Solutions to Extra Exercise 16

1. [a)]

We can calculate the premium by

$$\text{Premium} = \text{PV(benefit)} + \text{PV(expenses)}$$

where we have

$$\text{PV(benefit)} = 3S_0 e^{\int_0^{30} a(s)ds} e^{-\int_0^{35} r(s)ds} e^{-\int_0^{35} \mu_{30+s}ds}$$

and

$$\text{PV(expenses)} = c\int_0^{35} e^{-\int_0^s r(u)du} e^{-\int_0^s \mu_{x+u}du} ds$$

(b) We need to calculate

$$\begin{aligned} W_1(t) &= E\left(e^{\int_t^{30} a(s)ds} e^{-\int_t^{35} r(s)ds} | Y_t = 1\right) \\ &= E\left(e^{\int_t^{35} (a^*(s)-r(s))ds} | Y_t = 1\right) \end{aligned}$$

where $a^*(t) = a(t)\mathbf{1}_{\{t\le 30\}}$. We have

$$W_1(t-dt) = (1-r^{(1)}dt)(1+a^{(1)}\mathbf{1}_{\{t\le 30\}}dt)\left(W_1(t)(1-\lambda dt) + \lambda dt W_1(t)\right) + o(dt)$$

$$\begin{aligned} \frac{W_1(t-dt)-W_1(t)}{dt} &= -\left(r^{(1)} - a^{(1)}\mathbf{1}_{\{t\le 30\}} + \lambda\right) W_1(t) + \lambda W_2(t) + \frac{o(dt)}{dt} \\ \frac{dW_1(t)}{dt} &= (r^{(1)} - a^{(1)}\mathbf{1}_{\{t\le 0\}} + \lambda)W_1(t) - \lambda W_2(t) \\ \frac{dW_2(t)}{dt} &= (r^{(2)} - a^{(2)}\mathbf{1}_{\{t\le 30\}} + \lambda)W_2(t) - \lambda W_1(t) \end{aligned}$$

where $W_1(35) = W_2(35) = 1$. EPV of benefit is $3S_0 {}_{35}p_{30} W_1(0)$.

(c) $a(t)$ is deterministic, so we just need to calculate

$$\tilde{W}_1(t) = E\left(e^{-\int_t^{35} r(s)ds} | Y_t = 1\right)$$

$$\begin{aligned} \tilde{W}_1(t-dt) &= (1-r^{(1)}dt)\left((1-\lambda dt)W_1(t) + \lambda dt W_2(t)\right) + o(dt) \\ \frac{d\tilde{W}_1(t)}{dt} &= (r^{(1)} + \lambda)\tilde{W}_1(t) - \lambda\tilde{W}_2(t) \\ \frac{d\tilde{W}_2(t)}{dt} &= (r^{(2)} + \lambda)\tilde{W}_2(t) - \lambda\tilde{W}_1(t) \end{aligned}$$

where $\tilde{W}_1(35) = \tilde{W}_2(35) = 1$. EPV of benefit is $e^{\int_0^{30} a(s)ds} S_{035} p_{30} \tilde{W}_1(0)$.

To calculate EPV of expenses, we need

$$W_1^*(t) = E\left(\int_t^{35} e^{-\int_t^s r(u)du}{}_s p_{30} ds | Y_t = 1\right)$$

$$\begin{aligned}
W_1^*(t-dt) &= {}_t p_{30} dt + (1 - r^{(1)} dt)\left((1-\lambda dt) W_1^*(t) + \lambda dt W_2^*(t)\right) + o(dt) \\
\frac{dW_1^*(t)}{dt} &= -{}_t p_{30} + (r^{(1)} + \lambda) W_1^*(t) - \lambda W_2^*(t) \\
\frac{dW_2^*(t)}{dt} &= -{}_t p_{30} + (r^{(2)} + \lambda) W_2^*(t) - \lambda W_1^*(t)
\end{aligned}$$

where $W_1^*(35) = W_2^*(35) = 0$. The EPV of expenses is $cW_1^*(0)$.

(d) We need to calculate

$$L_1(t) = E\left(\int_t^{35} e^{\int_0^u (a(s) - r(s))ds}{}_u p_{30} \mu_{30+u} du | Y_t = 1\right)$$

$$\begin{aligned}
\frac{L_1(t-dt) - L_1(t)}{dt} &= {}_t p_{30} \mu_{30+t} - (r^{(1)} + \lambda - a^{(1)}) L_1(t) + \lambda L_2(t) + \frac{o(dt)}{dt} \\
\frac{dL_1(t)}{dt} &= -{}_t p_{30} \mu_{30+t} + (r^{(1)} + \lambda - a^{(1)}) L_1(t) - \lambda L_2(t) \\
\frac{dL_2(t)}{dt} &= -{}_t p_{30} \mu_{30+t} + (r^{(2)} + \lambda - a^{(2)}) L_2(t) - \lambda L_1(t)
\end{aligned}$$

where $L_1(35) = L_2(35) = 0$. EPV of benefit is $3S_0 L_1(0)$.