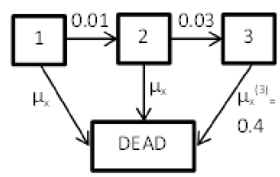
Solutions to Extra Exercise 11



1. We make use of the markov property of the process (given the present, past and future are independent).

$$
\begin{aligned}
& P(\text{in state 1 10 years ago and dead in 10 years time}|\text{in state 2 now}) \\
= \; & P(\text{in state 1 10 years ago}|\text{in state 2 now})P(\text{dead in 10 years}|\text{in state 2 now}) \\
= \; & P(\text{in state 1 10 years ago}|\text{in state 2 now}) \\
& \times (1 - P(\text{state 2 in 10 years}|\text{in state 2 now}) - P(\text{state 3 in 10 years}|\text{in state 2 now})) \\
= \; & \frac{P(\text{in state 2 when 35}|\text{in state 1 when 25})P(\text{in state 1 when 25}|\text{in state 1 when 0})}{P(\text{in state 2 when 35}|\text{in state 1 when 0})} \\
& \times (1 - P(\text{in state 2 at 45}|\text{in state 2 when 35}) - P(\text{in state 3 at 45}|\text{in state 2 when 35})) \\
= \; & \frac{p_{12}(25,35)p_{11}(0,25)}{p_{12}(0,35)} \times (1 - p_{22}(35,45) - p_{23}(35,45))
\end{aligned}
$$

So we need to find $p_{11}(u,v)$, $p_{12}(u,v)$, $p_{22}(u,v)$ and $p_{23}(u,v)$.

$$
\begin{aligned}
p_{11}(u,v) &= e^{-\int_u^v (0.01+\mu_{x+s})ds} \\
&= e^{-0.01(v-u)}{}_{v-u}p_{x+u} \\
p_{12}(u,v) &= \int_u^v p_{11}(u,s)0.01e^{-\int_s^v (0.03+\mu_{x+w})dw}ds \\
&= \int_u^v 0.01e^{-0.01(s-u)}e^{-\int_u^s (0.01+\mu_{x+w})dw}e^{-\int_s^v (0.03+\mu_{x+w})dw}ds \\
&= \int_u^v 0.01e^{-0.01(s-u)}e^{-0.03(v-s)}e^{-\int_u^v \mu_{x+w}dw}ds \\
&= \frac{1}{2}\left(e^{-0.01(v-u)} - e^{-0.03(v-u)}\right){}_{v-u}p_{x+u} \\
p_{22}(u,v) &= e^{-\int_u^v (0.03+\mu_{x+w})dw} \\
&= e^{-0.03(v-u)}{}_{v-u}p_{x+u} \\
p_{23}(u,v) &= \int_u^v p_{22}(u,s)0.03e^{-\int_s^v \mu^{(3)}_{x+w}dw}ds \\
&= \int_u^v e^{-0.03(s-u)}{}_{s-u}p_{x+u}0.03e^{-0.4(v-s)}ds
\end{aligned}
$$

Use maple to calculate

$$\begin{aligned} p_{11}(0,25) &= 0.7639 \\ p_{12}(0,35) &= 0.1712 \\ p_{12}(25,35) &= 0.0807 \\ p_{22}(35,45) &= 0.7183 \\ p_{23}(35,45) &= 0.0573 \end{aligned}$$

So the required probability is 0.0808.

2. We first assume that there is no time limit, the contract is valid for life. Since the annuity at 3 is not dependent on age, the whole thing is equivalent to getting a sum of $100000 + 50000\bar{a}_x^{(3)}$ upon transition from state 2 to 3, no matter when the transition occurs, where

$$\begin{aligned} \bar{a}_x^{(3)} &= \int_0^\infty e^{-0.05s}e^{-0.4s}ds \\ &= \frac{1}{0.45} \end{aligned}$$

So we have

$$\begin{aligned} \text{EPV(benefits)} &= (100000 + 50000\bar{a}_x^{(3)})\int_0^\infty e^{-0.05s}0.03p_2(s)ds \\ &= \left(100000 + \frac{50000}{0.45}\right)\int_0^\infty e^{-0.05s}0.03p_2(s)ds \end{aligned}$$

<u>Reserves</u>

$$\begin{aligned} V_3(t) &= \frac{50000}{0.45} \quad \text{at all times} \\ V_2(t) &= \left(100000 + \frac{50000}{0.45}\right)\int_t^\infty e^{-0.05(s-t)}0.03p_{22}(35+t, 35+s)ds \\ V_1(t) &= \left(100000 + \frac{50000}{0.45}\right)\int_t^\infty e^{-0.05(s-t)}0.03p_{12}(35+t, 35+s)ds \end{aligned}$$

Adjusting for the fact that policy expires at time 25 (if at state 1), we deduct EPV of excess of contract.

$$V_1^*(t) = V_1(t) - V_1(25)e^{-0.05\times(25-t)}p_{11}(35+t, 60)$$

The single premium will be

$$\begin{aligned} V_1^*(0) &= V_1(0) - V_1(25)e^{-0.05\times 25}p_{11}(35, 60) \\ &= 9354.3046 - 4707.8013 \times 0.2443 \\ &= 8204.92 \end{aligned}$$

3. We can get the reserves at time 10 using the formula above, but we need to adjust for the reserves

in state 1.

$$\begin{aligned}
V_3(10) &= \frac{50000}{0.45} \\
V_2(10) &= \left(100000 + \frac{50000}{0.45}\right) \int_t^{\infty} e^{-0.05(s-t)} 0.03 p_{22}(35+t, 35+s) ds \\
&= 67959.97 \\
V_1^*(10) &= V_1(10) - V_1(25) e^{-0.05\times 15} p_{11}(45, 60) \\
&= 7698.1502 - 4707.8013 \times 0.4155 \\
&= 5742.06
\end{aligned}$$

4. The single premium offered to a 45 year old in state 2 will be the reserve in state 2 at time 10: 67959.97.

   We can also solve Thiele's differential equations to get the reserves - see the other MAPLE worksheet. (There the mortality intensity in state 3, $\mu_x^{(3)} = 0.4exp(0.002x)$ instead of just 0.4, so solutions are slightly different.)