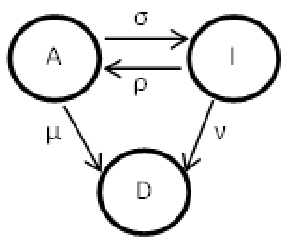
Solutions to Extra Exercise 9



We start in state A. $P_A(t) = P(X(t) = A|X(0) = A)$ and $P_I(t) = P(X(t) = I|X(0) = A)$. The forward equations are

$$\begin{aligned} p_A'(t) &= -(\sigma+\mu)p_A(t) + \rho p_I(t), \qquad p_A(0) = 1 \\ p_I'(t) &= -(\nu+\rho)p_I(t) + \sigma p_A(t), \qquad p_I(0) = 0 \end{aligned}$$

The general solution is

$$\begin{aligned} p_A(t) &= A_1 e^{-k_1 t} + B_1 e^{-k_2 t} \\ p_I(t) &= A_2 e^{-k_1 t} + B_2 e^{-k_2 t} \end{aligned}$$

where k_1 and k_2 are the solutions to the equation

$$(-(\sigma+\mu) - k)(-(\nu+\rho) - k) - \rho\sigma = 0$$

After getting k_1 and k_2, we can solve using the initial conditions

$$\begin{aligned} A_1 + B_1 &= 1 \\ -k_1 A_1 - k_2 B_1 &= -(\sigma+\mu) \end{aligned}$$

Solving the two equations we obtain A_1 and B_1. The same for A_2 and B_2.

For backward equations we denote $p_H^B(t) = P(X(u) = S|X(t) = H)$ and $p_S^B(t) = P(X(u) = S|X(t) = S)$ for $t < u$. If we want to find $P(X(u) = A|X(t) = A)$ and $P(X(u) = A|X(t) = I)$ we solve

$$\begin{aligned} {p'}_A^B(t) &= (\sigma+\mu)p_A^B(t) - \rho p_I^B(t), \qquad p_A^B(u) = 1 \\ {p'}_I^B(t) &= (\rho+\nu)p_I^B(t) - \sigma p_A^B(t), \qquad p_I^B(u) = 0 \end{aligned}$$

The characteristic equation is the same, so we will get the same answer as forward.