EXTRA EXERCISES ST305, A. DASSIOS

EXERCISE 0

Please calculate the following using the Danish mortality table G82M, MAPLE and the file life1.mws where appropriate.

A friend of yours was born on the 1st of October 1946 and he is now (1.10.2006) alive. What is the probability he will die within 10 years? (No need to use MAPLE for this one)

What is his expected remaining lifetime and what is its standard deviation?

Another friend of yours was born on the 1st of October 1936. You last spoke to him on the 1st of October 1996 and he was alive then. On the 1st of October 2006 you called him up to wish him happy birthday but you found out he was dead. What is the probability he died less than 5 years ago? (No need to use MAPLE for this one)

Let K be the time of his death with the 1st of October 1996 being time 0. Calculate E(K)

EXERCISE 1

Please use a force of interest of 4%, the Danish G82M table and MAPLE (loading the file life1.mws is helpful) to solve the following exercise.

A pension contract with a life insurance element commencing on the first of October 2006 provides the following benefits:

1. If the life is alive at the age of 65, a pension in the form of a life annuity of 30000 per annum payable continuously.

2. If the life dies before the age of 65 a lump sum of 100000 payable upon death.

The life pays an annual premium on the first of October of every year from 2006, but premiums cease after the age of 65 or at death if earlier. The life was born on the first of April of 1971. Calculate the annual premium.

Suppose now that the premium is payable continuously. Do you expect the premium to be higher or lower than before? Calculate it to check if your guess is right.

EXERCISE 2

Continuing from exercise 1 calculate the reserve that the office has to set aside on the 2nd of October of 2021 and the 2nd of April of 2041 for both models (continuous and discrete premiums). Comment on any differences.

Use both methods (prospective and retrospective) and check that the reserves are the same however you calculate them.

EXERCISE 3

Please do the following exercise, which is more difficult than the previous ones. You should make sure you have revised the relevant material on the term structure of interest rates from ST226.

Please use a the Danish G82M table and MAPLE (loading the file life1.mws is helpful) to solve the following exercise.

A pension contract with a life insurance element commencing on the first of October 2006 provides the following benefits:

1. If the life is alive at the age of 65, a pension in the form of a life annuity of 30000 per annum payable continuously.

2. If the life dies before the age of 65 a lump sum of 100000 payable upon death.

The life pays a continuous premium, but premiums cease after the age of 65 or at death if earlier. The life was born on the first of April of 1971. You have calculated the annual premium for a fixed interest rate in extra exercise 1 and the reserves to be set aside on the 2nd of October of 2021 and the 2nd of April of 2041.

Assume now a term structure of interest rates such that the vaue of a zero coupon bond with term t is given by

$$\exp\big\{-\big(0.07-0.04e^{-0.03t}\big)t\big\}.$$

Calculate the annual premium and the reserves to be set aside on the same dates. Once again use both prospective and retrospective formulae to see if the answers are the same.

EXERCISE 4

Using a force of interest of 0.04 and the Danish G82M mortality table, calculate for a life aged 65 the following:

1. The expected present value of a continuous life annuity guaranteed for 5 years and payable thereafter for life.

2. The expected present value of a continuous life annuity payable till 5 years after the time of death.

Explain why the two values differ and also explain how you would calculate the standard deviation of both quantities

EXERCISE 5

For a rather old category of lives the 1-year and 2-year survival probabilities are 0.94 and 0.87 respectively. The value of a life annuity payable annually in advance calculated at a force of interest of 0.04 is 7.52.

1. A particular life has passed a medical examination before purchasing the annuity and is therefore subject to a select mortality table, where for the first year the probability of death is only 80% of the corresponding probability for non-select lives and for the second year 90% of the corresponding probability for non-select lives. Thereafter there is no difference in mortality. Calculate the value of the life annuity described earlier for this life.

2. Assume now that for the first year the force of mortality is 80% of the corresponding force for non-select lives and for the second year 90% of the corresponding force for non-select lives. Thereafter there is no difference in mortality as before. Calculate the value of the life annuity described earlier. (No other details of the mortality table are given)

EXERCISE 6

Please use a force of interest of 0.04, the G82M mortality table and MAPLE to do calculations for this exercise. However, the point is to derive the formulae for each quantity asked correctly, so please do your calculations only after you have finished doing this.

Three partners A, B and C aged 40, 45 nd 50 take out a 20 year joint temporary life insurance contract. 1m is paid at the second death to the survivor, provided this has happened before the 20 year period has elapsed. Assume expenses that are 5% of premiums. The mortalities of the three lives are independent. Calculate the premium charged in the following three cases. 1. A single premium is charged up front. 2. A continuous premium is payable for as long as all partners are alive and for no more than 10 years. 3. A continuous premium is payable as long as two lives are alive and for no more than 10 years.

Suppose now at the end of the period you know that the sum assured was paid. What is the probability that life A was the survivor that got the money. What is the same probability for B and C?

Suppose that you actually know that the sum was paid 15 years exactly after the policy took effect. What are the same probabilities for A, B and C now?

EXERCISE 7

Consider two identical (same age and mortality) and independent lives 1 and 2, whose force of mortality is increasing with age. Prove the following:

1. The probability that they will die within k years of each other is larger than the probability that life 1 will die within k years.

2. The expected value of the interval between their two deaths is smaller than the expected remaining lifetime of life 1.

Do you find the two statements intuitive? Why?

EXERCISE 8

Consider two independent lives aged 40 and 45 that are subject to the Danish G82M mortality table. Use MAPLE to calculate the following:

The probability they will die within 10 years of each other. 2. The expected value of the time interval between their two deaths. Please note that although you are probably intelligent enough to make MAPLE calculate double integrals, this is strongly discouraged. There is an elegant way to calculate this as a single integral.

Suppose now you have three independent lives aged 40, 45 and 50. How would you calculate the expected value of the time interval between the first and the last of the three deaths?

EXERCISE 9

For the model with three states active, inactive and dead as laid out in section 7.3 (diagram in page 76) of you life book, assume constant transition intensities throughout and solve the backward and forward differential equations.

Please feel free to use the fact that the relevant probabilities add up to 1. You should of course get the same answers whether you solve the backward or the forward equations.

EXERCISE 10

Please answer the following questions. They are designed to make you think rather than do too much work. You might find question 2 hard, but please persist till you get an idea of what to do. You do not need to answer question 2 in order to do question 3.

Extra Exercise 10

1. An employee of a company can be working in one of their two sites. The rate of transition from site 1 to site 2 is λ_1 and the rate of transition from site 2 to site 1 is λ_2. The employee who is aged x is also subject to a force of mortality μ_{x+t} at time t regardless of where he is working. Let $p_1(t)$ denote the probability that he is at site 1 at time t and $p_2(t)$ denote the probability that he is at site 2 at time t. Derive the forward equations and show that

$$p_1(t) = \left(\frac{\lambda_1}{\lambda_1 + \lambda_2} \exp(-(\lambda_1 + \lambda_2)t) + \frac{\lambda_2}{\lambda_1 + \lambda_2} \cdot \right) {}_tp_x$$

and

$$p_2(t) = \left(\frac{\lambda_1}{\lambda_1 + \lambda_2} - \frac{\lambda_1}{\lambda_1 + \lambda_2} \exp(-(\lambda_1 + \lambda_2)t) \cdot \right) {}_tp_x$$

where

$${}_tp_x = \exp\left(-\int_0^t \mu_{x+s} ds \right).$$

You can of course use direct substitution to show this. However, there is a nice way to derive the probabilities without knowing the answer. Hint: Think of an *immortal* individual moving between the two sites.

2. Suppose now that $\lambda_1 = \lambda_2 = 0.1$, but also that when the employee is in site 2 he is subject to an extra force of mortality 0.008 (so the force of mortality while in site 2 is $0.008 + \mu_{x+t}$). The force of mortality while in site 1 is as in question 1. Show that

$$p_2(t) = 0.4996\left(e^{-0.00392t} - e^{-0.20408t}\right) {}_tp_x.$$

Note that this time you can not use direct substitution as the answer for $p_1(t)$ is not given to you, so you have to think of another way. Hint: Think of an *almost immortal* individual moving between the two sites.

3. Use the worksheet life3.mws and MAPLE (you can assume μ_{x+t} is the usual Danish G82M formula) to solve the forward equations arising in question 2 and thus confirm the formula in question 2 fro various values of t.

4. Continuing from question 2 suppose that the employee died at time $t = 10$. What is the probability he was at site 2 at the time? What would the answer be under the assumptions of question 1?

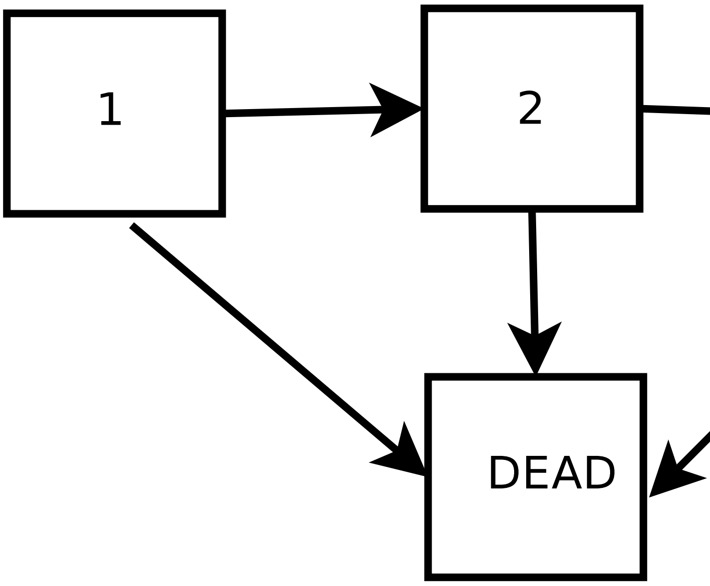
EXERCISE 11

Please do the attached exercise. You need MAPLE (life1.mws is sufficient) to calculate some integrals, but of course it is more important that you derive are write down the correct expressions rather than actually calculating them.

Please replace the force of mortality at state 3 with a constant 0.4 instead of 0.4exp(0.002x), This will make calculations a bit easier (it is still a hard exercise).

EXERCISE 11

The following (not very artistic) diagram illustrates the development of AIDS amongst class of lives. State 1 represents a healthy life, state 2 a life that is HIV positive but has not developed AIDS, state 3 represents full AIDS and state "dead" is self-explanatory.



The mortality force while at state 1 is given by the Danish G82M table; the same mortality force applies to lives in state 2. The mortality force for a life aged x suffering from AIDS ids given by $\mu_x^{(3)} = 0.4\exp(0.002x)$. The force of transition from state 1 to state 2 is 0.01 and the force of transition from 2 to 3 is 0.03. The force of interest is 0.05 per annum.

1. Suppose a life aged 35 is in state 2. What is the probability of the event: "the life was in state 1 10 years ago and will be dead in 10 years time" ? What makes the calculation easy?

2. The following policy is effected for a 35 year old life, currently at state 1. On transition from state 2 to state 3 an amount of 100000 is paid and while at state 3 a continuous income of 50000 per annum. The policy expires at the age of 60 if the life is still at state 1, but it continues being valid for life otherwise. There are no other benefits. The policy id financed by a single premium payable up front. Calculate the premium.

3. Derive integral formulae for the reserve at all times and all states. Calculate these reserves at time $t = 10$.

4. Suppose the policy were to be offered to a 45 year old life currently at state 2. Calculate the single premium.

EXERCISE 12

This is a continuation of extra exercise 10 whose results you should of course use. A number of calculated annuities and probabilities are provided. The idea is to use MAPLE as little as possible and hopefully not at all, Some of the information might not be relevant. A little hint: There is of course a way to calculate assurance functions from annuity functions.
Correction note: Please note that the value of the 35 year annuity at a force of 0.35 is 2.8418 (not 3.8418) Sorry about the typo, which makes some of the calculations strange.

Extra Exercise 12

The following calculations might be useful to you for this exercise. You might not need all of them.

$${}_5p_{30} = 0.991 \qquad {}_{35}p_{30} = 0.77$$

At a force of interest 0.05

$$\bar{a}_{30:\bar{5}|} = 4.4059 \qquad \bar{a}_{30:\overline{35}|} = 15.7932$$

At a force of interest 0.15

$$\bar{a}_{30:\bar{5}|} = 3.5044 \qquad \bar{a}_{30:\overline{35}|} = 6.5227$$

At a force of interest 0.25

$$\bar{a}_{30:\bar{5}|} = 2.8444 \qquad \bar{a}_{30:\overline{35}|} = 3.9663$$

At a force of interest 0.35

$$\bar{a}_{30:\bar{5}|} = 2.3535 \qquad \bar{a}_{30:\overline{35}|} = 3.8418$$

1. An employee of a company can be working in one of their two sites. The rate of transition from site 1 to site 2 is 0.1 and the rate of transition from site 2 to site 1 is 0.2 . The employee who is aged x is also subject to a force of mortality μ_{x+t} at time t regardless of where he is working. Let $p_1(t)$ denote the probability that he is at site 1 at time t and $p_2(t)$ denote the probability that he is at site 2 at time t. Derive the forward equations and show that

$$p_1(t) = \left(\frac{1}{3}\exp(-0.3t) + \frac{2}{3}\cdot\right) {}_tp_x$$

and

$$p_2(t) = \left(\frac{1}{3} - \frac{1}{3}\exp(-0.3t)\right) {}_tp_x$$

where

$${}_tp_x = \exp\left(-\int_0^t \mu_{x+s} ds\right).$$

The employee is currently at stage 1 and aged 30. The company is offering him the following benefits. On every transition from 1 to 2 a sum of 5000, a continuous allowance of 2000 per annum while at state 2 and a death benefit of 50000 provided death occurs while at state 2. All benefits will stop when the employee retires at the age of 65. Use a force of interest of 0.05 to calculate the expected present value of the benefits.

2. Suppose now that the allowance has a qualification period of 5 years. This means that the employee starts getting the allowance after he has styed at state 2 for an uninterrupted period of 5 years. This qualification period does not apply to any other benefits, Calculate the expected present value of the benefits.

3. Suppose now that all the benefits in question 1 (without a qualification period) apply to the first visit to site 2 only and no benefits are applicable for subsequent visit, Calculate the expected present value of the benefits if they apply to the first visit to site 2 only and nothing is paid after the employee leaves state 2. (Hint: Find the probability that the employee has not yet left state 1 and that he is in state 2 for his first visit first).

4. Suppose now that all the benefits in question 1 (with the qualification period) apply to the first visit to site 2 only and no benefits are applicable for subsequent visit, Calculate the expected present value of the benefits if they apply to the first visit to site 2 only and nothing is paid after the employee leaves state 2.

5. Provide expressions for the reserve at any time t at all possible states for the contracts in each one of the first four questions.

EXERCISE 13

This one has some questions that might be harder than they look.
Correction note: Question 4 obviously refers to question 3 rather than question 2.

Extra Exercise 13

1. An insurer has identified n possible causes of death. These are exhaustive (there is a category called other). The force of mortality due to cause i for a life aged x is given by $\mu_x^{(i)}$. A whole life assurance contract is in force under which a life will receive an amount b_i if she dies due to cause i. Provide expressions for the following:

   i. The expected death benefit and its variance.

   ii. The expected death benefit and its variance given that it occurs at time k.

   iii. The expected death benefit and its variance given that it occurs before time k.

   What is the distribution of the death benefit in its case. Use your results to provide an expression for the continuous premium payable while the life is alive.

2. Continuing from question 1, provide expressions for the expected value of the reserve at time t and its variance.

3. Suppose now that the company is offering a joint policy to two independent lives. The benefit is variable as in question 1 and is payable at the time of the first death and it depends on its cause. Provide expressions for the mean and the variance of the benefit and also for the mean and the variance given it occurs at time k. Use you results to calculate the value of the premium which is now payable upfront.

4. Repeat question 2 for the case where the benefit is payable at the time of the second death.

EXERCISE 14

An insurer is offering 30 year endowment assurance policies to lives aged 30. The sum of 100000 is payable at the end of the policy or at earlier death. The policies are financed by a continuous premium payable throughout. The policies might lapse at any time with a force 0.01 (the policyholder decides not to pay any more premiums). Should that happen during the first 20 years the company imposes a charge of 5% on the reserve and uses the remainder as a single premium for a pure endowment assurance payable at time 30. If the policy lapses during the last 10 years the charge is not applicable, so the whole reserve is used to finance the pure endowment. In either case there is no death benefit.

1, Using a force of interest of 0.05 the G82M table and MAPLE calculate the continuous premium. 2. Provide expressions for the reserve at all times and states. 3. What is the probability that a policyholder will get nothing at all? 4. Given that a policyholder is alive at time 30, calculate the expected value and the variance of the amount he receives.

EXERCISE 15

1. An office issued a with profits 20 year endowment assurance with a sum assured of 100000. The policyholder was then aged 40. The premium is payable continuously and is to be calculated using the Danish G82M table a force of interest of 0.04 and no expenses. The real force of interest is stochastic and can take the values 0.03 and 0.06. The transition rates between the two values are 0.5. The real mortality table is also the Danish G82M. The force of interest is 0.06 initially. The policyholder will receive all surplus in the form of a terminal bonus. Use MAPLE to predict the terminal bonus. Hint: You should use the equations suggested by normal exercise 21.

2. This is a lot more difficult. Suppose that we are at time t=5 and the force of interest has been 0.06 for the last 5 years (it still is). Use MAPLE to predict the terminal bonus.
Hint: It is not that simple to calculate the accumulated surplus. You have to solve the reserves equation in a forward way together with an equation for the retrospective accumulation of surplus.

EXERCISE 16

1. A pension plan provides a lump sum benefit at the age of 65 of 3 times the memeber's salary five years before retrirement. The menber who is entitled to the benefit is now aged 30. The force of interest is assumed to be deterministic and given by $r(t)$. The salary is also calculated as a deterministic function, where the salary of the member at time t is given by $S_0 \exp\Big(\int_0^t a(s) \mathrm{ds} \Big)$, where S_0 is the current salary. Assume a force of mortality μ_{30+t} at time t and expenses at a constant rate c. The plan is finaced by a single premium payable upfront.

   a) Write down an integral expression for the premium.

   b) Suppose now that the office assumes that $r(t)$ and $a(t)$ follow a stochastic model with two states $\Big(r^{(1)}, a^{(1)}\Big)$ and $\Big(r^{(2)}, a^{(2)}\Big)$. It is assumed that $(r(0), a(0)) = \Big(r^{(1)}, a^{(1)}\Big)$. The transition rates from one state to another are equal and given by λ. Explain how to calculate the premium.

   c) Repeat the previous part assuming that only $r(t)$ is stochastic (with two states as before), but $a(t)$ is deterministic.

2. Repeat the previous question for a death benefit that is three times the member's salary at death.

EXERCISE 17

This is similar to exercise 11 but not identical.

Extra Exercise 17

The following calculations might be useful to you for this exercise. You might not need all of them. They all refer to a froce of mortality μ_x.

$${}_{40}p_{20} = 0.845$$

At a force of interest 0.05

$$\bar{a}_{20} = 18.079 \qquad \bar{a}_{60} = 11.183$$

At a force of interest 0.1

$$\bar{a}_{\overline{20}|} = 9.784 \qquad \bar{a}_{60} = 7.634$$

At a force of interest 0.15

$$\bar{a}_{\overline{20}|} = 6.601 \qquad \bar{a}_{20:\overline{40}|} = 6.589$$

At a force of interest 0.2

$$\bar{a}_{\overline{20}|} = 4.969 \qquad \bar{a}_{60} = 4.480$$

At a force of interest 0.25

$$\bar{a}_{\overline{20}|} = 3.981 \qquad \bar{a}_{60} = 3.684$$

At a force of interest 0.6

$$\bar{a}_{\overline{20}|} = 1.664 \qquad \bar{a}_{60} = 1.620$$

At a force of interest 0.65

$$\bar{a}_{\overline{20}|} = 1.536 \qquad \bar{a}_{60} = 1.499$$

A life starts aged x in state 1. It can move to state 2 with transition force 0.1. Once at 2 it can move to state 3 with force 0.2. If can not go to state 1 from state 2 and it can not go to states 1 or 2 from state 3. While at states 1 or 2 the force of mortality is μ_x and while at state 3 it is $0.6 + \mu_x$. Let $p_i(t)$ represent the probability that the life is alive and in state i.

1. Show that

$$p_2(t) = (\exp(-0.1t) - \exp(-0.2t))\,{}_tp_x$$

and

$$p_3(t) = (0.4\exp(-0.1t) - 0.5\exp(-0.2t) + 0.1\exp(-0.6t))\,{}_tp_x$$

where

$${}_tp_x = \exp\left(-\int_0^t \mu_{x+s} ds\right).$$

2. Suppose now that $x = 20$. Find the probability that the life will never be in state 3.

3. The life is entitled to a death benefit of 50000 if death occurs while at states 2 or 3 (but not state 1) and a continuous benefit of 50000 per annum while at state 3. The policy has a term of 40 years, but if the life is either in state 2 or in state 3 at the time it continues for life. Calculate the expected present value of the benefits using a force of interest of 0.05 per annum.

EXERCISE 18

Two lives aged 30 and 35 are subject to the following mortality model. They are independently exposed to the Danish 82M table, In addition to this they are subject to an extra mortality force of 0.3/(100-t) which affects both lives simultaneously (they will both die together). If one of them is alive only, he or she is still subject to this force. Under the terms of a 30 year temporary contract their heirs are entitled to 100000 payable on the second death (if they die together it is still payable).

The policy is financed by a continuous premium payable while they are both alive for 30 years. Use MAPLE to calculate the premium. Calculate also the reserves the office should holds at various times and states and see if there is a problem with the contract. (Plotting the reserve for the case they are both alive might be a good idea).

Use a force of interest of 0.04 and assume no expenses.