mathbb{P}[-S > -s\,,\, T > t] \;\geq\; \mathbb{P}[-S > -s]\,\mathbb{P}[T > t]$$

for all s (or all $-s$, which is the same, of course) and all t. This is the same as

$$\mathbb{P}[S < s\,,\, T > t] \;\geq\; \mathbb{P}[S < s]\,\mathbb{P}[T > t]\,,$$

which is the same as

$$\mathbb{P}[T > t] \;-\; \mathbb{P}[S \geq s\,,\, T > t] \;\geq\; (1 \;-\; \mathbb{P}[S \geq s])\,\mathbb{P}[T > t]\,,$$

which is the same as

$$\mathbb{P}[S \geq s\,,\, T > t] \;\leq\; \mathbb{P}[S \geq s]\,\mathbb{P}[T > t]\,.$$

Due to the result in (b), this is the same as the asserted result.

(c) AS($-S,T$) means that

$$\mathbb{C}(g(-S,T),h(-S,T)) \geq 0$$

for all g and h that are increasing in both arguments. But this is equivalent to the asserted result.

(d) RTI($-S|T$) means that $\mathbb{P}[-S > -s \,|\, T > t]$ is increasing in t for fixed s. Rewriting

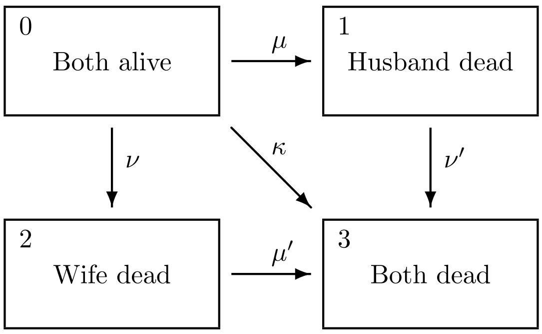
$$\mathbb{P}[-S > -s \,|\, T > t] \;=\; 1 \;-\; \mathbb{P}[S \geq s \,|\, T > t]$$

and recalling the result in (b), we arrive at the asserted result.

(e) The Markov model is sketched in the figure below. Only small amendments are needed in the calculations made in the theory ('depend-l.pdf'), but the results are a bit surprising. We will discuss the matter under the more general assumption that $\mu'_t \geq \mu_t$ and $\nu'_t \geq \nu_t$.

First the case $s \leq t$:

$$\begin{aligned}
\mathbb{P}[S > s \,|\, T > t] \quad &= \quad \frac{e^{-\int_0^t \mu+\nu+\kappa} + \int_s^t e^{-\int_0^\tau \mu+\nu+\kappa}\,\mu_\tau\, e^{-\int_\tau^t \nu'}\,d\tau}{e^{-\int_0^t \mu+\nu+\kappa} + \int_0^t e^{-\int_0^\tau \mu+\nu+\kappa}\,\mu_\tau\, e^{-\int_\tau^t \nu'}\,d\tau} \\
&= \quad 1 - \frac{\int_0^s e^{-\int_0^\tau \mu+\nu+\kappa-\nu'}\,\mu_\tau\,d\tau}{e^{-\int_0^t \mu+\nu+\kappa-\nu'} + \int_0^t e^{-\int_0^\tau \mu+\nu+\kappa-\nu'}\,\mu_\tau\,d\tau}\,.
\end{aligned}$$



We need to discuss this expression as a function of t, which appears only in the denominator. The derivative of the denominator is

$$e^{-\int_0^s \mu+\nu+\kappa-\nu'}(\nu'_t - \nu_t - \kappa_t)\,.$$

It follows that, in the presence of a positive κ_t, $\mathbb{P}[S > s \,|\, T > t]$ is not in general an increasing function function of t if $\nu'_t \geq \nu_t$. For $\nu'_t = \nu_t$ it is actually decreasing. We have thus already answered the question and need not look into the case $s < t$.

The second part of the question is now easily sorted out in the case $s < t$ by setting $\mu'_t = \mu_t + \kappa_t$ and $\nu'_t = \nu_t + \kappa_t$ in the result above. We find that $\mathbb{P}[S > s \,|\, T > t]$ is independent of t for $s < t$. Therefore, the RTI issue is so far unsettled and we need to investigate the case $s > t$:

$$\begin{aligned}\mathbb{P}[S > s \,|\, T > t] &= \frac{e^{-\int_0^s \mu+\nu+\kappa} + \int_t^s e^{-\int_0^\tau \mu+\nu+\kappa}\,\nu_\tau\, e^{-\int_\tau^s \mu+\kappa}\,d\tau}{e^{-\int_0^t \mu+\nu+\kappa} + \int_0^t e^{-\int_0^\tau \mu+\nu+\kappa}\,\mu_\tau\, e^{-\int_\tau^t \nu+\kappa}\,d\tau} \\ &= e^{-\int_t^s \kappa}\,\mathbb{P}^*[S > s \,|\, T > t]\,,\end{aligned}$$

where $\mathbb{P}^*$ denotes probability under the independence hypothesis $\mu'_t = \mu_t$ and $\nu'_t = \nu_t$ (see expression in 'depend-l.pdf'). This is an increasing function of t, and we have proved RTI$(S|T)$.

Exercise 15

Pure verification - just insert the appropriate expressions on the right hand side. Any combination of cash bonus at rate $\tilde{b}_t = \alpha_t\, c_t$ and additional death benefit of $\tilde{\tilde{b}}_t = (1-\alpha_t)\, c_t/\mu_{x+t}$, $0 \le \alpha \le 1$, produces a right hand side equal to the left hand side.

Exercise 16

$\tilde{b}_n$ must solve

$$\int_0^n e^{-\int_0^\tau (r_u+\mu_{x+u})\,du} c_\tau\, d\tau \;=\; e^{-\int_0^n (r_u+\mu_{x+u})\,du} \tilde{b}_n\,, \tag{5}$$

hence

$$\tilde{b}_n \;=\; \int_0^n e^{\int_\tau^n (r_u+\mu_{x+u})\,du} c_\tau\, d\tau\,. \tag{6}$$

Exercise 17

Relation (7) becomes

$$c_t = \Delta r\, V_t^* \;+\; \Delta\mu(b_t - V_t^*)\,.$$

Here are some examples of time t prognosis of future bonuses, assuming that the insured will survive the term of the contract:
1. Rate of cash bonus payments at time $u \in (t, n)$ is just c_u defined above.
2. Present value of future cash bonuses:

$$\int_t^n e^{-(r^*+\Delta r)(\tau - t)}\, c_\tau\, d\tau\,,$$

3. Value of terminal bonus (not discounted):

$$\int_0^n e^{\int_\tau^n (\mu^*_{x+s} + \Delta\mu + r^* + \Delta r)\, ds}\, c_\tau \, d\tau \, .$$

Exercise 18

All that is needed is to put $m+n$ in the role of n and work with the general formulas. Here V^*_t is given by

$$\begin{aligned}
\frac{d}{dt} V^*_t &= (\mu^*_{x+t} + r^*)\, V^*_t + \pi \, , \qquad 0 < t < m \, , \\
\frac{d}{dt} V^*_t &= (\mu^*_{x+t} + r^*)\, V^*_t - 1 \, , \qquad m < t < m+n \, .
\end{aligned}$$

This is the only place where the particulars of the contract matter: Thiele's differential equation is needed for the computation of V^*_t alongside that of c_t.

Since $V^*_t > 0$ for all $t \in (m, m+n)$, also $c_t > 0$ throughout this time interval.

Exercise 19

Expenses can be treated as benefits in addition to those specified in the contract (see Chapter 5). We need the differential equation for the first order gross reserve,

$$\frac{d}{dt} V^{*'}_t = V^{*'}_t \, r^* + \pi - \beta^* \pi - \gamma^* b - \mu^*_{x+t} (b - V^{*'}_t) \tag{7}$$

(with side condition $V^{*'}_{n-} = b$), and the equivalence relationship,

$$V^{*'}_0 = -\alpha^* b \, ,$$

which determines π. The discounted mean surplus per policy at time t is now

$$\begin{aligned}
S_t = & -(\alpha' + \alpha'' b) \\
& + \int_0^t e^{-\int_0^\tau (r_s + \mu_{x+s})\, ds} (\pi - \mu_{x+\tau}\, b - \beta'_\tau - \beta''_\tau \pi' - \gamma'_\tau - \gamma''_\tau b - \gamma'''_\tau V^{*'}_\tau)\, d\tau \\
& - e^{-\int_0^t (r_s + \mu_{x+s})\, ds} V^{*'}_t \, .
\end{aligned}$$

It is seen that

$$S_0 = -(\alpha' + \alpha'' b) - V^{*'}_0 = \alpha^* b - (\alpha' + \alpha'' b) \, ,$$

which is the surplus arising immediately upon issue of the contract due to prudent first order assumptions about the initial cost. It is positive (and indeed prudent) if

$$\alpha^* b > (\alpha' + \alpha'' b) \, ,$$

which means that the first order initial cost is set on the safe side. (This cannot be achieved for all $b > 0$ if $\alpha' > 0$; one then has to assume that b is greater than a certain minimum, which is certainly the case in practice.)

The dynamics of the surplus is

$$\begin{aligned} dS_t &= e^{-\int_0^t (r_s+\mu_{x+s})\,ds}(\pi - \mu_{x+t}\, b - \beta_t' - \beta_t''\pi' - \gamma_t' - \gamma_t''b - \gamma_t'''V_t^{*'}\,)\,dt \\ &\quad + e^{-\int_0^t (r_u+\mu_{x+u})\,du}\,(r_t + \mu_{x+t})\,V_t^{*'} \;-\; e^{-\int_0^t (r_u+\mu_{x+u})\,du}\,dV_t^{*'}\,. \end{aligned}$$

Inserting $dV_t^{*'} \;=\; \frac{d}{dt}V_t^{*'}\,dt$ from (7), we gather

$$dS_t = e^{-\int_0^t (r_s+\mu_{x+s})\,ds}\,c_t\,dt\,,$$

where

$$\begin{aligned} c_t &= (r_t - r^*)V_t^{*'} \;+\; (\beta^*\pi' - \beta_t' - \beta_t''\pi') \\ &\quad + (\gamma^*V_t^{*'} - \gamma_t' - \gamma_t''b - \gamma_t'''V_t^{*'}) \;+\; (\mu_{x+t}^* - \mu_{x+t})(b - V_t^{*'}) \end{aligned}$$

is the mean contribution to surplus per survivor at time t. This contribution decomposes into gains stemming from safety loadings in the various first order elements – interest, expenses of β type, expenses of γ type, and mortality – and how these elements can be set on the safe side. Just as for the initial cost, there is a problem with the safety loading on expenses of β and γ type: if e.g. $\gamma_t' > 0$, then there will inevitably be a loss on the γ expenses for small t since the gross reserve starts from a negative value. This loss has to be compensated by setting other first order elements sufficiently to the safe side to make c_t (or at least S_t) non-negative for all $t \in (0, n)$.

Exercise 20

This is a trivial one, and the same goes for the conditional expected value of any random variable that depends only on the state of Y at some fixed future time. Starting from

$$W_e(t) \;=\; (1 - \lambda_{e\,\cdot}\,dt)W_e(t + dt) \;+\; \sum_{\ell;f\neq e} \lambda_{ef}\,dt\,W_f(t + dt) \;+\; o(dt)\,,$$

we get

$$W_e(t) - W_e(t + dt) \;=\; -\lambda_{e\,\cdot}\,W_e(t + dt) \;+\; \sum_{\ell;f\neq e} \lambda_{ef}\,dt\,W_f(t + dt) \;+\; o(dt)\,,$$

and, dividing by dt and letting $dt \searrow 0$, we arrive at the answer. The side conditions are obvious (as always).

Exercise 21

We start with W_e' and supply details (to be precise, add a term $o(dt)$ on the right of the two expressions given for W'_t and W''_t in the exercise):

$$\begin{aligned}
W'_e(t) &= (1-\lambda_{e\,\cdot}dt)\mathbb{E}\left[e^{r_t\,dt}W'_{t+dt}\,\middle|\,Y_\tau = e\,,\,t\le\tau\le\tau+d\tau\right] \\
&\quad + \sum_{f;f\neq e}\lambda_{ef}\,dt\,\mathbb{E}\left[e^{r_t\,dt}W'_{t+dt}\,\middle|\,Y_t = e\,,\,Y_{t+dt}=f\right]\;+\;o(dt) \\
&= (1-\lambda_{e\,\cdot}dt)e^{r^e\,dt}W'_e(t+dt) + \sum_{f;f\neq e}\lambda_{ef}\,dt\,e^{O(dt)}W'_f(t+dt)\;+\;o(dt)\,,
\end{aligned}$$

where $O(dt)$ signifies a term of order dt (i.e. such that $O(dt)/dt$ is bounded as $dt\searrow 0$). Inserting the Taylor expansions

$$\begin{aligned}
e^{r^e\,dt} &= 1+r^e dt+o(dt)\,, \\
W'_e(t+dt) &= W'_e(t)+\frac{d}{dt}W'_e(t)\,dt+o(dt)\,, \\
e^{O(dt)} &= 1+O(dt)\,, \\
W'_f(t+dt) &= W'_f(t)+O(dt)\,,
\end{aligned}$$

multiplying out, gathering all $o(dt)$ terms and rearranging a bit, one obtains the differential equations for the functions W'_e.

Next, the W''_e, a bit more sketchy and gathering $o(dt)$ terms currently as they arise without further mentioning:

$$\begin{aligned}
W''_e(t) &= (1-\lambda_{e\,\cdot}dt)\,(W'_e(t)\,(r^e-r^*)\,V^*_t\,dt\;+\;W''_e(t+dt)) \\
&\quad + \sum_{f;f\neq e}\lambda_{ef}\,dt\,W''_e(t)\;+\;o(dt)\,.
\end{aligned}$$

Proceeding as above, we obtain the differential equations for the functions W''_e.

The side conditions are obvious.

Exercise 22
Goes along the lines of Exercise 18.

Exercise 23
(a) Same thing again. Introduce

$$W_t\;=\;\int_t^n e^{-\int_t^\tau r_s\,ds}(r_\tau-r^*)V^*_\tau\,d\tau\,,$$

and write

$$W_t\;=\;(r_t-r^*)V^*_t\,dt\;+\;e^{-r_t\,dt}W_{t+dt}\;+\;o(dt)\,.$$

Apply the direct backward construction to $W_e(t)\;=\;\mathbb{E}[W_t\,|\,Y_t=e]$. Start from

$$\begin{aligned}
W_e(t) &= (1-\lambda_{e\,\cdot}dt)\Big((r^e-r^*)V^*_t\,dt\;+\;e^{-r^e\,dt}W_e(t+dt)\Big) \\
&\quad + \sum_{f;f\neq e}\lambda_{ef}\,dt\,W_f(t)\;+\;o(dt)\,,
\end{aligned}$$

and do a small piece of paper-work to arrive at

$$\frac{d}{dt}W_e(t) \,=\, W_e(t)\,r^e \,-\, (r^e - r^*)\,V_t^* \,-\, \sum_{f;f\neq e} \lambda_{ef}(W_f(t) - W_e(t))\,.$$

Side conditions are: $W_e(n-) = 0$, $e = 1, \ldots, J^Y$.

(b) Basically the same exercise as (a).

(c) For discounted cash bonuses use the general formula for higher order moments of present values of payment streams with state-dependent payment intensity and interest intensity.

Forget about the variance of terminal bonus (too cumbersome).

Exercise 26

We need to find $\mathbb{E}[S(n) \vee g]$. Start as usual from the conditional expected value of the random variable $S(n) \vee g$, given everything that is known by time t, $\mathcal{F}_t = \{N(\tau);\, 0 \leq \tau \leq t\}$:

$$\mathbb{E}[S(n) \vee g \,|\, \mathcal{F}_t] \,=\, \mathbb{E}\left[S(t) e^{\alpha(n-t)+\beta(N(n)-N(t))} \vee g \,\middle|\, \mathcal{F}_t \right]\,.$$

Here we have separated out what pertains to the past (known under the conditioning) and what pertains to the future (remains random under the conditioning). Due to the independent increments property of the Poisson process, it is seen that we can work with the function

$$W(s,t) \,=\, \mathbb{E}\left[s\, e^{\alpha(n-t)+\beta(N(n)-N(t))} \vee g \right]\,.$$

Preparing for a backward construction, write

$$W(s,t) \,=\, \,=\, \mathbb{E}\left[s\, e^{\alpha dt+\beta(N(t+dt)-N(t))} e^{\alpha(n-t-dt)+\beta(N(n)-N(t+dt))} \vee g \right]\,,$$

and proceed as usual, conditioning on what happens in $(t, t+dt)$:

$$\begin{aligned}
W(s,t) &= (1 \,-\, \lambda\, dt)\, W\left(s e^{\alpha dt}, t+dt\right) \,+\, \lambda\, dt\, W\left(s e^{\alpha dt+\beta}, t+dt\right) \,+\, o(dt) \\
&= W\left(s e^{\alpha dt}, t+dt\right) \,-\, \lambda\, dt\, W(s,t) \,+\, \lambda\, dt\, W\left(s e^{\beta}, t\right) \,+\, o(dt) \\
&= W(s \,+\, s\,\alpha\, dt\,,\, t+dt) \,-\, \lambda\, dt\, W(s,t) \,+\, \lambda\, dt\, W\left(s e^{\beta}, t\right) \,+\, o(dt) \\
&= W(s,t) \,+\, \frac{\partial}{\partial s} W(s,t)\, s\, \alpha\, dt \,+\, \frac{\partial}{\partial t} W(s,t)\, dt \\
&\quad - \lambda\, dt\, W(s,t) \,+\, \lambda\, dt\, W\left(s e^{\beta}, t\right) \,+\, o(dt)\,.
\end{aligned}$$

We arrive at

$$\frac{\partial}{\partial t} W(s,t) \,=\, -\frac{\partial}{\partial s} W(s,t)\, s\, \alpha \,-\, \lambda \left(W\left(s e^{\beta}, t\right) - W(s,t) \right)\,,$$

which is to be solved subject to the condition

$$W(s,n) \;=\; s \vee g\,.$$

Remark: We could have written

$$\pi \;=\; \mathbb{E}\left[e^{\beta N(n)} \vee g e^{-\alpha\, n}\right]\; e^{(\alpha - r)\, n}\; {}_n p_x\,,$$

and, redefining $W(s,t)$ accordingly, essentially get rid of $e^{-\alpha\, t}$. We have chosen the present approach since it gives us an opportunity to see the different roles of the (non-stochastic) smooth function $e^{-\alpha\, t}$ and the (stochastic) jump process $N(t)$.