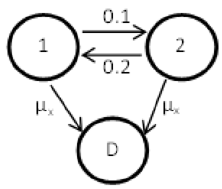
Solutions to Extra Exercise 12



1. We use the same method as in extra exercise 10, considering an immortal individual. Then

$$\begin{aligned} p_1(t) &= p_1(t)\,{}_tp_x \\ p_2(t) &= p_2^*(t)\,{}_tp_x \end{aligned}$$

From the result in exercise 10,

$$\begin{aligned} p_1^*(t) &= \frac{1}{3}e^{-0.3t} + \frac{2}{3} \\ p_2^*(t) &= -\frac{1}{3}e^{-0.3t} + \frac{1}{3} \end{aligned}$$

We have the following

$$\begin{aligned} 5000\int_0^{35} p_1(t)0.1e^{-0.05t}dt &= 5000\int_0^{35}\left(\frac{1}{3}e^{-0.3t} + \frac{2}{3}\right)0.1e^{-0.05t}\,{}_tp_x dt \\ &= \frac{500}{3}\left(\int_0^{35} e^{-0.35t}\,{}_tp_x dt + 2\int_0^{35} e^{-0.05t}\,{}_tp_x dt\right) \\ &= \frac{500}{3}\left(\bar{a}^{0.35}_{30:\overline{35}|} + 2\bar{a}^{0.05}_{30:\overline{35}|}\right) \\ &= \frac{500}{3}(2.8418 + 15.7932) \\ &= 3105.83 \end{aligned}$$

$$\begin{aligned} 2000\int_0^{35} p_2(t)e^{-0.05t}dt &= \frac{2000}{3}\int_0^{35}(-e^{-0.3t} + 1)e^{-0.05t}\,{}_tp_x dt \\ &= \frac{2000}{3}\left(\bar{a}^{0.05}_{30:\overline{35}|} - \bar{a}^{0.35}_{30:\overline{35}|}\right) \\ &= \frac{2000}{3}(15.7932 - 2.8418) \\ &= 8634.27 \end{aligned}$$

$$\begin{aligned}
50000\int_0^{35} p_2(t)e^{-0.05t}\mu_{30+t}dt &= \frac{50000}{3}\int_0^{35}(1-e^{-0.3t})e^{-0.05t}{}_tp_{30}\mu_{30+t}dt \\
&= \frac{50000}{3}\left(\bar{A}^{1\quad 0.05}_{30:\overline{35}|} - \bar{A}^{1\quad 0.35}_{30:\overline{35}|}\right) \\
\bar{A}^{1\quad 0.05}_{30:\overline{35}|} &= 1-0.05\bar{a}^{0.05}_{30:\overline{35}|} - {}_{35}p_{30}e^{-0.05\times 35} \\
&= 1-0.05(15.7932)-0.77e^{-0.05\times 35} \\
&= 0.0765 \\
\bar{A}^{1\quad 0.35}_{30:\overline{35}|} &= 1-0.35(2.8418)-0.77e^{-0.35\times 35} \\
&= 0.00537
\end{aligned}$$

The expected present value of the benefits is the sum of the above which is 12925.60.

2. To get the benefit, the life has to be in state 2 at t, $t-5$ and all times $s \in [t-5, t]$.

$$\begin{aligned}
2000\int_5^{35} e^{-0.05t}p_2(t-5)p_{\overline{22}}(t-5,t)dt &= 2000\int_5^{35} e^{-0.05t}\left(\frac{1}{3}-\frac{1}{3}e^{-0.3(t-5)}\right){}_{t-5}p_x e^{-0.2\times 5}{}_5p_{x+t-5}dt \\
&= \frac{2000e^{-1}}{3}\int_5^{35} e^{-0.05t}\left(1-e^{-0.3(t-5)}\right){}_tp_x dt \\
&= \frac{2000e^{-1}}{3}\left(\left(\bar{a}^{0.05}_{30:\overline{35}|}-\bar{a}^{0.05}_{30:\overline{5}|}\right)-e^{0.3\times 5}\left(\bar{a}^{0.35}_{30:\overline{35}|}-\bar{a}^{0.35}_{30:\overline{5}|}\right)\right) \\
&= \frac{2000e^{-1}}{3}(15.7932-4.4059-e^{0.3\times 5}(2.8418-2.3535)) \\
&= 2256.06
\end{aligned}$$

EPV of all benefits will now be 6547.39.

3. The occupation probability of not having left state 1 is

$$p_{\overline{11}}(t) = e^{-0.1t}{}_tp_x$$

The probability that he is at 2 at time t and it is the first visit is

$$\begin{aligned}
\tilde{p}_{12}(t) &= \int_0^t p_{\overline{11}}(u)\times 0.1\times p_{\overline{22}}(t-u)du \\
&= \int_0^t e^{-0.1u}{}_up_x 0.1e^{-0.2(t-u)}{}_{t-u}p_{x+u}du \\
&= {}_tp_x\int_0^t e^{-0.1u}0.1e^{-0.2(t-u)}du \\
&= {}_tp_x(e^{-0.1t}-e^{-0.2t})
\end{aligned}$$

Now we calculate the EPV of the benefits:

$$\begin{aligned}
5000\int_0^{35} p_{\overline{11}}(t)0.1e^{-0.05t} &= 5000\int_0^{35} 0.1e^{-0.1t}e^{-0.05t}{}_tp_x dt \\
&= 500\bar{a}^{0.15}_{30:\overline{35}|} \\
&= 500\times 6.5227 \\
&= 3261.35
\end{aligned}$$

$$
\begin{aligned}
2000\int_0^{35}\tilde{p}_{12}(t)e^{-0.05t}dt &= 2000\int_0^{35}(e^{-0.1t}-e^{-0.2t})e^{-0.05t}{}_tp_x dt \\
&= 2000\left(\bar{a}^{0.15}_{30:\overline{35}|}-\bar{a}^{0.25}_{30:\overline{35}|}\right) \\
&= 2000(6.5227-3.9663) \\
&= 5112.8
\end{aligned}
$$

$$
\begin{aligned}
50000\int_0^{35}\tilde{p}_{12}(t)\mu_{30+t}e^{-0.05t}dt &= 50000\int_0^{35}(e^{-0.1t}-e^{-0.2t})e^{-0.05t}{}_tp_{30}\mu_{30+t}dt \\
&= 50000\left(\bar{A}^{1\ \ \ \ 0.15}_{30:\overline{35}|}-\bar{A}^{1\ \ \ \ 0.25}_{30:\overline{35}|}\right) \\
&= 50000\left((1-0.15\times 6.5227-0.77e^{-0.15\times 35})\right. \\
&\quad \left.-(1-0.25\times 3.9663-0.77e^{-0.25\times 35})\right) \\
&= 462.57
\end{aligned}
$$

EPV of benefits is 8836.72.

4. The EPV for the allowance now becomes

$$
\begin{aligned}
2000\int_5^{35}e^{-0.05t}\tilde{p}_{12}(t-5)p_{\overline{22}}(t-5,t)dt &= 2000\int_5^{35}e^{-0.05t}(e^{-0.1(t-5)}-e^{-0.2(t-5)})_{t-5}p_{30}e^{-0.2\times 5}{}_5p_{30+t-5} \\
&= 2000e^{-1}\int_5^{35}e^{-0.05t}(e^{-0.1(t-5)}-e^{-0.2(t-5)})_tp_{30}dt \\
&= 2000e^{-1}\left(e^{0.1\times 5}\left(\bar{a}^{0.15}_{30:\overline{35}|}-\bar{a}^{0.15}_{30:\overline{5}|}\right)-e^{0.2\times 5}\left(\bar{a}^{0.25}_{30:\overline{35}|}-\bar{a}^{0.25}_{30:\overline{5}|}\right)\right) \\
&= 2000e^{-1}\left(e^{0.5}(6.5227-3.5044)-e(3.9663-2.8444)\right) \\
&= 1417.58
\end{aligned}
$$

So EPV of benefits is 5141.5.

5. For the first model,

$$
\begin{aligned}
V_i(t) &= 5000\int_t^{35}e^{-0.05(u-t)}p_{i1}(t,u)0.1du+2000\int_t^{35}e^{-0.05(u-t)}p_{i2}(t,u)du \\
&\quad +50000\int_t^{35}e^{-0.05(u-t)}p_{i2}(t,u)\mu_{x+u}du
\end{aligned}
$$

for $i=1,2$. We need to find $p_{ij}(t,u)$ for $i=2$.

$$
\begin{aligned}
p_{11}(t,u) &= p^*_{11}(t,u)_{u-t}p_{x+t} \\
p_{ij}(t,u) &= p^*_{ij}(t,u)_{u-t}p_{x+t}, \qquad \text{in general, for } i,j=1,2
\end{aligned}
$$

Due to time homogeneity of immortal,

$$
\begin{aligned}
p^*_{11}(t,u) &= p^*_1(0,u-t) \\
&= \frac{1}{3}e^{-0.3(u-t)}+\frac{2}{3} \\
p^*_{12}(t,u) &= p^*_2(0,u-t) \\
&= \frac{1}{3}(1-e^{-0.3(u-t)})
\end{aligned}
$$

We define $m_1(t) = P(\text{life at 1 at t}|\text{at 2 at 0})$ and $m_2(t) = P(\text{life at 2 at t}|\text{at 2 at 0})$ as the transition probabilities starting in state 2.

$$\begin{aligned} p^*_{21}(t,u) &= p^*_{21}(0,u-t) = m_1(u-t) \\ p^*_{22}(t,u) &= p^*_{22}(0,u-t) = m_2(u-t) \end{aligned}$$

Using the same method of considering an immortal individual and solving the differential equations, we can get expressions for $m_1(t)$ and $m_2(t)$

$$\begin{aligned} m_1(t) &= \frac{2}{3}(1-e^{-0.03t}) \\ m_2(t) &= \frac{1}{3}+\frac{2}{3}e^{-0.03t} \end{aligned}$$

For the third model where only the first visit counts, we have

$$\begin{aligned} V_1(t) &= 5000\int_t^{35} e^{-0.05(u-t)}0.1e^{-0.1(u-t)}{}_{u-t}p_{x+t}du \\ &\quad +2000\int_t^{35} e^{-0.05(u-t)}(e^{-0.1(u-t)}-e^{-0.2(u-t)})_{u-t}p_{x+t}du \\ &\quad +50000\int_t^{35} e^{-0.05(u-t)}(e^{-0.1(u-t)}-e^{-0.2(u-t)})_{u-t}p_{x+t}\mu_{x+u}du \\ V_2(t) &= 0+2000\int_t^{35} e^{-0.05(u-t)}e^{-0.2(u-t)}{}_{u-t}p_{x+t}du+50000\int_t^{35} e^{-0.05(u-t)}e^{-0.2(u-t)}{}_{u-t}p_{x+t}\mu_{x+u}du \end{aligned}$$

With qualification period (model 2), we only look at the term for the allowance since all the other terms will remain the same as in model 1.

$$V_1(t) = 2000\int_t^{35} e^{-0.05(u-t)}p_{12}(t,u-5)e^{-0.2\times 5}{}_5p_{x+u-5}du$$

with the provision that $p_{12}(t,u-5) = 0$ for $u \leq 5$. And in state 2,

$$V_2(t,w) = \begin{cases} \begin{array}{c} 2000\int_t^{35} e^{-0.05(u-t)}p_{\overline{22}}(t,u)du \\ +2000\int_t^{35} p_{\overline{22}}(t,u)0.2\int_u^{35} e^{-0.05(s-t)}p_{12}(u,s-5)e^{-0.2\times 5}{}_5p_{x+s-5}dsdu \end{array} & \text{if } w \geq 5 \\ \begin{array}{c} 2000\int_{t+5-w}^{35} e^{-0.05(u-t)}p_{\overline{22}}(t,u)du \\ +2000\int_t^{35} p_{\overline{22}}(t,u)0.2\int_u^{35} e^{-0.05(s-t)}p_{12}(u,s-5)e^{-0.2\times 5}{}_5p_{x+s-5}dsdu \end{array} & \text{if } w < 5 \end{cases}$$

where w is the time elapsed since we last entered state 2. Moreover, if any arguments is negative, $p_{12}(-) = 0$.