# Solutions to Extra Exercise 14

1. We have the Thiele's equation

$$\frac{dV_t}{dt} = (0.05 + \mu_{x+t})V_t + \pi - 100000\mu_{x+t} + 0.01(0.05V_t)\mathbf{1}_{\{t<20\}}, \qquad V_0 = 0$$

   To solve this:

$$\begin{aligned} \frac{dV_t}{dt} &= (0.05 + \mu_{x+t} + 0.0005\mathbf{1}_{\{t<20\}})V_t + \pi - 100000\mu_{x+t} \\ \frac{dV_t}{dt} - a(t)V_t &= \pi - 100000\mu_{x+t} \end{aligned}$$

   where

$$a(t) = 0.05 + \mu_{x+t} + 0.0005\mathbf{1}_{\{t<20\}}$$

   Multiply $e^{-\int_0^t a(s)ds}$ to both sides.

$$\begin{aligned} \frac{d}{dt}\left(V_t e^{-\int_0^t a(s)ds}\right) &= (\pi - 100000\mu_{x+t})e^{-\int_0^t a(s)ds} \\ V_t e^{-\int_0^t a(s)ds} &= \int_0^t (\pi - 100000\mu_{x+u})e^{-\int_0^u a(s)ds} du \end{aligned}$$

   Since $V_{30} = 100000$,

$$100000e^{-\int_0^{30} a(s)ds} = \int_0^{30} (\pi - 100000\mu_{x+u})e^{-\int_0^u a(s)ds} du$$

   Solve for $\pi$ to get $\pi = 1663$.

2. i. If the policy has not lapsed,

$$V_t = e^{\int_0^t a(s)ds} \int_0^t (\pi - 100000\mu_{x+u})e^{-\int_0^u a(s)ds} du$$

   ii. If the policy has lapsed,

$$\begin{aligned} V_t^{(lapse)} &= 0.95V_w e^{\int_w^t (0.05+\mu_{x+s})ds} = 0.95V_w \frac{e^{0.05(t-w)}}{{}_{t-w}p_{x+w}} && \text{if } w < 20 \\ &= V_w \frac{e^{0.05(t-w)}}{{}_{t-w}p_{x+w}} && \text{if } w > 20 \end{aligned}$$

   $V_{30}^{lapse}$ is the amount the policyholder will receive if he survives.

3. This is the probability that the policyholder will lapse and die before time 30.

$$\begin{aligned}
P(\text{lapse and die}) &= \int_0^{30} {}_tp_x e^{-0.01t} 0.01(1 - {}_{30-t}p_{x+t})dt \\
&= \int_0^{30} 0.01 {}_tp_x e^{-0.01t} dt - {}_{30}p_x \int_0^{30} 0.01 e^{-0.01t} dt \\
&= \int_0^{30} {}_tp_x e^{-0.01t} 0.01 dt - {}_{30}p_x (1 - e^{-0.01(30)}) \\
&= 0.01 \bar{a}^{0.01}_{x:\overline{30|}} - {}_{30}p_x (1 - e^{-0.01(30)}) \\
&= 0.0277
\end{aligned}$$

4. The expected value of the amount he receives given that the policyholder is alive at time 30 is

$$\left( \int_0^{\infty} (0.95 \mathbf{1}_{\{t<20\}} + \mathbf{1}_{\{t>20\}}) V_t \frac{e^{0.05(30-t)}}{{}_{30-t}p_{x+t}} 0.01 e^{-0.01t} {}_tp_x dt + 100000 {}_{30}p_x e^{-0.01(30)} \right) \times \frac{1}{{}_{30}p_x}$$