Solutions to Extra Exercise 15

1. First we calculate the premium using the first order basis ($r^* = 0.04$).

$$\pi \int_0^{20} e^{-0.04t} {}_tp_x dt = 100000 \left(\int_0^{20} e^{-0.04t} {}_tp_x \mu_{x+t} dt + {}_tp_x e^{-0.04\times 20} \right)$$

The terminal bonus is

$$\int_0^n e^{\int_s^n (r_u+\mu_{x+u})du}(r_s - r^*)V_s^* ds = \tilde{W}_t \int_0^t e^{\int_s^t (r_u+\mu_{x+u})du}(r_s - r^*)V_s^* ds + W_t^*$$

where

$$\begin{aligned} \tilde{W}_t &= e^{\int_t^n (r_u+\mu_{x+u})du} \\ W_t^* &= \int_t^n e^{\int_s^n (r_u+\mu_{x+u})du}(r_s - r^*)V_s^* ds \\ &= \int_t^n \tilde{W}_s (r_s - r^*)V_s^* ds \end{aligned}$$

We need to calculate the following

$$\begin{aligned} \tilde{V}_i(t) &= E(\tilde{W}_t | r_t = r^{(i)}) \\ \tilde{V}_i^{**}(t) &= E(W_t^* | r_t = r^{(i)}) \end{aligned}$$

where $r^{(1)} = 0.03$, $r^{(2)} = 0.06$, and $r^* = 0.04$.

$$\begin{aligned} \tilde{V}_1(t-dt) &= (1 + r^{(1)} + \mu_{x+t})dt)((1 - \lambda_{12}dt)\tilde{V}_1(t) + \lambda_{12}\tilde{V}_2(t)) + o(dt) \\ \frac{d\tilde{V}_1(t)}{dt} &= -(r^{(1)} + \mu_{x+t})\tilde{V}_1(t) + \lambda_{12}(V_1(t) - V_2(t)) \\ \frac{d\tilde{V}_2(t)}{dt} &= -(r^{(2)} + \mu_{x+t})\tilde{V}_2(t) + \lambda_{12}(V_2(t) - V_1(t)) \end{aligned}$$

$$\begin{aligned} V_1^{**}(t-dt) &= \tilde{V}_1(t)(r^{(1)} - r^*)V_t^* dt + (1 - \lambda_{12}dt)V_1^{**}(t) + \lambda_{12}dt V_2^{**}(t) + o(dt) \\ \frac{dV_1^{**}(t)}{dt} &= -\tilde{V}_1(t)(r^{(1)} - r^*)V_t^* + \lambda_{12}(V_1^{**}(t) - V_2^{**}(t)) \\ \frac{dV_2^{**}(t)}{dt} &= -\tilde{V}_2(t)(r^{(2)} - r^*)V_t^* + \lambda_{21}(V_2^{**}(t) - V_1^{**}(t)) \end{aligned}$$

where

$$\frac{dV_t^*}{dt} = (r + \mu_{x+t})V_t^* + \pi - \mu_{x+t}^* b_t$$

Solve the 5 equations together subject to the conditions $\tilde{V}_i(20) = 1$, $V_i^{**}(20) = 0$, and $V_{20}^* = 100000$ for $i = 1, 2$. The answer we need is $V_2^{**}(0)$. (See maple worksheet).

2. First we find the accumulation

$$U_t = \int_0^t e^{\int_s^t (r_u + \mu_{x+u}) du} (r_s - r^*) V_s^* ds$$

We solve the following two equations together

$$\begin{aligned}
\frac{dV_t^*}{dt} &= (r + \mu_{x+t}) V_t^* + \pi - \mu_{x+t}^* b_t \\
\frac{dU_t}{dt} &= (r^{(2)} + \mu_{x+t}) U_t + (r^{(2)} - r^*) V_t^*
\end{aligned}$$

such that $V_0^* = 0$, $U_0 = 0$. We obtain $U(5)$, the accumulation up to time 5. To get the prediction, we solve the above 5 equations again and the prediction of terminal bonus would be $\tilde{V}_2(5)U(5) + V_2^{**}(5)$. (See maple worksheet).