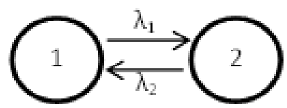
Solutions to Extra Exercise 10

1. We start at 1. Think of an immortal individual moving between the two sites 1 and 2.



The forward equations are

$$\begin{aligned}\tilde{p}_1'(t) &= -\lambda_1\tilde{p}_1(t) + \lambda_2\tilde{p}_2(t) \qquad \text{where } \tilde{p}_2(t) = 1 - \tilde{p}_1(t) \\ &= -\lambda_1\tilde{p}_1(t) + \lambda_2(1 - \tilde{p}_1(t)) \qquad \text{s.t. } \tilde{p}_1(0) = 1\end{aligned}$$

$$\begin{aligned}\tilde{p}_1'(t) + (\lambda_1 + \lambda_2)p_1(t) &= \lambda \\ \frac{d}{dt}\left(\tilde{p}_1 e^{(\lambda_1+\lambda_2)t}\right) &= \lambda_2 e^{(\lambda_1+\lambda_2)t} \\ \tilde{p}_1 e^{(\lambda_1+\lambda_2)t} - 1 &= \lambda_2 \int_0^t e^{(\lambda_1+\lambda_2)s} ds \\ &= \frac{\lambda_2}{\lambda_1+\lambda_2}\left(e^{(\lambda_1+\lambda_2)t} - 1\right)\end{aligned}$$

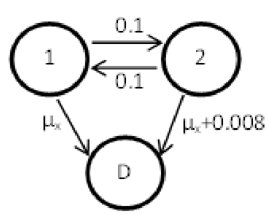
So

$$\tilde{p}_1(t) = \frac{\lambda_2}{\lambda_1+\lambda_2} + \frac{\lambda_1}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)t}$$

Now, since μ_x is the same for states 1 and 2, death is independent of other movements and we have

$$\begin{aligned}p_1(t) &= \tilde{p}_1(t) \times P(\text{life is alive at t}) \\ &= \tilde{p}_1(t)\,{}_tp_x \\ p_2(t) &= \tilde{p}_2(t)\,{}_tp_x \\ &= (1 - \tilde{p}_1(t))\,{}_tp_x\end{aligned}$$

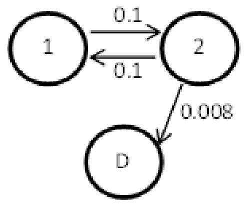
2. Now, the model becomes



The forward equations are

$$\begin{aligned}
p_1'(t) &= -(0.1+\mu_{x+t})p_1(t)+0.1p_2(t), \qquad p_1(0)=1 \\
p_2'(t) &= -(0.1+\mu_{x+t}+0.008)p_2(t)+0.1p_1(t), \qquad p_2(0)=0
\end{aligned}$$

But now, direct substitution does not work as we do not have $p_1(t)$. Think of an almost immortal individual.



The forward equations are

$$\begin{aligned}
\tilde{p}_1'(t) &= -0.1\tilde{p}_1(t)+0.1\tilde{p}_2(t), \qquad \tilde{p}_1(0)=1 \\
\tilde{p}_2'(t) &= -0.108\tilde{p}_2(t)+0.1\tilde{p}_1(t), \qquad \tilde{p}_2(0)=0
\end{aligned}$$

The general solution is $\tilde{p}_1(t)=A_1e^{-k_1t}+B_1e^{-k_2t}$ where k_1 and k_2 are solutions to

$$(-0.1-k)(-0.108-k)-0.1\times 0.1 \quad = \quad 0$$

Solving, we get $k_1=-0.00392$ and $k_2=-0.20408$. So

$$\begin{aligned}
\tilde{p}_1(t) &= A_1e^{-0.00392t}+B_1e^{-0.20408t} \\
\tilde{p}_2(t) &= A_2e^{-0.00392t}+B_2e^{-0.20408t}
\end{aligned}$$

Since $\tilde{p}(0)=1$, we have

$$\begin{aligned}
A_1+B_1 &= 1 \quad \text{and} \\
\tilde{p}_1'(0) &= -0.1 \\
-0.00392A_1-0.20408B_1 &= -0.1
\end{aligned}$$

Solving, we get $A_1=0.51998$ and $B_1=0.48002$. Similarly we solve for $A_2=0.4996$ and $B_2=-0.4996$. Since normal death is independent of all other movements as above, we get the answer

$$p_2(t)=0.4996(e^{-0.00392t}-e^{-0.20408t})_tp_x$$

3. Solutions in MAPLE worksheet.
4. Assuming a life of age 35, for the second model, the required probability is

$$\begin{aligned}
P(\text{in site 2 at time t=10}|\text{died at t=10}) &= \frac{P(\text{in site 2 at time t=10, died at t=10})}{P(\text{died at t=10})} \\
&= \frac{p_2(10)(\mu_{35+10}+0.008)dt}{p_2(10)(\mu_{35+10}+0.008)dt+p_1(10)\mu_{35+10}dt} \\
&= \frac{0.004991}{0.004991+0.002394} \\
&= 0.676
\end{aligned}$$

For the first model, the probability is

$$\begin{aligned}
\frac{p_2(10)\mu_{35+10}dt}{p_2(10)\mu_{35+10}dt + p_1(10)\mu_{35+10}dt} &= \frac{p_2(10)}{p_2(10) + p_1(10)} \\
&= \frac{\tilde{p}_2(10)}{\tilde{p}_2(10) + \tilde{p}_1(10)} \\
&= \tilde{p}_2(10)
\end{aligned}$$

Note that since death is independent of other movements, the question could have asked what is the probability we are at 2 at time 10.