Exercises to ST305 2005/2006

Exercise 1
Let $Z(t)$, $t \geq 0$ be a time-continuous Markov chain on a finite state space $\mathcal{J} = \{1, \ldots, J\}$, starting in state 1 at time 0; $Z(0) = 1$. Assume that it possesses transition intensities, and adopt standard notation for basic entities: intensities of transition $\mu_{jk}(t)$, transition probabilities $p_{jk}(t, u)$, and probabilities of uninterrupted sojourns $p_{\overline{jj}}(t, u)$.

(a) Let $t_0 \leq t_1 \leq \cdots \leq t_r \leq t_{r+1}$ be times in $[0, \infty)$ and let $j_0, j_1, \ldots, j_r, j_{r+1}$ be states in $\mathcal{J}$. Express the following probabilities in terms of the basic entities:

$$\mathbb{P}\left[\bigcap_{i=0}^{r+1} Z(t_i) = j_i\right] \; ;$$

$$\mathbb{P}\left[\bigcap_{i=1}^{r} Z(t_i) = j_i \,\middle|\, Z(t_0) = j_0,\, Z(t_{r+1}) = j_{r+1}\right] . \tag{1}$$

(b) Let s and t be fixed times such that $s < t$, and let i and j be fixed states. Use the result in (1) (if you got it right) to show that, conditional on $Z(s) = i$ and $Z(t) = j$, the process $Z(\tau)$, $\tau \in [s, t]$, is a Markov chain.

(c) Suppressing the dependence on s, t and i, j from the notation, denote the conditional Markov chain by $\tilde{Z}(\tau)$, $\tau \in [s, t]$, and denote its transition probabilities and intensities by $\tilde{p}_{gh}(\tau, \vartheta)$ and $\tilde{\mu}_{gh}(\tau)$, respectively. Determine these probabilities and intensities.

(d) What is the limit of $\tilde{\mu}_{gh}(\tau)$ as τ tends to t? Distinguish between the cases where (i) g and h are both different from the "destination state" j, (ii) $g \neq j$ and $h = j$, (iii) $g = j$ and $h \neq j$. Give a verbal explanation of the results.

(e) Now specialize to the disability model in Figure 7.3, and condition on $Z(0) = a$ and $Z(t) = i$. Write out the expression for $\tilde{\sigma}(\tau)$ and the value of $\tilde{\mu}(\tau)$ (trivial). Find an explicit expression for $\tilde{\sigma}(\tau)$ in the case with constant intensities and no recovery, and discuss it with particular attention to the limiting value as τ tends to t.

Exercise 2

Let N be a counting process and consider a process S defined by

$$S_t = \exp(at + bN_t) \, ,$$

where a and b are constants. Use Itô's formula to show that the dynamics of S is given by

$$dS_t = S_{t-}\left(a\,dt + (e^b - 1)dN_t\right) .$$

In integral form

$$S_t = 1 + \int_0^t S_{\tau-}\left(a\,d\tau + (e^b - 1)dN_\tau\right) .$$

Assume now that N is a Poisson process with intensity λ. By the independent increments property of N, we can heuristically conclude that $S_{\tau-}$ and dN_τ are independent, so

$$\mathbb{E}[S_{\tau-}dN_\tau] = \mathbb{E}[S_{\tau-}]\mathbb{E}[dN_\tau] .$$

Use this together with $\mathbb{E}[S_{\tau-}] = \mathbb{E}[S_\tau]$ to obtain an integral equation for $\mathbb{E}[S_t]$. Solve it to find

$$\mathbb{E}[S_t] = \exp\left(at + \left(e^b - 1\right)\lambda t\right) .$$

Verify the result by direct calculation using that N_t has a Poisson distribution with parameter λt.

Exercise 3

Working under the independence hypothesis, find an expression for the mortality intensity of the life length of the last-survivor status $\overline{x_1 \ldots x_r}$. Do this by direct reasoning and also by brute force calculating

$$\mu_{\overline{x_1 \ldots x_r}}(t) = -\frac{d}{dt} \ln_t p_{\overline{x_1 \ldots x_r}} .$$

Exercise 4

Figure 1 shows a Markov model apt to describe insurance policies involving three lives lives (x), (y), and (z). We adopt established terminology and denote by (x) a single life aged x upon inception of the policy and by T_x the remaining life length of this individual. Similarly T_y and T_z denote the remaining life lengths of the single life statuses (y) and (z). Probabilities and expected values can be calculated as integrals w.r.t. the joint density of the random triple (T_x, T_y, T_z), but it is easier to work with standard methods for general Markov chain model.

The *joint life* status (xyz) exists as long as all three individuals members are alive. Its remaining lifetime is $T_{xyz} = \min\{T_x, T_y, T_z\}$. The survival function of the joint life is

$${}_tp_{xyz} = \mathbb{P}\left[T_x > t \cap T_y > t \cap T_x > t\right] .$$

The mortality intensity of the joint-life status at policy duration t is denoted by $\mu_{xyz}(t)$.

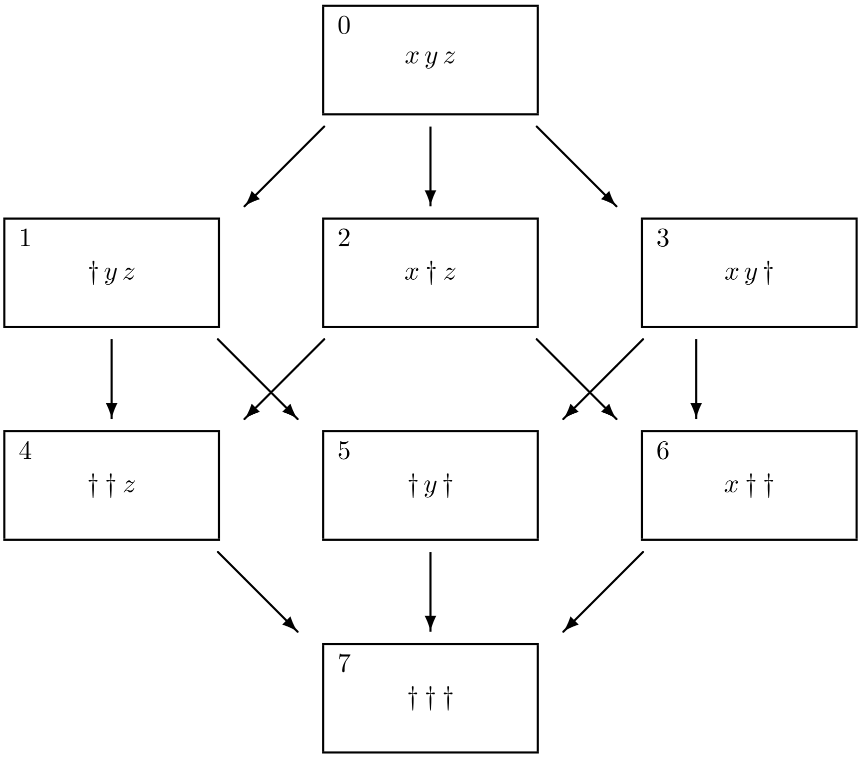

Figure 1: Flow-chart for a policy on three lives.

The *last survivor* status $\overline{xyz}$ exists as long as at least one individual is alive. Its remaining lifetime is $T_{\overline{xyz}} = \max\{T_x, T_y, T_z\}$. The survival function of this status is

$${}_tp_{\overline{xyz}} = \mathbb{P}\left[T_x > t \cup T_y > t \cup T_x > t\right] \ .$$

Its mortality intensity at policy duration t is denoted by $\mu_{\overline{xyz}}(t)$.

The q *survivors* status $\overset{q}{\overline{xyz}}$ exists as long as there are at least q survivors, $q = 1, 2, 3$. Its survival function and mortality intensity are denoted by ${}_tp_{\overset{q}{\overline{xyz}}}$ and $\mu_{\overset{q}{\overline{xyz}}}(t)$. (Of course, $\overset{3}{\overline{xyz}} = (xyz)$ and $\overset{1}{\overline{xyz}} = \overline{xyz}$.)

The expected present values of the standard n-year benefits to $\overset{q}{\overline{xyz}}$, the life endowment of 1, the life annuity of 1 per year, the term insurance with sum 1, and the endowment insurance with sum 1, are denoted by ${}_nE_{\overset{q}{\overline{xyz}}}$, $\bar{a}_{\overset{q}{\overline{xyz}}\,\overline{n}|}$, $\bar{A}_{\overset{1}{\overset{q}{\overline{xyz}}}\,\overline{n}|}$, and $\bar{A}_{\overset{q}{\overline{xyz}}\,\overline{n}|}$, respectively.

(a) Find expressions for ${}_tp_{\overset{q}{\overline{xyz}}}$ and $\mu_{\overset{q}{\overline{xyz}}}(t)$, $q = 1, 2, 3$, in terms of the transition probabilities and intensities in the Markov chain model.

(b) Assume that $x = y = z = 30$, $n = 30$, $r = \ln(1.045)$, and that all individuals follow the G82M mortality law. Use the program 'prores1' to compute ${}_nE_{\frac{\,q\,}{xyz}}$, $\bar{a}_{\frac{\,q\,}{xyz}\overline{n|}}$, $\bar{A}_{\overset{1}{\frac{\,q\,}{xyz}}\overline{n|}}$, and $\bar{A}_{\frac{\,q\,}{xyz}\overline{n|}}$, $q = 1, 2, 3$.

Exercise 5

Contingent insurance functions. There exists a huge variety of multi-life products, and some of them are rather exotic and occupy more space in actuarial textbooks than their practical relevance could justify. We shall look at some examples with payments dependent of the order of deaths within the group, which we for the sake of concreteness assume comprises two lives (x) and (y), and possibly also a third life (z). Working with continuous time, find expressions for and, using the scenario in Exercise 4b, numerical values for the expected present value at time 0 of the following n-year benefits:

(a) An annuity payable at rate 1 per year to (y) after the death of (x). This is called a reversionary annuity and the expected present value is denoted by $\bar{a}_{x|y\,:\overline{n|}}$ or ${}_n\bar{a}_{x|y}$.

(b) A life assurance of 1 payable upon the death of (y) if (x) is already dead. This is an example of a (for obvious reasons) so-called a contingent assurance and the expected present value of this one is denoted by ${}_n\bar{A}_{x\overset{2}{y}}$.

(c) A life insurance of 1 payable upon the death of (y) if (x) is still alive, ${}_n\bar{A}_{x\overset{1}{y}}$

(d) A life insurance of 1 payable upon the death of (y) if (x) is already dead and (z) is still alive, ${}_n\bar{A}_{\underset{3}{x\overset{2}{y}z}}$

Exercise 6

This is an example of an insurance policy with payments that are 'path-dependent', that is, dependent on the past history of the policy. Consider two independent lives (x) and (y) with remaining life lengths T_x and T_y, respectively.

(a) Assume that the benefit is an assurance of 1 payable at time T_y if $2T_x < T_y < n$ and that premium is payable at constant rate π until time $\min(T_x, T_y, n/2)$, where n is the term of the contract (fixed). Determine the equivalence premium π.

(b) Propose a method for computing the premium numerically. (Hint: One possibility is to treat ${}_{t/2}p_x$ as a survival function ${}_t\tilde{p}_x$ with intensity $\tilde{\mu}_{x+t}$, which you would need to express in terms of μ, and then solve a Thiele differential equation numerically.)

(c) Determine the reserve at any time t, assuming that the insurer currently knows the complete past history of the two lives. You need to distinguish between various cases, whether (y) is alive or dead, whether t is before or after time $n/2$, and whether x is alive or dead and, if dead, when. Is the reserve always non-negative?

(d) What is the variance of the present value of the benefit?

Exercise 7

Another example of 'path-dependent' payments:

(a) Consider two independent lives (x) and (y). Find the expected present value at time 0 of a life annuity payable continuously at rate 1 from time T_{xy} until time $\max(T_{xy} + 20\,,\, T_{\overline{xy}} + 10)$. In words, payments start at the time of the first death and continues thereafter for a term of 20 years or until 10 years after the death of the survivor, whichever is the longer period.

(b) Suppose premium is payable continuously at level rate π from time 0 until time T_{xy}. What is the reserve for this policy? This question is difficult and will be addressed later, but you can start thinking about it.

Exercise 8

Third example with 'path-dependent' payments: Consider three independent lives (x), (y), and (z). Find the expected present value of a sum insured of 1 payable at time T_x if $T_x < \min(T_z, T_y + 20)$. State in words what this contractual benefit is.

Exercise 9
We refer to the disability model.

(a) Consider an x years old insured who enters an insurance scheme at time 0. The probability $p_{\overline{aa}}(0,t) = \exp(-\int_0^t (\mu_{x+s} + \sigma_{x+s})\,ds)$ can be viewed as the probability $p_{aa}^{(0)}(0,t)$ of being active at time t after having been disabled 0 times. Derive forward differential equations for the probability $p_{ai}^{(1)}(0,t)$ of being disabled for the first time at time t and for the probability $p_{aa}^{(1)}(0,t)$ of being active at time t after having been disabled once.

(b) Find the probability of being disabled for the first time at time t and that the disability has lasted for at least q years.

(c) At time 0 an active person aged x buys a disability pension insurance with the following terms: The benefit is a pension payable at level rate 1 during the first disability, but only after it has lasted for at least q years (the *qualifying period*). Premium is payable at level rate π as long as the insured is active and has not yet been disabled, but not after time $n - q$, where n is the contract period ($n > q$). Determine the premium π by the principle of equivalence, assuming that the interest rate r is constant. Find the reserve at time $t < n - q$ for an insured who is disabled for the first time and is currently receiving the disability benefit.

Exercise 10
(a) Using your Turbo Pascal program with the Danish basis, compute the net premium rate π and the net premium reserve V_t (at selected times t) for an endowment insurance with age at entry $x = 30$, term $n = 30$, sum insured $b_t = b_n = b = 1$, and premium payable continuously at constant rate throughout the contract period.

(b) Compute the gross premium rate π' and the gross premium reserve V_t' assuming that administration expenses consist of a lump sum cost of $0.003{+}0.001\, b$ at time 0, costs incurring continuously at rate $0.0001 + 0.01\,\pi' + 0.005\,V_t'$ at any time t in the insurance period, a cost of 0.002 due immediately upon possible payment of the death benefit, and a cost of 0.0001 due at time n upon possible payment of the endowment benefit. Compare with the quantities net of expenses found in Item (a).

Exercise 11
We refer to Section 7.8 in the book.

(a) Use the direct backward argument to derive differential equations for the state-wise expected discount factors

$$W_e(t) = \mathbb{E}[e^{-\int_t^n r(s)\,ds} \mid Y(t) = e]\,,$$

$t \in [0, n)$. What are the side conditions?

(b) Consider the model in Fig. 7.6 and take $n = 5$. Fill in appropriate statements in the program 'prores2.pas' to make it compute the state-wise expected discount factors $W_e(0) = \mathbb{E}\left[\exp\left(-\int_0^5 r(s)\,ds\right)\middle| Y(0) = e\right]$, $e = 1, 2, 3$.

(c) Repeat the computation in Item (b) with intensities of the form $\lambda_{ef} = k\,\lambda_{ef}^{\circ}$, where the λ_{ef}° are the values specified in Fig. 7.6 and $k = 0, 0.5, 2, 5, 10$. What do the results tell you about the effect of increasing the frequency of changes in the interest rate?

(d) Prove that $\lim_{k\nearrow\infty} W_e(0) = e^{-0.05\times 5}$ for $e = 1, 2, 3$, as suggested by the results in Item (c). (One may prove that $\int_0^5 r(s)\,ds$ converges in probability to 0.25 by calculating its expected value and its variance and using Chebychev's inequality.)

Exercise 12
We refer here to Section 7.9 in the book.

(a) Prove the rather obvious statements PQD(T,T), AS(T,T), and RTI$(T|T)$.

(b) Negative dependence in the PQD sense: Prove that PQD$(-S,T)$ is equivalent to

$$\mathbb{P}[S > s\,,\, T > t] \leq \mathbb{P}[S > s]\,\mathbb{P}[T > t] \text{ for all } s \text{ and } t.$$

(c) Negative dependence in the AS sense: Prove that AS$(-S,T)$ is equivalent to $\mathrm{Cov}(g(S,T),h(S,T)) \geq 0$ for all real-valued functions g and h that are decreasing in S and increasing in T (and for which the covariance exists).

(d) Negative dependence in the RTI sense: Prove that RTI$(-S|T)$ is equivalent to RTD$(S|T)$.

(e) Consider the model in Figure 7.4, and let $\mu = \mu'$ and $\nu = \nu'$ so that S and T are independent.

Now, add a cause of simultaneous death (due to 'catastrophe') with intensity $\mu_{03} = \kappa$. Does it follow that RTI$(S|T)$?

Assume instead, maybe more reasonably, that catastrophe risk is present independently of the state of the marriage: $\mu_{01}(t) = \mu(t)$, $\mu_{02}(t) = \nu(t)$, $\mu_{03}(t) = \kappa(t)$, $\mu_{13}(t) = \nu(t) + \kappa(t)$, $\mu_{23}(t) = \mu(t) + \kappa(t)$. Prove that RTI$(S|T)$, hence PQD$(S,T)$.

Exercise 13
We refer to Section 10.6 in the lecture notes.

(a) Let $\mathcal{H}'_t$ and $\mathcal{H}''_t$ be sigma-algebras such that $\mathcal{H}'_t \supset \mathcal{H}''_t$, which means that the latter represents more summary information than the former. Prove that the variance of $U_{\mathbf{H}'}(t)$ is no less than the variance of $U_{\mathbf{H}''}(t)$ and that a similar statement holds also for the prospective reserves $V_{\mathbf{H}'}(t)$ and $V_{\mathbf{H}''}(t)$. (Use a well-known rule about the variance derived from the tower rule of iterated expectations.)

(b) In the disability model with recovery, consider a disability pension payable at constant rate b in disabled state against premium payable at constant rate c in active state. Let the interest rate be constant. Find the state-wise retrospective reserves at time t in all three states 'active', 'disabled', and 'dead'.

Exercise 14
Continuing the single life example in Paragraph 10.6.B:

(a) Let $\mathbf{H}' = \{\mathcal{H}'_t\}_{t\geq 0}$, where $\mathcal{H}'_t = \mathcal{H}'_{t\wedge s}$ is the life history up to time $t \wedge s$ for a fixed time $s \in [0, n]$. Find $U_{\mathbf{H}'}(t)$ and $V_{\mathbf{H}'}(t)$ for all $t \in [0, n]$.

(b) Find the state-wise retrospective and prospective reserves for an n-year pure life endowment against single premium at time 0.

(c) In the framework of the disability model in Fig 7.3 in the book, consider an n-year term insurance against level premium in active state (waiver of premium during disability). Find the retrospective reserve at time t, given that the policy is in state a and that it was in state i at time $s\,(< t)$.

WITH PROFIT CONTRACTS; SURPLUS, BONUS, GUARANTEES. UNIT-LINKED CONTRACTS

A. Preliminaries. We are going to restate the theory of surplus and bonus and related problems in the framework of the simple, still fairly general, single life contract and add material on interest guarantees and unit-linked insurance.

The terms of the contract are set out in the expression for the prospective reserve,

$$V_t = \int_t^n e^{-\int_t^\tau (r_u + \mu_{x+u})\,du} (\mu_{x+\tau}\, b_\tau - \pi_\tau)\, d\tau \;+\; e^{-\int_t^n (r_u+\mu_{x+u})\,du}\, b_n\,,$$

and the equivalence relation

$$V_0 = \pi_0\,, \tag{2}$$

where π_0 is the lump sum premium payment collected upon the inception of the policy (it may be 0, of course). The equivalence relation (2) can be cast as

$$\pi_0 \;+\; \int_0^t e^{-\int_0^\tau (r_u+\mu_{x+u})\,du} (\pi_\tau - \mu_{x+\tau}\, b_\tau)\, d\tau - e^{-\int_0^t (r_u+\mu_{x+u})\,du} V_t = 0\,. \tag{3}$$

The first two terms in (3) are the expected present value at time 0 of premiums less benefits up to and including time t. The second term is minus the expected present value at time 0 of benefits less premiums after time t. Thus, the equivalence principle ensures that, at any time, net incomes in the past provide precisely the amount needed to meet net liabilities in the future.

Thiele's differential equation is

$$\frac{d}{dt}\, V_t = r_t\, V_t + \pi_t - \mu_{x+t}\,(b_t - V_t)\,, \tag{4}$$

with the boundary condition

$$V_{t-} = b_n\,. \tag{5}$$

B. With profit contracts (participating policies); Surplus and Bonus. Insurance policies are long term contracts, with time horizons wide enough to capture significant variations in interest and mortality. Therefore, at time 0 when the contract is written with benefits and premiums binding to both parties, the future development of (r_t, μ_{x+t}), $t > 0$, is uncertain, and it is impossible to foresee which premium level will satisfy (3) and establish equivalence in the end. If it should turn out that, due to adverse development of interest and mortality, premiums are insufficient to cover benefits, then there is no way the insurance company can avoid a loss; it cannot reduce the benefits and it cannot increase the premiums since these were irrevocably set out in the contract at time 0. The only way the insurance company can prevent such a loss, is to charge a premium 'on the safe side', high enough to be adequate under all likely scenarios. Then, if everything goes well, a surplus will accumulate. This surplus belongs to the insured and is to be repaid as so-called *bonus*, e.g. as increased benefits or reduced premiums.

The usual way of setting premiums to the safe side is to base the calculation of the premium level and the reserves on a provisional *first order basis*, (r^*_t, μ^*_{x+t}), $t > 0$, which represents a worst case scenario and leads to higher premium and reserves than are likely to be needed. We follow common practice and take the first order interest rate to be constant, r^*. (From a mathematical point of view this is just a matter of notation.) The reserve based on the prudent first order assumptions is called the *first order reserve*, and we denote it by V^*_t. It satisfies Thiele's differential equation

$$\frac{d}{dt}\,V^*_t = r^*\,V^*_t + \pi_t - \mu^*_{x+t}(b_t - V^*_t)\,, \tag{6}$$

subject to the natural side condition $V^*_{n-} = b_n$. The premiums are determined so as to satisfy the first order equivalence relation $V^*_0 = \pi_0$.

Taking our stand at a given time t after the inception of the policy, the development of interest and mortality in the past, $(r_\tau, \mu_{x+\tau})$, $\tau \le t$, is now known and can be invoked in an updated calculation of the net incomes up to time t. The future development of interest and mortality, $(r_\tau, \mu_{x+\tau})$, $\tau > t$, remains uncertain, however, so for the assessment of future liabilities one must stick to the conservative first order basis, $(r^*, \mu^*_{x+\tau})$, $\tau > t$. Thus, instead of the balance equation (3), which cannot be set up since it involves unknown future rates of interest and mortality, we have the following expression for the discounted *mean surplus per policy* at time t:

$$S_t = \pi_0\, + \int_0^t e^{-\int_0^\tau (r_u + \mu_{x+u})\,du}(\pi_\tau - \mu_{x+\tau}\,b_\tau)\,d\tau - e^{-\int_0^t (r_u+\mu_{x+u})\,du}V^*_t\,. \tag{7}$$

If the factual interest and mortality in the past were more favourable than the pessimistic first order basis, then this surplus is positive.

C. Emergence of surplus. To see how the surplus is emerges, we need to study the dynamics of S_t. Differentiating (7), using (6), and rearranging terms, we obtain

$$\frac{d}{dt} S_t = e^{-\int_0^t (r_u + \mu_{x+u})\, du}\, c_t,$$

where

$$c_t = (r_t - r^*)\, V_t^* + (\mu_{x+t}^* - \mu_{x+t})(b_t - V_t^*)\,. \tag{8}$$

Obviously, c_t is the rate at which surplus emerges per survivor and per time unit at time t. Interpret the two terms on the right hand side of (8) as surplus emerging from safety margins in the interest rate and in the mortality rate, respectively. We can reasonably say that a first order element is set on the safe side if the corresponding contribution to the surplus is positive. By inspection of the first term on the right of (8), we see that r^* is on the safe side as long as it is less than the true r_t (provided that the first order reserve is positive, as it should be for any meaningful contract). By inspection of the second term on the right of (8), we see that the sign of the sum at risk by death, $b_t - V_t^*$, determines how to set first order mortality to the safe side: If the sum at risk is positive (e.g. term assurance or endowment assurance), then μ_{x+t}^* is on the safe side if it is bigger than μ_{x+t}. If the sum at risk is negative (as is the case for e.g. a pure endowment, a deferred annuity, or some other savings insurance with $b_t = 0$), then μ_{x+t}^* is on the safe side if it is less than μ_{x+t}.

D. Redistribution of surplus as bonus. The word *bonus* is Latin and means 'good'. In insurance terminology it denotes various forms of repayments to the policyholders of that part of the company's surplus that stems from good performance of the insurance portfolio, a sub-portfolio, or the individual policy. In the present context of life insurance it denotes the repayments of surplus stemming from favourable development of interest and mortality. Let us denote such repayments by $\tilde{b}$ in general. For the sake of concreteness, suppose bonuses are paid back continuously at rate $\tilde{b}_t$ per survivor for $0 < t < n$ and possibly with a lump sum $\tilde{b}_n$ per survivor at time $t = n$. By statute, surplus is to be repaid in its entirety, which means that equivalence is to be re-established on basis of the true interest and mortality conditions when these are ultimately known at the term of the contract:

$$\begin{aligned} \int_0^n e^{-\int_0^\tau (r_u + \mu_{x+u})\, du} c_\tau \, d\tau &= \int_0^n e^{-\int_0^\tau (r_u + \mu_{x+u})\, du}\, \tilde{b}_\tau \, d\tau \\ &\quad + e^{-\int_0^n (r_u + \mu_{x+u})\, du} \tilde{b}_n\,. \end{aligned} \tag{9}$$

In the following Paragraphs E - G we will study some commonly used bonus schemes.

E. Cash Bonus. This means that surplus is being repaid continually as it emerges, i.e. $\tilde{b}_t = c_t$, $0 < t < n$, and $\tilde{b}_n = 0$. It may for instance take the form of a premium deductible payable at rate c_t as long as the insured is alive during the contract period. In the case of a term assurance contract it could reasonably take the form of an additional payment $\hat{b}_t$ upon death at time $t \in (0, n)$, and a natural choice is $\hat{b}_t = c_t / \mu_{x+t}$.

Exercise 15
Verify that the two cash bonus schemes described above comply with the ultimate equivalence requirement (9). Construct a scheme that is a combination of the two proposed here.

F. Terminal Bonus. This means that surplus is repaid as a lump sum $\tilde{b}_n$ to survivors at the end of the term, and $\tilde{b}_t = 0$, $0 < t < n$.

Exercise 16
Determine $\tilde{b}_n$ by (9).

G. Purchase of Additional Insurance. Under this bonus scheme the surplus is spent on purchase of additional insurance. Additional insurance is written on the first order basis and will therefore also generate surplus, which in its turn will be used for further purchase of additional insurance, and so on. The scheme is non-trivial and requires a bit of theoretical reasoning:

For a policy in force at time t let V_t^{*+} denote the expected present value, on the first order basis, of future benefits only;

$$V_t^{*+} = \int_t^n e^{-\int_t^\tau (r^* + \mu^*_{x+u})\,du} \mu^*_{x+\tau}\, b_\tau \, d\tau \;+\; e^{-\int_t^n (r^* + \mu^*_{x+u})\,du}\, b_n \,.$$

It satisfies the Thiele's differential equation

$$\frac{d}{dt}\, V_t^{*+} = r^*\, V_t^{*+} - \mu^*_{x+t}(b_t - V_t^{*+})\,, \tag{10}$$

with natural side condition $V_{n-}^{*+} = b_n$.

The quantity V_t^{*+} is the single premium payable at time t if the insured then were to purchase an additional insurance for the balance of the term, with the same benefits as in the original contract. Spending the surplus $c_t\,dt$ generated in $[t, t+dt)$ as a single premium for additional benefits of the form specified in the original contract, will buy the insured a fraction $q_t\,dt$ of future benefits given by

$$c_t = q_t\, V_t^{*+}\,. \tag{11}$$

At any time $t \in (0, n)$ the death benefits from the original contract and from the additional benefits purchased during $(0, t]$ total

$$(1 + Q_t)\, b_t\,, \tag{12}$$

where

$$Q_t = \int_0^t q_\tau \, d\tau\,. \tag{13}$$

Likewise, the total endowment benefit at the term of the contract is

$$(1 + Q_n)\, b_n\,. \tag{14}$$

At time t the total surpluses from the original contract and the additional benefits purchased during $(0, t]$ emerge at rate

$$c_t = (r_t - r^*)\,(V_t^* + Q_t V_t^{*+})\; +\; (\mu^*_{x+t} - \mu_{x+t})\,\big((1 + Q_t)b_t - V_t^* - Q_t V_t^{*+}\big)\;. \tag{15}$$

Now all elements needed are in place, and a dynamic computation will deliver the solution. First, at the time of the inception of the contract, the functions V_t^* and V_t^{*+} and the equivalence premium π_t are determined by use of the program 'prores1.pas' (or 'prores2.pas'). The computation goes backwards starting from the side conditions $V^*_{t-} = b_n$ and $V^{*+}_{t-} = b_n$. Then, as time passes and surpluses are being observed and redistributed, one computes simultaneously the functions V_t^* and V_t^{*+} (again) and the random function Q_t as solutions to the differential equations (6), (10), and (rewrite (11))

$$\frac{d}{dt} Q_t = \frac{1}{V_t^{*+}} c_t\,, \tag{16}$$

with c_t given by (15). The computation goes forwards, starting from time $t = 0$ with the initial conditions

$$V_0^* = 0\,,$$
$$V_0^{*+} = V_0^{*+}$$

(picked from the first computation), and

$$Q_0 = 0\,.$$

Having determined Q, the benefits under this bonus scheme are now given by (12) and (14).

H. Prognostication of bonus. At regular times (typically annually) the customer receives a statement of his policy account, informing about bonus earned from surplus in the past and also predicting future bonuses based on a qualified guess as to the future development of interest and mortality.

Exercise 17
Outline such a statement with those pieces of information for the standard contract considered so far, including the relevant formulas, and basing the prognosis of future surplus on the assumption that $r_\tau = r^* + \Delta r$ and $\mu_{x+\tau} = \mu^*_{x+\tau} - \Delta\mu$ for some given positive Δr and $\Delta\mu$.

Exercise 18
Apply the present theory to a pension insurance policy for which benefits are an m year deferred life annuity payable at level rate 1 per year in n years, and premiums are payable at level rate during the deferred period. Write out all

relations and formulas that differ from the corresponding ones above. Will surplus emerge also in the benefit period $[m, m+n)$?

Exercise 19
Extend the theory so as to include expenses. Consider an endowment insurance with constant sum insured b and constant gross premium rate π' (no down payment π'_0 at time 0). Assume that true expenses incur with a lump sum $\alpha' + \alpha'' b$ at time 0 and thereafter continuously at rate $\beta'_t + \beta''_t \pi' + \gamma'_t + \gamma''_t b + \gamma'''_t V^{*'}_t$ at time $t \in (0, n)$ as long as the policy is in force. Here $V^{*'}_t$ denotes the gross premium reserve on first order basis. First order assumptions specify that expenses incur with a lump sum $\alpha^* b$ at time 0 and thereafter continuously at constant rate $\beta^* \pi' + \gamma^* b$. Discuss how the first order elements can be set on the safe side. Observe that there is a lump sum contribution to surplus at time 0.

I. Stochastic interest. The uncertain development of the second order elements can be built into the model by describing the interest rate and the (parameters of the) mortality rate as stochastic processes. To keep things simple, we will focus on interest, which is the more important of the two, and assume that the mortality is perfectly predicted by the first order basis: $\mu^*_{x+t} = \mu_{x+t}$. This means that contributions to surplus stem only from interest gains, so that

$$c_t = (r_t - r^*) V^*_t \,. \tag{17}$$

As a simple, but flexible, model for stochastic interest, we will assume that $\{r_t\}_{t \geq 0}$ is generated by the Markov chain model in Section 7.8 of the 'lifebook'. To save space, we will write Y_t and r_t instead of $Y(t)$ and $r(t)$ and, since subscripts are now used for the time variable t, denote the state-wise interest rate by r^e.

The statement of account, which is regularly sent to the insured, usually comes with a prognosis of future bonuses on the insurance. Such a prognosis must be based on a qualified guess about the future development of the factual valuation basis – in our simplified situation about r. This guess may be exogenous to the model, e.g. based on combined opinions of experts in the finance department of the company. Having adopted a stochastic model for r, the insurer can make an endogenous, model-based forecast of future bonus payments. Thus, consider a policy which is still in force at time t, and suppose the insurer wants to inform the insured about the conditional expected value of future bonuses, given that the current interest rate is $r_t = r^e$ (which means $Y_t = e$, assuming that all r^e are different). We will consider a few examples.

J. Cash bonus: The rate at which bonus will be paid at some fixed future time u, provided the insured is then alive, is

$$W = (r_u - r^*) V^*_u \,.$$

At time $t < u$, given $r_t = r^e$, W is predicted by its conditional expected value

$$W_e(t) = \mathbb{E}[W \mid Y_t = e] \,.$$

Exercise 20
Show that the functions $W_e(t)$ are the solution to the differential equations

$$\frac{d}{dt}W_e(t) = \sum_{\ell;f\neq e} \lambda_{ef}(W_e(t) - W_f(t))\,,$$

subject to the conditions

$$W_e(u) = (r^e - r^*)V_u^*\,,$$

$e = 1, \ldots, J^Y$.

K. Terminal bonus: By (9), bonus payable as a lump sum at the term of the contract n, provided the insured is then alive, is

$$\begin{aligned} W &= \int_0^n e^{\int_\tau^n (r_s+\mu_{x+s})\,ds}(r_\tau - r^*)V_\tau^*\,d\tau \\ &= W_t' \int_0^t e^{\int_\tau^t (r_s+\mu_{x+s})\,ds}(r_\tau - r^*)V_\tau^*\,d\tau + W_t''\,, \end{aligned}$$

where

$$\begin{aligned} W_t' &= e^{\int_t^n (r_s+\mu_{x+s})\,ds}\,, \\ W_t'' &= \int_t^n e^{\int_\tau^n (r_s+\mu_{x+s})\,ds}(r_\tau - r^*)V_\tau^*\,d\tau\,. \end{aligned}$$

The random variables W_t' and W_t'', which are unknown at time t, are predicted by

$$\begin{aligned} W_e'(t) &= \mathbb{E}[W_t' \mid Y_t = e]\,, \\ W_e''(t) &= \mathbb{E}[W_t'' \mid Y_t = e]\,. \end{aligned}$$

Exercise 21
Writing

$$\begin{aligned} W_t' &= e^{(r_t+\mu_{x+t})\,dt}\,W_{t+dt}'\,, \\ W_t'' &= W_t'\,(r_t - r^*)\,V_t^*\,dt \,+\, W_{t+dt}''\,, \end{aligned}$$

show that the functions $W_e'(t)$ and $W_e''(t)$ are the solution to the differential equations

$$\begin{aligned} \frac{d}{dt}W_e'(t) &= -(r_e + \mu_{x+t})\,W_e'(t) + \sum_{f;f\neq e} \lambda_{ef}(W_e'(t) - W_f'(t))\,, \\ \frac{d}{dt}W_e''(t) &= -W_e'(t)(r^e - r^*)V_t^* + \sum_{f;f\neq e} \lambda_{ef}(W_e''(t) - W_f''(t))\,, \end{aligned}$$

subject to the conditions

$$W'_e(n-) = 1\,,$$
$$W''_e(n-) = 0\,,$$

$e = 1, \ldots, J^Y$.

L. Additional benefits: At a fixed future time u bonus is paid as a multiple Q_u of the contractual benefits provided the insured is then alive. At time t we decompose Q_u into Q_t, which is known, and $Q_u - Q_t$, which is unknown, and we need to predict the latter. Recalling (11) and (17), start from the differential equation

$$\frac{d}{dt}Q_t = (r_t - r^*)\,(\frac{V_t^*}{V_t^{*+}} + Q_t)$$

and use the technique with integrating factor to obtain

$$Q_u \;=\; W'_t\,Q_t \;+\; W''_t\,,$$

where

$$\begin{aligned} W'_t &= e^{\int_t^u (r_s - r^*)\,ds}\,, \\ W''_t &= \int_t^u e^{\int_\tau^u (r_s - r^*)\,ds}(r_\tau - r^*)\frac{V_\tau^*}{V_\tau^{*+}}\,d\tau\,. \end{aligned}$$

Exercise 22
Derive differential equations for the state-wise predictions $W'_e(t)$ and $W''_e(t)$ of W'_t and W''_t.

Exercise 23
(a) Predict the discounted future cash bonuses given survival to n,

$$\int_t^n e^{-\int_t^\tau r_s\,ds}(r_\tau - r^*)V_\tau^*\,d\tau\,.$$

(b) Predict expected discounted future cash bonuses,

$$\int_t^n e^{-\int_t^\tau (r_s + \mu_{x+s})\,ds}(r_\tau - r^*)V_\tau^*\,d\tau\,.$$

(c) Find differential equations for the conditional variance, given $Y_t = e$, of the future cash bonuses. You may try your hand also on the conditional variance of future terminal bonus.

As we have said before, the Markov model proposed here can hardly be 100 per cent realistic. Now, the usefulness of a model depends on its purpose. The sole purpose of the interest rate model is to provide the insured with a reasonable guess as to his future prospects of bonus, and for that purpose a rough model can certainly be adequate. Anyway, at the end of the day the bonus payments will be determined entirely by the factual interest rate and will not depend on the assumptions in our model.

M. Guaranteed interest. Recall the basic rules of the 'with profit' insurance contract: On the one hand, any surplus is to be redistributed to the insured. On the other hand, benefits and premiums set out in the contract cannot be altered to the insured's disadvantage. This means that negative surplus, should it occur, cannot result in negative bonus. Thus, the with profit policy comes with an interest rate guarantee to the effect that bonus is to be paid as if factual interest were no less than first order interest, roughly speaking. For instance, cash bonus is to be paid at rate

$$(r_t - r^*)_+ V_t^*$$

per survivor at time t, hence the insurer has to cover

$$(r^* - r_t)_+ V_t^* \,. \tag{18}$$

Similarly, terminal bonus at time n (typical for e.g. a pure endowment benefit) is to be paid to survivors as a lump sum

$$\left(\int_0^n e^{\int_\tau^n (r_s + \mu_{x+s})\, ds} (r_\tau - r^*) V_\tau^* \, d\tau \right)_+$$

per survivor at time n, hence the insurer has to cover

$$\left(\int_0^n e^{\int_\tau^n (r_s + \mu_{x+s})\, ds} (r^* - r_\tau) V_\tau^* \, d\tau \right)_+ \,. \tag{19}$$

(We write $a_+ = \max(a, 0) = a \vee 0$.)

An interest guarantee of this kind represents a liability on the part of insurer. It cannot be offered for free, of course, but has to be compensated by a premium. This can certainly be done without violating the rules of the participating policy game, which lay down that premiums and benefits be set out in the contract at time 0. Thus, for simplicity, suppose a single premium is to be collected at time 0 for the guarantee. The question is, how much should it be?

Being brought up with the principle of equivalence, we might think that the expected discounted value of the liability is an agreeable candidate for the premium. However, the rationale of the principle of equivalence, which was to make premiums and benefits balance on the average in an infinitely large portfolio, does not apply to financial risk. Interest rate variations cannot be eliminated by increasing the size of the portfolio; all policy-holders are faring together in one and the same boat on their once-in-a-lifetime voyage through the troubled waters of their chapter of economic history. This risk cannot be averaged out in the same way as the risk associated with the lengths of the individual lives. None the less, in lack of anything better, let us find the expected discounted value of the interest guarantee, and just anticipate here that this actually would be the correct premium in an extended model specifying a so-called complete financial market. Those who are familiar with basic arbitrage theory know what this means. Those who are not should just imagine that, in addition to the bank account with the interest rate r_t, there are some other investment opportunities, and that any future financial claim can be duplicated perfectly by investing a certain amount at time 0 and thereafter just selling and buying available assets without any further infusion of capital. The initial amount required to perform this duplicating investment strategy is, quite naturally, the price of the claim. It turns out that this price is precisely the expected discounted value of the claim, only under a different probability measure than the one we have specified in our physical model. With these reassuring phrases, let us proceed to find the expected discounted value of the interest guarantee.

(a) Cash bonus with guarantee given by (18): Given that $r_0 = r^e$ (say), the price of the total claims under the guarantee, averaged over an infinitely large portfolio, is

$$\mathbb{E}\left[\int_0^n e^{-\int_0^\tau r}(r^* - r_\tau)_+ V_\tau^* \,{}_\tau p_x \, d\tau \,\middle|\, r_0 = r^e\right] . \tag{20}$$

A natural starting point for creating some useful differential equations by the backward construction is the 'price of future claims under the guarantee' in state e at time t,

$$W_e(t) = \mathbb{E}\left[\int_t^n e^{-\int_t^\tau r}(r^* - r_\tau)_+ V_\tau^* \,{}_\tau p_x \, d\tau \,\middle|\, r_t = r^e\right] , \tag{21}$$

$e = 1, \ldots, J^Y$, $0 \le t \le n$. The price in (20) is precisely $W_e(0)$.

Conditioning on what happens in the time interval $(t, t + dt]$ and neglecting terms of order $o(dt)$ that will disappear in the end anyway, we find

$$W_e(t) = (1 - \lambda_{e\cdot} dt)\left((r^* - r^e)_+ V_t^* \,{}_t p_x \, dt + e^{-r^e dt} W_e(t + dt)\right) + \sum_{f;\, f \ne e} \lambda_{ef} dt \, W_f(t) \, .$$

From here we easily arrive at the differential equations

$$\frac{d}{dt} W_e(t) = -(r^* - r^e)_+ V_t^* \,{}_t p_x + r^e W_e(t) - \sum_{f;\, f \ne e} \lambda_{ef} dt \,(W_f(t) - W_e(t)) \, , \tag{22}$$

which are to be solved subject to the conditions

$$W_e(n-) = 0 \, . \tag{23}$$

(b) Terminal bonus at time n given by (19): Given $r_0 = r^e$, the price of the claim under the guarantee, averaged over an infinitely large portfolio, is

$$\mathbb{E}\left[e^{-\int_0^n (r_s + \mu_{x+s})\, ds} \left(\int_0^n e^{\int_\tau^n (r_s + \mu_{x+s})\, ds} (r^* - r_\tau) V_\tau^* \, d\tau \right)_+ \,\middle|\, r_0 = r^e \right]$$

$$= \mathbb{E}\left[\left(\int_0^n e^{-\int_0^\tau (r_s + \mu_{x+s})} (r^* - r_\tau) V_\tau^* \, d\tau \right)_+ \middle| r_0 = r^e \right] . \tag{24}$$

Let us try and copy the method of Item (a) and look at the 'price of the claim at time t', which should be the conditional expected discounted value of the claim, given what we know at the time:

$$\begin{aligned} &\mathbb{E}\left[\left(\int_0^n e^{-\int_0^\tau (r_s + \mu_{x+s})\, ds} (r^* - r_\tau) V_\tau^* \, d\tau \right)_+ \middle| r_\tau;\, 0 \le \tau \le t \right] \\ &= e^{-\int_0^\tau (r_s + \mu_{x+s})\, ds} \mathbb{E}\left[\left(U_t + \int_t^n e^{-\int_t^\tau (r_s + \mu_{x+s})\, ds} (r^* - r_\tau) V_\tau^* \, d\tau \right)_+ \middle| r_\tau;\, 0 \le \tau \le t \right] , \end{aligned} \tag{25}$$

where

$$U_t = e^{\int_0^t (r_s + \mu_{x+s})\, ds} \int_0^t e^{-\int_0^\tau (r_s + \mu_{x+s})\, ds} (r^* - r_\tau) V_\tau^* \, d\tau \, .$$

The conditional expected value (25) is more involved than the one in (21) since it depends effectively on the past history of interest rate through U_t. We can, therefore, not hope to end up with the same simple type of problem as in Item (a) above and in all other situations encountered so far, where we essentially had to determine the conditional expected value of some function depending only on the future course of the interest rate. Which was easy since, by the Markov property, we could look at state-wise conditional expected values $W_e(t)$, $e = 1, \ldots, J^Y$, say. These are deterministic functions of the time t only and can be determined by solving ordinary differential equations.

Let us proceed and see what happens. Due to the Markov property (conditional independence between past and future, given the present) the conditional expected value in (25) is a function of t, r_t and U_t. Consider its value for given $U_t = u$ and $r_t = r_e$,

$$W_e(t, u) = \mathbb{E}\left[\left(u + \int_t^n e^{-\int_t^\tau (r_s + \mu_{x+s})\, ds} (r^* - r_\tau) V_\tau^* \, d\tau \right)_+ \middle| r_t = r^e \right] .$$

Use the backward construction:

$$\begin{aligned} &W_e(t, u) = \\ &(1 - \lambda_{e\cdot} dt) \mathbb{E}\left[\left(u + (r^* - r^e) V_t^* \, dt + e^{-(r^e + \mu_{x+t})\, dt} \int_{t+dt}^n e^{-\int_{t+dt}^\tau (r_s + \mu_{x+s})\, ds} (r^* - r_\tau) V_\tau^* \, d\tau \right)_+ \middle| r_{t+dt} = r \right. \\ &+ \sum_{f;\, f \ne e} \lambda_{ef} dt \, W_f(t, u) = \\ &(1 - \lambda_{e\cdot} dt) e^{-(r^e + \mu_{x+t})\, dt} W_e \left(t + dt, e^{(r^e + \mu_{x+t})\, dt} \left(u + (r^* - r^e) V_t^* \, dt \right) \right) + \sum_{f;\, f \ne e} \lambda_{ef} dt \, W_f(t, u) \, . \end{aligned}$$

Insert here $e^{\pm(r^e+\mu_{x+t})\,dt} = 1 \pm (r^e + \mu_{x+t})dt + o(dt)$,

$$
\begin{aligned}
&W_e\left(t+dt, e^{(r^e+\mu_{x+t})\,dt}\,(u + (r^* - r^e)V_t^*\,dt)\right) = \\
&W_e\,(t+dt,\; u \,+\, u\,(r^e + \mu_{x+t})\,dt + (r^* - r^e)V_t^*\,dt) = \\
&W_e(t,u) + \frac{\partial}{\partial t}W_e(t,u)\,dt + \frac{\partial}{\partial u}W_e(t,u)(u\,(r^e + \mu_{x+t}) + (r^* - r^e)V_t^*)\,dt + o(dt)\,,
\end{aligned}
$$

and proceed in the usual manner to arrive at the partial differential equations

$$
\frac{\partial}{\partial t}W_e(t,u)+(u\,(r^e+\mu_{x+t})+(r^*-r^e)V_t^*)\frac{\partial}{\partial u}W_e(t,u)-(r^e+\mu_{x+t})\,W_e(t,u)+\sum_{f;\,f\neq j}\lambda_{ef}\,(W_f(t,u)-W_e(t,u)) = 0
$$

These are to be solved subject to the conditions

$$
W_e(n-,u) = u_+\,,
$$

$e = 1,\ldots,J^Y$.

Since the functions we are interested in involved both t and U_t, we are lead to state-wise functions in two arguments and, therefore, quite naturally end up with partial differential equations for those.

N. Unit linked insurance. We have been discussing the participating (or with profit) policy, characteristic of which is that benefits and premiums are set out in nominal amounts in the contract at time 0. Thus, For the fairly general contract described in the introduction to this note, the functions b_t and π_t would be deterministic, not dependent on the development of the interest rate over the term of the contract. Introduce

$$
U_t = e^{\int_0^t r_u\,du}\,,
$$

which is the value at time t of a unit deposited in the investment portfolio at time 0. We may call it the price index of the investment portfolio. Recast the equivalence relation (3) as

$$
-\pi_0 \,+\, \int_0^n U_\tau^{-1}\,{}_\tau p_x(\mu_{x+\tau}\,b_\tau - \pi_\tau)\,d\tau \,+\, U_n^{-1}\,{}_n p_x\,b_n = 0\,. \tag{26}
$$

With b_t and π_t fixed at time 0 there is no way one can make them fulfill (26) for all possible future courses of the interest rate process. Depending on the economic development there will be inequality in the one or the other direction. The financial risk thus introduced is hedged (hopefully perfectly) by setting premiums on a prudent first order basis, i.e. replacing the unknown r_t in (26) by some r^* set to the 'safe side'.

An alternative scheme for management of financial risk in life insurance is known as *unit linked insurance* (also called *variable life insurance*). The idea of this concept is to link benefits and premiums to the performance of the investment portfolio, that is, let contractual payments be inflated by the index U instead of being fixed nominal amounts.

Under a *perfect unit linked* contract we would have $b_t = U_t b_t^\circ$ and $\pi_t = U_t\, \pi_t^\circ$ for some 'baseline' benefits b_t° and premiums π_t°, $t \in [0,n]$, determined at time 0. Inserting this into (26) gives

$$-\pi_0 \;+\; \int_0^n {}_\tau p_x (\mu_{x+\tau}\, b_\tau^\circ - \pi_\tau^\circ)\, d\tau \;+\; {}_n p_x\, b_n^\circ = 0\,. \tag{27}$$

We see that, for a given baseline benefit function b_t°, the equivalence relation can be fulfilled by a suitable choice of baseline premium rate π_t°. The future course of the interest rate process has disappeared from the relation upon discounting the indexed payments and, thus, the problem with financial risk has been resolved by the perfect unit linked device.

However, in practice unit-linked contracts are usually not perfect in the sense described above. Typically, only the benefits are linked to the investment index, whereas premiums are not. Furthermore, the unit linked contract is typically equipped with a guarantee specifying that the benefit cannot fall below a certain pre-specified nominal minimum. Such modifications to the perfect linking re-introduce financial risk, of course.

Before we return to mathematics, we dare to suggest that guarantees, whether they apply to benefits under unit linked contracts or to interest under with profit contracts, are remains of the social security concern that traditionally was paramount in life insurance. They introduce a discrimination between various forms of saving; unlike those who invest in stocks, bonds, or real estate, those who invest in life or pension insurance are granted the privilege of gaining from booms without loosing from recessions. However, parity can be restored by letting the insured pay for the guarantee. Thus we proceed to determine its right price.

For an example, let us try and determine the single premium payable at time 0 for a term insurance with sum $b_t = (U_t \vee g)$ at time $t \in (0,n)$, where g is the guaranteed minimum sum insured specified at time 0. The premium is

$$\pi = \mathbb{E}\left[\int_0^n e^{-\int_0^\tau r}\left(e^{\int_0^\tau r} \vee g\right) {}_\tau p_x \mu_{x+\tau}\, d\tau\right] = \mathbb{E}\left[\int_0^n \left(1 \vee e^{-\int_0^\tau r} g\right) f_\tau\, d\tau\right]\,,$$

where we have abbreviated

$$f_t = {}_t p_x \mu_{x+t}\,.$$

Following the recipe in Item (b) of Paragraph M, consider the 'price of future claims at time t',

$$\begin{aligned}&\mathbb{E}\left[\int_t^n e^{-\int_t^\tau r}\left(e^{\int_0^\tau r} \vee g\right) f_\tau\, d\tau \,\middle|\, r_\tau;\, 0 \le \tau \le t\right]\\ &= \mathbb{E}\left[\int_t^n \left(U_t \vee e^{-\int_t^\tau r} g\right) f_\tau\, d\tau \,\middle|\, r_\tau;\, 0 \le \tau \le t\right]\,.\end{aligned} \tag{28}$$

Arguing as before, the expression in (28) is a function of t, r_t and U_t. Consider its value at time t for given $U_t = u$, and $r_t = r^e$,

$$W_e(t,u) = \mathbb{E}\left[\int_t^n \left(u \vee e^{-\int_t^\tau r} g\right) f_\tau\, d\tau \,\middle|\, r_t = r^e\right]\,.$$

When $r_0 = r^e$, the premium we seek is $W_e(t, 1)$

Now use the backward construction, this time leaving details aside:

$$\begin{aligned}
&W_e(t,u) = \\
&(1-\lambda_{e\cdot}dt)\mathbb{E}\left[(u \vee g)\, f_t\, dt + e^{-r^e dt}\int_{t+dt}^{n}\left(e^{r^e dt}u \vee e^{-\int_{t+dt}^{\tau} r}g\right) f_\tau\, d\tau \middle|\, r_{t+dt} = r^e\right] \\
&+ \sum_{f;\, f\neq e} \lambda_{ef} dt\, W_f(t,u) \\
&= (1-\lambda_{e\cdot}dt)\left((u \vee g)\, f_t\, dt + e^{-r^e dt} W_e(t+dt, e^{r^e dt}u)\right) + \sum_{f;\, f\neq e} \lambda_{ef} dt\, W_f(t,u)\,.
\end{aligned}$$

Insert $e^{\pm r^e\, dt} = 1 \pm r^e dt + o(dt)$ and

$$\begin{aligned}
W_e(t+dt, e^{r^e dt}u) &= W_e(t+dt, u + ur^e dt) + o(dt) \\
&= W_e(t,u) + \frac{\partial}{\partial t}W_e(t,u)\, dt + \frac{\partial}{\partial u}W_e(t,u)\, u\, r^e\, dt + o(dt)\,,
\end{aligned}$$

and fill in some details to arrive at the partial differential equations

$$(u \vee g) f_t - r^e W_e(t,u) + \frac{\partial}{\partial t}W_e(t,u) + \frac{\partial}{\partial u}W_e(t,u)\, u\, r_e + \sum_{f;\, f\neq e} \lambda_{ef}\left(W_f(t,u) - W_e(t,u)\right) = 0\,.$$

These are to be solved subject to the conditions

$$W_e(n-, u) = 0\,,$$

$e = 1, \ldots, J^Y$.

O. Salary dependent premiums and benefits. The employees of a firm are enrolled in a pension scheme with salary dependent premiums and benefits. Consider an employee (x), who enters the scheme x years old at time 0 (say), retires at pensionable age 65 at time $m = 65 - x$, earns salary at rate $S(t)$ per time unit at any time $t < m$, and will receive pension continuously at level rate Q (yet to be determined) for n years after retirement. Let us first work under the assumption that the interest rate r is constant and known for the entire term of the contract up to time $m + n$.

(a) We will first consider a 'defined contributions' scheme under which a fixed proportion of the salary is used as premium for additional pension benefits. It will turn out that equivalence is automatically attained regardless of the development of the salary.

In any small time interval $[t, t + dt)$, $t < m$, the insured earns $S(t)\, dt$. A fixed proportion $\pi S(t)\, dt$, $0 < \pi < 1$, of this salary is used as a single premium for a pension of $q(t)\, dt$ per time unit in the time interval $[m, m + n]$. By the principle of equivalence, q_t is given by

$$\pi\, S(t)\, dt = q(t)\, dt\; {}_{m-t|n}\bar{a}_{x+t}\,,$$

that is,

$$q(t) \;=\; \pi\, \frac{S(t)}{{}_{m-t|n}\bar{a}_{x+t}} \;=\; \pi\, \frac{S(t)}{{}_{m-t}E_{x+t}\; \bar{a}_{x+m\,\overline{n}|}}\,.$$

The total rate of pension per time unit purchased by a survivor at time m is

$$Q = \int_0^m q(\tau)\, d\tau = \pi \int_0^m \frac{S(\tau)}{{}_{m-\tau}E_{x+\tau}\; \bar{a}_{x+m\,\overline{n}|}}\, d\tau\,.$$

It should be fairly obvious that the equivalence requirement is fulfilled by this scheme since, no matter how much or little salary the insured will earn and no matter if it can be predicted or not at the outset, the benefits are entirely determined by the salary-dependent contributions. Let us, however, just check: For any given salary function, S, the expected discounted premiums are

$$\pi \int_0^m e^{-\int_0^\tau (r+\mu_{x+s})\, ds}\, S(\tau)\, d\tau = \pi \int_0^m {}_{\tau}E_x\, S(\tau)\, d\tau\,,$$

and the expected discounted benefits are

$$Q\; {}_{m|n}\bar{a}_x = \pi \int_0^m \frac{S(\tau)}{{}_{m-\tau}E_{x+\tau}\; \bar{a}_{x+m\,\overline{n}|}}\, d\tau\; {}_mE_x\; \bar{a}_{x+m\overline{n}|} = \pi \int_0^m {}_{\tau}E_x\, S(\tau)\, d\tau\,,$$

where we have used the well-known identity ${}_mE_x = {}_{\tau}E_x\; {}_{m-\tau}E_{x+\tau}$.

(b) Suppose now that, instead of letting the contributions determine the benefits, the benefits are linked to the salary whereas the premiums are not. More specifically, suppose pension is payable continuously at rate $0.75S(m)$ (i.e. 75% of the salary rate at the time of retirement) for n years after retirement, and that premium is payable at a prefixed level rate π while active (both contingent on survival, of course). To determine the premium level π at time 0, we now need to make assumptions about the future development of the salary. Let us also abandon the unrealistic assumption that the future development of the interest rate is known:

Assume that the economy is governed by a continuous time Markov chain $Y(t)$, $t \geq 0$, with state space $\mathcal{J} = \{1, \ldots, J\}$, constant intensities of transition λ_{jk}, $j \neq k$, and initial state $Y(0) = i$, say. At any time $t \geq 0$ the accumulation factor $U(t)$ of the investment portfolio is given by

$$U(t) = \exp\left(\int_0^t r(s)\, ds\right)\,, \qquad r(s) = \sum_j I_j(s) r_j\,, \tag{29}$$

and the salary rate is given by

$$S(t) = \exp\left(\int_0^t a(s)\, ds\right)\,, \qquad a(s) = \sum_j I_j(s) a_j\,.$$

Here, $I_j(t) = 1[Y(t) = j]$, and the r_j and the a_j are known, fixed numbers; r_j is the interest rate and a_j is the rate at which salary increases per time unit and per unit of salary when the economy is in state j.

Exercise 24
Determine the premium rate π that makes expected discounted benefits less premiums equal to 0 at time 0; Construct differential equations for benefits and for premiums and specify appropriate side conditions.

Exercise 25
A 30 years old buys a pure life endowment of 1 in $n = 30$ years against premium payable continuously at level rate as long as the policy is in force. The first order basis specifies G82M mortality and interest at instantaneous rate 0.02. The second order basis specifies the same mortality G82M and stochastic interest as given in Fig 2.

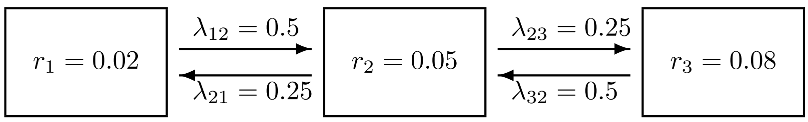

Figure 2: Sketch of a simple Markov chain interest model.

(a) Suppose surpluses are to be repaid immediately as cash bonus. Compute the state-wise expected present values (under second order interest) at time 0 of future bonuses, given that the insured survives 30 years. Use the program 'prores2.pas'. To compute the first order reserve, it may be a good idea to define an auxiliary economy state that does not communicate with the states in Fig. 2.

(b) Suppose instead that that surpluses are currently spent on purchase of additional benefits. Compute the state-wise expected values at time 0 of the additional benefit Q_{30}.

Exercise 26
Recall Exercise 2. Let the price at time t of a stock be

$$S(t) = e^{\alpha t + \beta N(t)},$$

where $N(t)$ is a Poisson process with intensity λ and α and β are constants. The money market account bears interest at constant spot rate r.

At time 0 our a life aged x purchases an n-year unit linked pure life endowment with sum insured $S(n) \vee g$ against a single premium π. Here g is the guaranteed minimum sum insured introduced to protect the insured against poor performance of the stock; if β is negative (in which case α should certainly be greater that r), then a Poisson event at time t represents a sudden drop in the stock price from $S(t-)$ to $S(t) = S(t-)e^{\beta}$ (a crash in the stock market if the absolute value of β is big). Combining basic principles in finance (no arbitrage) and insurance (equivalence), π should be the expected discounted value of the claim under a suitable probability measure (equivalent martingale measure for the market and physical measure for the life length):

$$\pi \;=\; \mathbb{E}\left[e^{-rn}\,(S(n) \vee g)\, 1[T_x > n]\right] \;=\; \mathbb{E}\left[S(n) \vee g\right]\, e^{-rn}\, {}_n p_x\,.$$

Work out a partial differential equation that can be used for numerical computation of π.

Exercise 27
(a) Describe in words the insurance product for which the premium rate is

$$\frac{\bar{A}_x + \bar{A}_y - \bar{A}_{xy}}{\bar{a}_{xy}}\,. \tag{30}$$

(b) Show how to compute the premium in (a) and the state-wise reserves at all times by filling in appropriate statements in the following excerpt of the program 'prores1.pas'. Assume that both lives are subject to the same mortality law with intensity $\mu_t = 0.0005 + 0.00008\, e^{0.09t}$ at age t, that $x = 35$, $y = 30$, and that the annual interest rate is 5%. (You will have to specify a sufficiently long term of the policy.)

```
(* SPECIFY NON-NULL PAYMENTS AT TIME t ! *)
ca[ ] :=     ; bi[ , ] :=     ; bi[ , ] :=     ;

(* SPECIFY MAXIMUM ORDER OF MOMENTS AND NUMBER OF STATES
! *)
q :=    ;  (*moments*)
Jmin :=    ;  (*first state*)
Jmax :=    ;  (*last state*)

(*SPECIFY NON-NULL LIFE ENDOWMENTS AT TERM OF CONTRACT,
be[j] and ce[j], AND AT TIME 0, b0 and c0. PUT c0 := 1 IF ALL OTHER
PREMIUMS ARE 0 AND ONLY MOMENTS OF BENEFITS ARE WANTED
! *)
be[ ] :=    ; (*endowments at term of contract*)
c0 :=    ; b0 :=    ;

(*SPECIFY AGES UPON INCEPTION OF CONTRACT x,y etz, TERM t,
AND INTEREST RATES ! *)
x :=    ; y :=    ;
t :=    ;  (*term*)
```

r := ln(1 +   );   (*interest rate*)

(* SPECIFY TRANSITION INTENSITIES FOR POLICY Z ! *)
(*TWO LIVES:*) alpha[0,1] :=   ;
gamma[0,1] :=   ;
beta[0,1] :=   ;
alpha[0,2] :=   ;
gamma[0,2] :=   ;
beta[0,2] :=   ;
alpha[1,3] :=   ;
gamma[1,3] :=   ;
beta[1,3] :=   ;
alpha[2,3] :=   ;
gamma[2,3] :=   ;
beta[2,3] :=   ;

(c) Suppose now that the remaining life lengths of (x) and (y) are positive quadrant dependent. Show that both the numerator and the denominator in (30) would be underestimated if they were calculated under the usual independence hypothesis.

Exercise 28
Referring to Paragraph N on pages 19-21 in '305exerc', assume now that the benefits are an n-year life annuity payable at rate $e^{\int_0^t r_u\,du} \vee e^{a\,t}$ at time $t \in (0, n)$, where r is the interest process and a is a fixed guaranteed minimum interest rate.

(a) Assume that the interest rate process r is of Markov chain type as described in Paragraph N. Derive the partial differential equations from which the expected present value of the benefits can be determined. Instead of starting from the time t analogue of this expected present value as in Paragraph N, you should start from the martingale associated with the random variable as we have been doing in the lectures.

(b) Assume instead that r is a Poisson-driven Ornstein-Uhlenbeck process with dynamics

$$dr_t = \alpha\,(\rho - r_t)\,dt + \sigma\,dN_t\,,$$

starting from a known value r_0 at time 0. The parameters α, ρ, and σ are positive constants, and N is a Poisson process with intensity λ. Derive the partial differential equation from which the expected present value of the benefits can be determined. (The backward argument still works because the processes involved are piece-wise deterministic, jumping only at isolated points of time. In the present model, r makes a jump of σ when N jumps by 1.)

Exercise 29
At time 0 a firm introduces a pension scheme that comprises all current and future employees. Let $\ell_t(x)\,dx$ be the number of employees at ages between x and $x + dx$ at time t. Assume that, in any time interval $(t, t + dt)$ after time 0, the firm hires $h_t(y)\ dy\,dt$ new employees aged between y and $y + dy$. At any time $t \geq 0$, the mortality rate at age x in this population is $\mu_t(x)$.

(a) Explain the expressions

$$\ell_t(x) = \begin{cases} \int_{t-x}^{t} h_\tau(x - t + \tau)\, e^{-\int_{x-t+\tau}^{x} \mu_{s-x+t}(s)\, ds}\, d\tau & , \quad x < t\,, \\ \ell_0(x - t)\, e^{-\int_{x-t}^{x} \mu_{s-x+t}(s)\, ds} + \int_0^t h_\tau(x - t + \tau)\, e^{-\int_{x-t+\tau}^{x} \mu_{s-x+t}(s)\, ds}\, d\tau & , \quad x \geq t\,. \end{cases}$$

(b) At any time $t > 0$, every person aged $x \leq 65$ is to contribute premium at rate c_t per time unit and every person aged $x > 65$ is to receive pension at rate b_t per time unit. The total fund of the scheme is invested in an asset portfolio that bears interest at rate r_t per time unit at time t. Write down the differential equation for the dynamics of the total fund at time t, denoted by U_t. Explain how the differential equation can be used for calculation of future U_t under various scenarios for the mortality μ and interest r (not controllable), recruitment policy h (partly controllable), and contributions and benefits c, b, and z (controllable).

Exercise 30

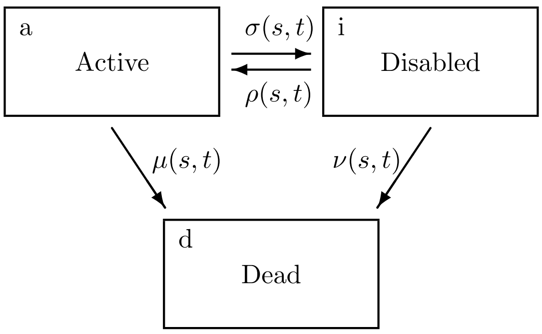

Figure 3: Semi-Markov model for a policy with states 'active', 'disabled', and 'dead'.

The model sketched in Fig. 3 is apt to describes the development of a policy with payments dependent of the state of health of the insured. For any policy duration $t \geq 0$, let $Z(t)$ denote the state of the insured at time t, and let $S(t)$

denote the state duration at that time (the time elapsed since the latest entry into the current state). The transition intensities depend on t and the current value s of $S(t)$ (the insured's age is suppressed in the notation).

At time 0 an active person purchases an n year life endowment with sum b_a in active state and sum b_i in disabled state, against premium payable at level rate c while active (premium is waived during disability). Assume that the interest rate r is constant and known.

(a) Derive backward differential equations for the state-wise reserves $V_j(s,t)$, $j \in \{a, i\}$, seen as functions of (s,t) in the domain $0 \le s \le t \le n$, and specify their side conditions.

(b) Suppose the contract is modified such that waiver of premium in disabled state is granted only after a qualifying period of q years. What are then the differential equations for the reserves?

(c) Explain that, in the special case $b_a = 0$, $b_i = 1$, $c = 0$, $r = 0$, the reserves $V_j(s,t)$ reduce to the transition probabilities

$$p_{ji}(s,t,n) = \mathbb{P}[Z(n) = i \,|\, Z(t) = j,\, S(t) = s]\,.$$

(d) Prove that, in the case with no recovery ($\rho(s,t) \equiv 0$),

$$p_{ii}(s,t,n) = e^{-\int_t^n \nu(s+\tau-t,\tau)\, d\tau}\,.$$

Do this in two ways: First, verify that the expression on the right satisfies the backward differential equation for $p_{ii}(s,t,n)$. Second, construct a forward differential equation for $p_{ii}(s,t,u)$ seen as a function of u for fixed s and t.

Exercise 31

At time 0 an x years old purchases an insurance policy with the following terms. The benefits are an m year deferred n year life annuity payable continuously at rate b at time $t \in (m, m+n)$, and the premiums are an n year life annuity payable continuously at level rate π at time $t \in (0,m)$. The premium rate π is determined by the principle of equivalence using a 1st order basis with constant interest rate r^* and mortality rate μ^*_{x+t}, $t \ge 0$. Denote the 1st order reserve at time t by V^*_t. The empirical interest and mortality rates (the experience basis) are denoted by r_t and μ_{x+t}, respectively.

(a) Write down Thiele's differential equation for V^*_t and its side condition.

(b) The discounted mean surplus per policy at time t is defined as

$$S_t = \int_0^t e^{-\int_0^\tau (r_u+\mu_{x+u})\,du} \left(\pi\, 1_{(0,m)}(\tau) - b\, 1_{(m,m+n)}(\tau)\right) d\tau - e^{-\int_0^t (r_u+\mu_{x+u})\,du} V^*_t\,.$$

Give a verbal motivation of this notion of surplus.

(c) Show that

$$\frac{d}{dt} S_t = e^{-\int_0^t (r_u + \mu_{x+u})\, du}\, c_t,$$

where

$$c_t = (r_t - r^* + \mu_{x+t} - \mu^*_{x+t})\, V^*_t$$

is the rate at which surplus emerges per survivor and per time unit at time t. Interpret the two terms in the expression for c_t.

(d) Discuss how to choose the 1st order basis on the safe side such that c_t is positive for all t.

(e) Suppose surpluses emerging during the deferred period are used as a single premium for additional pensions to survivors at time m, while surpluses emerging during the benefit period are paid back immediately as cash bonus. Find an expression for the rate at which pensioners receive benefits and bonuses at any time $t \in (m, m+n)$.