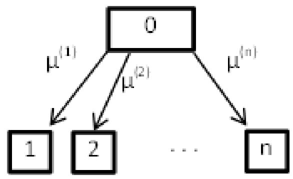
Solutions to Extra Exercise 13



1. i. Denote $\mu_x = \sum_{i=1}^{n} \mu_x^{(i)}$. Benefits $B = b_i$ if death due to i.

$$\begin{aligned} E(B) &= \sum_{i=1}^{n} b_i P(\text{death is due to i}) \\ &= \sum_{i=1}^{n} b_i \int_0^\infty e^{-\int_0^t \mu_{x+s} ds} \mu_{x+t}^{(i)} dt \\ &= \sum_{i=1}^{n} b_i \int_0^\infty {}_tp_x \mu_{x+t}^{(i)} dt \end{aligned}$$

$$\begin{aligned} E(B^2) &= \sum_{i=1}^{n} b_i^2 P(\text{death is due to i}) \\ &= \sum_{i=1}^{n} b_i^2 \int_0^\infty {}_tp_x \mu_{x+t}^{(i)} dt \end{aligned}$$

$$Var(B) = E(B^2) - E(B)^2$$

ii. Given death occurs at time k,

$$\begin{aligned} E(B|T=k) &= \sum_{i=1}^{n} b_i P(\text{death is due to i}|T=k) \\ &= \sum_{i=1}^{n} b_i \frac{\mu_{x+k}^{(i)} {}_kp_x}{\mu_{x+k} {}_kp_x} \\ &= \sum_{i=1}^{n} b_i \frac{\mu_{x+k}^{(i)}}{\mu_{x+k}} \end{aligned}$$

$$\begin{aligned} E(B^2|T=k) &= \sum_{i=1}^{n} b_i^2 \frac{\mu_{x+k}^{(i)}}{\mu_{x+k}} \\ Var(B^2|T=k) &= E(B^2|T=k) - E(B|T=k)^2 \end{aligned}$$

iii. Given death occurs before time k,

$$\begin{aligned} E(B|T<k) &= \sum_{i=1}^{n} b_i P(\text{death is due to i}|T<k) \\ &= \sum_{i=1}^{n} b_i \frac{\int_0^k {}_tp_x \mu_{x+t}^{(i)} dt}{\int_0^k {}_tp_x \mu_{x+t} dt} \end{aligned}$$

$$\begin{aligned} E(B^2|T<k) &= \sum_{i=1}^{n} b_i^2 \frac{\int_0^k {}_tp_x \mu_{x+t}^{(i)} dt}{\int_0^k {}_tp_x \mu_{x+t} dt} \\ Var(B^2|T<k) &= E(B^2|T<k) - E(B|T<k)^2 \end{aligned}$$

The continuous premium payable while the life is alive will be

$$\begin{aligned} \pi &= \frac{EPV(benefits)}{\int_0^\infty {}_tp_x e^{-rt} dt} \\ &= \frac{\sum_{i=1}^n \int_0^\infty e^{-rt} {}_tp_x \mu_{x+t}^{(i)} b_i dt}{\int_0^\infty {}_tp_x e^{-rt} dt} \\ &= \frac{\int_0^\infty e^{-rt} {}_tp_x \left(\sum_{i=1}^n \mu_{x+t}^{(i)} b_i \right) dt}{\int_0^\infty {}_tp_x e^{-rt} dt} \end{aligned}$$

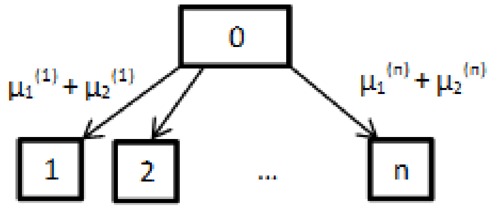
2. i. First note that

$$\begin{aligned} \bar{a}_{x+t} &= \left(\frac{1 - e^{-T_{x+t}}}{r} \right) \\ &= \int_t^\infty e^{-r(u-t)} {}_{u-t}p_{x+t} \mu_{x+u} du \end{aligned}$$

Then

$$\begin{aligned} E(\text{Reserves at t}) &= \int_t^\infty e^{-r(u-t)} {}_{u-t}p_{x+t} \sum_{i=1}^n \mu_{x+u}^{(i)} b_i - \pi \bar{a}_{x+t} \\ &= \int_t^\infty e^{-r(u-t)} {}_{u-t}p_{x+t} \sum_{i=1}^n \mu_{x+u}^{(i)} \left(b_i - \pi \frac{1 - e^{-r(u-t)}}{r} \right) du \\ &= \int_t^\infty e^{-r(u-t)} {}_{u-t}p_{x+t} \mu_{x+u} \sum_{i=1}^n \frac{\mu_{x+u}^{(i)}}{\mu_{x+u}} \left(b_i - \frac{1 - e^{-r(u-t)}}{r} \right) du \end{aligned}$$

$$E(R^2) = \int_t^\infty e^{-2r(u-t)} {}_{u-t}p_{x+t} \mu_{x+u} \sum_{i=1}^n \frac{\mu_{x+u}^{(i)}}{\mu_{x+u}} \left(b_i - \frac{1 - e^{-r(u-t)}}{r} \right)^2 du$$



3. We have $\mu_x + \mu_y = \sum_{i=1}^{n} \left(\mu_x^{(i)} + \mu_y^{(i)} \right)$.

$$
\begin{aligned}
P(B = b_i) &= \int_0^\infty e^{-\int_0^t \mu_{x+u}+\mu_{y+u} du} \left(\mu_{x+t}^{(i)} + \mu_{y+t}^{(i)} \right) dt \\
&= \int_0^\infty {}_tp_x {}_tp_y \left(\mu_{x+t}^{(i)} + \mu_{y+t}^{(i)} \right) dt
\end{aligned}
$$

So

$$
\begin{aligned}
E(B) &= \sum_{i=1}^{n} b_i \int_0^\infty {}_tp_x {}_tp_y \left(\mu_{x+t}^{(i)} + \mu_{y+t}^{(i)} \right) dt \\
E(B^2) &= \sum_{i=1}^{n} b_i^2 \int_0^\infty {}_tp_x {}_tp_y \left(\mu_{x+t}^{(i)} + \mu_{y+t}^{(i)} \right) dt
\end{aligned}
$$

Given that benefit is paid at time k,

$$
\begin{aligned}
P(B = b_i | T = k) &= \frac{{}_kp_x {}_kp_y \left(\mu_{x+k}^{(i)} + \mu_{y+k}^{(i)} \right)}{{}_kp_x {}_kp_y \left(\mu_{x+k} + \mu_{y+k} \right)} \\
&= \frac{\mu_{x+k}^{(i)} + \mu_{y+k}^{(i)}}{\mu_{x+k} + \mu_{y+k}} \\
&= \frac{\mu_{x+k}^{(i)}}{\mu_{x+k}} \qquad\qquad if \quad \mu_{x+k}^{(i)} = \mu_{y+k}^{(i)}
\end{aligned}
$$

$$
\begin{aligned}
E(B|T = k) &= \sum_{i=1}^{n} b_i \frac{\mu_{x+k}^{(i)} + \mu_{y+k}^{(i)}}{\mu_{x+k} + \mu_{y+k}} \\
E(B^2|T = k) &= \sum_{i=1}^{n} b_i^2 \frac{\mu_{x+k}^{(i)} + \mu_{y+k}^{(i)}}{\mu_{x+k} + \mu_{y+k}}
\end{aligned}
$$

Premium payable upfront is

$$
\pi = \int_0^\infty e^{-rt} {}_tp_x {}_tp_y \sum_{i=1}^{n} b_i \left(\mu_{x+t}^{(i)} + \mu_{y+t}^{(i)} \right) dt
$$

4. Benefit is payable now at the time of the second death. Let $T^{(2)}$ be the time of the second death.

$$
\begin{aligned}
P(T^{(2)} \le t) &= (1 - {}_tp_x)(1 - {}_tp_y) \\
&= 1 - {}_tp_x - {}_tp_y + {}_tp_x {}_tp_y
\end{aligned}
$$

$$
\begin{aligned}
P(B=b_i) &= \int_0^\infty \left({}_tp_x\mu_{x+t}^{(i)}(1-{}_tp_y)+{}_tp_y\mu_{y+t}^{(i)}(1-{}_tp_x)\right)dt \\
&= 2\int_0^\infty {}_tp_x(1-{}_tp_x)\mu_{x+t}^{(i)} \qquad\qquad if \quad \mu_{x+t}^{(i)}=\mu_{y+t}^{(i)}
\end{aligned}
$$

$$
\begin{aligned}
P(B=b_i|T=k) &= \frac{{}_tp_x\mu_{x+t}^{(i)}(1-{}_tp_y)+{}_tp_y\mu_{y+t}^{(i)}(1-{}_tp_x)}{\sum_{i=1}^n\left({}_tp_x\mu_{x+t}^{(i)}(1-{}_tp_y)+{}_tp_y\mu_{y+t}^{(i)}(1-{}_tp_x)\right)} \\
&= \frac{\mu_{x+t}^{(i)}}{\mu_{x+t}} \qquad\qquad if \quad \mu_{x+t}^{(i)}=\mu_{y+t}^{(i)}
\end{aligned}
$$

The premium upfront,

$$
\pi=\int_0^\infty e^{-rt}\sum_{i=1}^n\left({}_tp_x\mu_{x+t}^{(i)}(1-{}_tpy)+{}_tp_y\mu_{y+t}^{(i)}(1-{}_tp_x)\right)dt
$$