Let $V(t)$ and $U(t)$ be the accumulations of the cash and unit funds respectively. We work in units of 100000. Since there will be money in the cash fund at time 30, we will have $U(30) < 1$ and therefore $U(t) < 1$ for all t. The accumulated values of the two funds at 30 will be such that $V(30) + U(30) = 1$. We have

$$\frac{dV(t)}{dt} = \mu_{30+t}(1 - U(t)) + 0.08\pi - \mu_{30+t}(1 - U(t)) + (0.02 - \mu_{30+t} + \mu_{30+t})V(t) =$$

$$= 0.08 + 0.02V(t).$$

Solving this subject to $V(0) = 0$, we have

$$V(t) = 0.08\pi \int_0^t \exp(0.02(t - s))\mathrm{ds} = 0.08\pi \frac{\exp(0.02t) - 1}{0.02}.$$

Also

$$\frac{dU(t)}{dt} = \pi - \mu_{30+t}(1 - U(t)) - 0.08\pi + 0.04U(t) = 0.92\pi - \mu_{30+t} + (0.04 + \mu_{30+t})U(t)$$

with $U(0) = 0$. Then,

$U(t) = 0.92\pi \int_0^t \exp\left(\int_s^t (0.04 + \mu_{30+u})du\right)\mathrm{ds} - \int_0^t \mu_{30+s}\exp\left(\int_s^t (0.04 + \mu_{30+u})du\right)\mathrm{ds} = 0.92\pi \frac{\exp(0.04t)}{{}_tp_{40}} \int_0^t \exp\left(-\int_0^s (0.04 + \mu_{30+u})du\right)\mathrm{ds} - \frac{\exp(0.04t)}{{}_tp_{30}} \int_0^t \mu_{30+s}\exp\left(-\int_0^s (0.04 + \mu_{30+u})du\right)\mathrm{ds} =$

$$0.92\pi \frac{\exp(0.04t)}{{}_tp_{30}} \bar{a}_{30:\overline{t|}} - \frac{\exp(0.04t)}{{}_tp_{30}} \left(1 - 0.04\bar{a}_{30:\overline{t|}} - \frac{{}_tp_{30}}{\exp(0.03t)}\right) =$$

$$0.92\pi \frac{\exp(0.04t)}{{}_tp_{30}} \bar{a}_{40:\overline{t|}} - \frac{\exp(0.04t)}{{}_tp_{30}} \left(1 - 0.03\bar{a}_{30:\overline{t|}}\right) + 1$$

and therefore

$$0.08\pi \frac{\exp(0.02 \times 30) - 1}{0.02} + 0.92\pi \frac{\exp(0.04 \times 30)}{{}_{30}p_{30}} \bar{a}_{30:\overline{30|}}$$

$$- \frac{\exp(0.04 \times 30)}{{}_{30}p_{30}} \left(1 - 0.04\bar{a}_{30:\overline{30|}}\right) + 1 = 1$$

and so

$$0.08\pi \frac{\exp(0.02 \times 30) - 1}{0.02} + 0.92\pi \frac{\exp(0.04 \times 30)}{0.8452} 16.8054 = \frac{\exp(0.04 \times 30)}{0.8452}(1 - 0.04 \times 16.8054)$$

and hence

$$(3.2884 + 60.7338)\pi = 1.2876$$

and

$$\pi = 0.02011$$

so the annual premium rate is 2011.

* The accumulation is

$$0.92 \times 0.02011 \times \frac{\exp(0.04 \times 20)}{{}_{20}p_{30}} \bar{a}_{30:\overline{20|}} - \frac{\exp(0.04 \times 20)}{{}_{20}p_{30}} \left(1 - 0.04 \bar{a}_{30:\overline{20|}}\right) + 1$$

but

$$\bar{a}_{30:\overline{20|}} = \bar{a}_{30:\overline{30|}} - {}_{20}p_{30} \exp(-0.04 \times 20) \bar{a}_{50:\overline{10|}} = 16.8054 - 0.9353 \exp(-0.8) 7.9187 =$$

$$13.4775$$

and the accumulation is

$$0.92 \times 0.02011 \times \frac{\exp(0.04 \times 20)}{0.9353} 13.4775 - \frac{\exp(0.04 \times 20)}{0.9558} (1 - 0.04 \times 13.4775) + 1 = 0.38771.$$

It is 38771.

Let us now consider part 3.

We have

$$\frac{dV(t)}{dt} = -\mu_{30+t}(1 - U(t)) + (\mu_{30+t} + 0.02)(1 - U(t)) + 0.02V(t) =$$

$$= 0.02V(t) + 0.02(1 - U(t)).$$

Solving this subject to $V(0) = 0$, we have

$$V(t) = 0.02 \int_0^t \exp(0.02(t - s)) \mathrm{ds} - 0.02 \int_0^t \exp(0.02(t - s)) U(s) \mathrm{ds}$$

Also

$$\frac{dU(t)}{dt} = \pi - (\mu_{30+t} + 0.02)(1 - U(t)) + 0.04U(t) = \pi - (\mu_{30+t} + 0.02) + (\mu_{30+t} + 0.06)U(t).$$

with $U(0) = 0$. Then,

$$U(t) = (\pi - 0.02) \int_0^t \exp\left(\int_s^t (0.06 + \mu_{30+u}) du\right) \mathrm{ds} - \int_0^t \mu_{30+s} \exp\left(\int_s^t (0.06 + \mu_{30+u}) du\right) \mathrm{ds}$$

Define now

$$W_1(t) = \int_0^t \exp\left(\int_s^t (0.06 + \mu_{30+u}) du\right) \mathrm{ds}$$

$$W_2(t) = \int_0^t \mu_{30+s} \exp\left(\int_s^t (0.06 + \mu_{30+u}) du\right) \mathrm{ds}$$

and then

$V(t) = 0.02 \int_0^t \exp(0.02(t - s)) \mathrm{ds} - 0.02(\pi - 0.02) \int_0^t \exp(0.02(t - s)) W_1(s) \mathrm{ds} + 0.02 \int_0^t \exp(0.02(t - s)) W_2(s) \mathrm{ds}.$

Define

$$Z_1(t) = \int_0^t \exp\left(0.02(t-s)\right) W_1(s)\mathrm{ds}$$

and

$$Z_2(t) = \int_0^t \exp\left(0.02(t-s)\right) W_2(s)\mathrm{ds}.$$

We can calculate the integrals by solving the following equations

$$\frac{dW_1(t)}{dt} = (0.06 + \mu_{30+t})W_1(t) + 1$$

$$\frac{dW_2(t)}{dt} = (0.06 + \mu_{30+t})W_2(t) + \mu_{30+t}$$

$$\frac{dZ_1(t)}{dt} = 0.02Z_1(t) + W_1(t)$$

$$\frac{dZ_2(t)}{dt} = 0.02Z_2(t) + W_2(t)$$

All to be solved subject to $W_1(0) = W_2(0) = Z_1(0) = Z_2(0)$. (see MAPLE worksheet) Note that we can solve 1&3 and 2&4 seperately. We use $W_1(30)$, $W_2(30)$, $Z_1(30)$, $Z_2(30)$ as the values of our integrals. Therefore

$$0.02\frac{\exp\left(0.02 \times 30\right) - 1}{0.02} - 0.02(\pi - 0.02) \times 1147.63 + 0.02 \times 3.4712 + (\pi - 0.02) \times 96.365 - 0.3761 = 1$$

and so $\pi - .02 = 0.066$ and the premium is 2660.

The accumulation of the unit fund at 20 is

$$660W_1(20) - 100000W_2(20) = 660 \times 40.553 - 11662 = 15103.$$

Not much goes in the unit fund; too conservative resulting to a high premium. See also some more results on the MAPLE worksheet.