ST305 - Actuarial Mathematics: Life

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1 Introduction

1.1 Actuarial Notations

Consider one individual. Define the following:

$$\begin{array}{rcl} T_x & : & \text{remaining lifetime of a life aged } x \\ \mu_{x+t} & : & \text{force of mortality} \\ {}_tp_x & : & \text{survival probability} \\ {}_tq_x & : & \text{probability of death within time } [0,t] \\ f(t) & : & \text{density of } T_x \end{array}$$

We have the following expressions:

$$\begin{aligned} {}_tp_x &= e^{-\int_0^t \mu_{x+s} ds} \\ {}_tq_x &= 1 - {}_tp_x = P(T_x \le t) \\ f(t) &= \mu_{x+t} e^{-\int_0^t \mu_{x+s} ds} = \mu_{x+t} {}_tp_x \\ E(T_x) &= \int_0^\infty t f(t) dt = \int_0^\infty t {}_tp_x \mu_{x+t} dt \\ &= \int_0^\infty P(T_x > t) dt = \int_0^\infty {}_tp_x dt \end{aligned}$$

1.2 Some insurance contracts

1.2.1 Actuarial principle - Expected present value

A payment of 1 at time t is worth $e^{-rt} = v^t$.
A payment of 1 made at time T_x is worth $e^{-rT_x} = v^{T_x}$ now. (A random quantity)
The value of a contract is the expected present value of its payments, in this case $E(e^{-rT_x})$. Below are some examples of insurance contracts and their expected present value:

1. **Whole life assurance**
   A payment of 1 made at time T_x. Payment is worth e^{-rT_x} now.

$$E[e^{-rT_x}] = \int_0^\infty e^{-rt} {}_tp_x \mu_{x+t} dt = \bar{A}_x$$

2. **Pure endowment**
   A payment of 1 is made at time n if life is alive. Payment is worth $e^{-rn}\mathbf{1}_{\{T_x>n\}}$ now.

$$\begin{aligned} E\left[e^{-rn}\mathbf{1}_{\{T_x>n\}}\right] &= e^{-rn}P(T_x>n) \\ &= e^{-rn}{}_np_x \\ &= e^{-rn}e^{-\int_0^n \mu_{x+t}dt} \\ &= e^{-\int_0^n (r+\mu_{x+t})dt} = A_{x:\overline{n}|}^{\;\;\,1} = {}_nE_x \end{aligned}$$

3. **Temporary Life Assurance**
   A payment of 1 immediately on death if that occurs before time n. Present value is $e^{-rT_x}\mathbf{1}_{\{T_x\le n\}}$.

$$\begin{aligned} E\left[e^{-rT_x}\mathbf{1}_{\{T_x\le n\}}\right] &= \int_0^n e^{-rt}f(t)dt = \int_0^n e^{-rt}\mu_{x+t}{}_tp_x dt \\ &= \int_0^n e^{-rt}\mu_{x+t}e^{-\int_0^t \mu_{x+s}ds}dt = \bar{A}^1_{x:\overline{n}|} \end{aligned}$$

4. **Endowment Assurance**
   A payment of 1 either immediately upon death or at time n, whichever comes first. Present value is $e^{-rn}\mathbf{1}_{\{T_x>n\}} + e^{-rT_x}\mathbf{1}_{\{T_x\le n\}} = e^{-r(T_x\wedge n)}$.

$$\begin{aligned} E\left[e^{-r(T_x\wedge n)}\right] &= A_{x:\overline{n}|}^{\;\;\,1} + \bar{A}^1_{x:\overline{n}|} \\ &= \bar{A}_{x:\overline{n}|} \end{aligned}$$

1.2.2 Variances

For the above examples, we have

1. **Whole life assurance**

$$\begin{aligned} \text{Present value} &= e^{-rT_x} \\ 2^{nd}\text{ moment} &= E\left[e^{-2rT_x}\right] \\ &= \bar{A}_x \text{ at force of interest } 2r \\ \text{Variance} &= \bar{A}_x \text{ at } 2r - \left(\bar{A}_x\right)^2 \text{ at } r \end{aligned}$$

2. **Pure endowment**

$$\begin{aligned} \text{Present value} &= e^{-rn}\mathbf{1}_{\{T_x>n\}} \\ 2^{nd}\text{ moment} &= E\left[e^{-2rn}\mathbf{1}_{\{T_x>n\}}\right] \\ &= e^{-2rn}{}_np_x \\ &= A_{x:\overline{n}|}^{\;\;\,1} \text{ at } 2r \\ \text{Variance} &= A_{x:\overline{n}|}^{\;\;\,1} \text{ at } 2r - \left(A_{x:\overline{n}|}^{\;\;\,1}\right)^2 \text{ at } r \end{aligned}$$

3. **Temporary Life Assurance**

$$\begin{aligned} \text{Present value:} &= e^{-rT_x}\mathbf{1}_{\{T_x\le n\}} \\ 2^{nd}\text{ moment:} &= E\left[e^{-2rT_x}\mathbf{1}_{\{T_x\le n\}}\right] \\ &= \bar{A}^1_{x:\overline{n}|} \text{ at } 2r \\ \text{Variance} &= \bar{A}^1_{x:\overline{n}|} \text{ at 2r} - \left(\bar{A}^1_{x:\overline{n}|}\right)^2 \text{ at r} \end{aligned}$$

4. **Endowment Assurance**

$$\begin{aligned}
\text{Present value} &= e^{-r(T_x \wedge n)} \\
2^{nd} \text{ moment} &= E\left[e^{-2r(T_x \wedge n)}\right] \\
&= \bar{A}_{x:\overline{n}|} \text{ at } 2r \\
\text{Variance} &= \bar{A}_{x:\overline{n}|} \text{ at } 2r - \left(\bar{A}_{x:\overline{n}|}\right)^2 \text{ at } r
\end{aligned}$$

1.3 Annuities

Annuity certain: $a_{\overline{n}|}$, $\ddot{a}_{\overline{n}|}$, $\bar{a}_{\overline{n}|}$

1. **Whole Life annuity (Continuous):**
   1 per annum payable continuously till death.

$$\begin{aligned}
\text{Present value} &= \bar{a}_{\overline{T_x}|} = \frac{1 - e^{-rT_x}}{r} \\
\text{Expected value} &= \bar{a}_x = E\left[\bar{a}_{\overline{T_x}|}\right] = \frac{1 - \bar{A}_x}{r} \Rightarrow \bar{A}_x + r\bar{a}_x = 1 \\
\text{Variance} &= Var\left(\frac{1 - e^{-rT_x}}{r}\right) \\
&= \frac{1}{r^2} Var(e^{-rT_x}) \\
&= \frac{1}{r^2}\left(\bar{A}_x \text{ at } 2r - \left(\bar{A}_x\right)^2 \text{ at } r\right)
\end{aligned}$$

   Alternatively, we also have

$$\begin{aligned}
\bar{a}_{\overline{T_x}|} &= \int_0^\infty \mathbf{1}_{\{T_x > t\}} e^{-rt} dt \\
\bar{a}_x &= E\left[\int_0^\infty \mathbf{1}_{\{T_x > t\}} e^{-rt} dt\right] \\
&= \int_0^\infty P(T_x > t) e^{-rt} dt \\
&= \int_0^\infty {}_t p_x e^{-rt} dt
\end{aligned}$$

2. **Whole Life annuity (Discrete):**

$$\begin{aligned}
\ddot{a}_{\overline{T_x}|} &= \sum_{j=0}^{\infty} e^{-rj} \mathbf{1}_{\{T_x > j\}} \\
\ddot{a}_x &= \sum_{j=0}^{\infty} e^{-rj} {}_j p_x \\
a_x &= \sum_{j=1}^{\infty} e^{-rj} {}_j p_x \\
\ddot{a}_x &\approx \bar{a}_x + \frac{1}{2} \text{ for whole life annuities}
\end{aligned}$$

3. **Temporary annuity (Continuous):**
   1 per annum payable continuously till death or time n, whichever comes first.

$$\begin{aligned}
\text{Present value} &= \bar{a}_{\overline{T_x \wedge n}|} = \frac{1 - e^{-r(T_x \wedge n)}}{r} \\
\text{Expected value} &= \bar{a}_{x:\overline{n}|} = \frac{1 - \bar{A}_{x:\overline{n}|}}{r} \Rightarrow \bar{A}_{x:\overline{n}|} + r\bar{a}_{x:\overline{n}|} = 1 \\
\text{Variance} &= Var\left(\frac{1 - e^{-r(T_x \wedge n)}}{r}\right) \\
&= \frac{1}{r^2} Var\left(e^{-r(T_x \wedge n)}\right) \\
&= \frac{1}{r^2}\left(\bar{A}_{x:\overline{n}|} \text{ at 2r} - \left(\bar{A}_{x:\overline{n}|}\right)^2 \text{ at } r\right)
\end{aligned}$$

   We also have

$$\begin{aligned}
\bar{a}_{\overline{T_x \wedge n}|} &= \int_0^n \mathbf{1}_{\{T_x > t\}} e^{-rt} dt \\
\bar{a}_{x:\overline{n}|} &= \int_0^n {}_t p_x e^{-rt} dt
\end{aligned}$$

4. **Temporary annuity (Discrete):**

$$\begin{aligned}
\ddot{a}_{x:\overline{n}|} &= \sum_{j=0}^{n-1} e^{-rj} {}_j p_x \\
a_{x:\overline{n}|} &= \sum_{j=1}^{n} e^{-rj} {}_j p_x
\end{aligned}$$

1.4 Principle of Equivalence

Expected P.V. of premiums = EPV(benefits) + EPV(expenses)
Ignoring expenses for the time being, EPV(premiums) = EPV(benefits)

Example: Suppose we have a whole life assurance financed by a life annuity of P per annum, then

$$\begin{aligned}
P\bar{a}_x &= \bar{A}_x \\
P &= \frac{\bar{A}_x}{\bar{a}_x}
\end{aligned}$$

If the premium is only payable till time n, then

$$\begin{aligned}
P\bar{a}_{x:\overline{n}|} &= \bar{A}_x \\
P &= \frac{\bar{A}_x}{\bar{a}_{x:\overline{n}|}}
\end{aligned}$$

Example: For a temporary life assurance, we have

$$\begin{aligned}
P\bar{a}_{x:\overline{n}|} &= \bar{A}^1_{x:\overline{n}|} \\
P &= \frac{\bar{A}^1_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}}
\end{aligned}$$

If premium is payable annually in advance,

$$P = \frac{\bar{A}^1_{x:\overline{n|}}}{\ddot{a}_{x:\overline{n|}}}$$

Example: Deferred annuity (pensions):
A whole life (or temporary) annuity commencing at a fixed time m provided the policyholder is alive then.

$$\begin{aligned} \text{Expected present value} = {}_{m|}\bar{a}_{x:\overline{n|}} &= e^{-rm}{}_m p_x \bar{a}_{x+m:\overline{n|}} \\ &= \bar{a}_{x:\overline{m+n|}} - \bar{a}_{x:\overline{m|}} \\ &= \bar{a}_x - \bar{a}_{x:\overline{m|}} \text{ (for } n = \infty) \end{aligned}$$

For a simple pension contract where premium is paid till time m and life annuity starts thereafter, we have

$$P = \frac{{}_{m|}\bar{a}_x}{\bar{a}_{x:\overline{m|}}}$$

2 Reserves

Money set aside to finance the rest of the contract. Two kinds of reserves:

1. **Prospective Reserves**: Value at time t of future liability minus future income = EPV(future benefits) - EPV(future premiums)

2. **Retrospective Reserves**: Net accumulation of money already received.
   If interest rates remain the same, prospective and retrospective reserves should equal.

Example: Consider a pure endowment, financed by a single premium P payable in advance.

$$P = v^n {}_n p_x = A_{x:\overline{n|}}^{\;\;1} = {}_nE_x$$

At time t,

$$\begin{aligned} \text{Prospective reserve, } V_t &= v^{n-t}{}_{n-t}p_{x+t} \\ \text{Retrospective reserve, } V_t^{(R)} &= Pe^{rt} \times \frac{1}{{}_tp_x} \\ &= Pe^{\int_0^t (r+\mu_{x+s})ds} \\ &= e^{-\int_0^n (r+\mu_{x+s})ds} e^{\int_0^t (r+\mu_{x+s})ds} \\ &= e^{-\int_t^n (r+\mu_{x+s})ds} \\ &= e^{-\int_0^{n-t} (r+\mu_{x+t+s})ds} \\ &= V_t \end{aligned}$$

The "accumulation" of P at time t is $Pe^{\int_0^t (r+\mu_{x+s})ds}$.

2.1 Thiele's Differential Equation

At any time, a premium π_t is being paid continuously and there is a death benefit b_t. Consider a small time interval $(t, t + dt)$.

Retrospective argument

$$\begin{aligned} V_{t+dt} &= V_t + r_t V_t dt + \pi_t dt + \mu_{x+t} dt(V_t - b_t) + o(dt) \\ V_{t+dt} - V_t &= (r + \mu_{x+t})V_t dt + \pi_t dt - \mu_{x+t} b_t dt + o(dt) \\ \frac{dV_t}{dt} &= (r + \mu_{x+t})V_t + \pi_t - \mu_{x+t} b_t \end{aligned}$$

Solve subject to $V_0 = 0$, since there is no premium upfront.

Prospective argument

$$\begin{aligned} V_t &= (1 - r_t dt)(1 - \mu_{x+t} dt)V_{t+dt} - \pi_t dt + \mu_{x+t} dt b_t + o(dt) \\ \frac{dV_t}{dt} &= (r_t + \mu_{x+t})V_t + \pi_t - \mu_{x+t} b_t \end{aligned}$$

subject to $V_{n-} = B$, the terminal benefit. In the case of whole life assurance, the condition is $\lim_{t\to\infty} V_t = 0$.

Solutions:
Retrospective

$$V_t = \int_0^t e^{\int_s^t (r_u + \mu_{x+u})du} \left(\pi_s - \mu_{x+s} b_s\right) ds$$

Prospective

$$V_t = \int_t^n e^{-\int_t^s (r + \mu_{x+u})du} \left(\mu_{x+s} b_s - \pi_s\right) ds + Be^{-\int_t^n (r_s + \mu_{x+s})ds}$$

Example: Pure endowment - single premium

Retrospective reserve:

$$\begin{aligned} V_{t+dt} &= V_t + V_t r dt + V_t \mu_{x+t} dt \\ \frac{V_{t+dt} - V_t}{dt} &= \frac{V_t r dt + V_t \mu_{x+t} dt}{dt} + \frac{o(dt)}{dt} \end{aligned}$$

As $dt \to 0$, $\frac{dV_t}{dt} = (r + \mu_{x+t})V_t$, $V_0 = P$. Solving, $V_t = Pe^{\int_0^t (r+\mu_{x+s})ds}$.

Prospective reserve:

$$\begin{aligned} V_t &= (1 - rdt)(1 - \mu_{x+t} dt)V_{t+dt} \\ e^{-rdt} &\approx 1 - rdt + o(dt) \\ e^{-\int_t^{t+dt} \mu_{x+s} ds} &\approx 1 - \mu_{x+t} dt + o(dt) \\ \frac{V_t - V_{t+dt}}{dt} &= \frac{-(r + \mu_{x+t})dt}{dt} V_{t+dt} + \frac{o(dt)}{dt} \end{aligned}$$

Letting $dt \to 0$, $\frac{dV_t}{dt} = (r + \mu_{x+t})V_t$, $V_{n-} = 1$. Solving, $V_t = e^{-\int_0^{n-t} (r + \mu_{x+t+s})ds}$ = retrospective.

Example: Pure endowment with premium P payable continuously.

$$P = \frac{v^n {}_n p_x}{\bar{a}_{x:\overline{n}|}}$$

Retrospective reserve:

$$\begin{aligned}
V_{t+dt} &= V_t + V_t r dt + \mu_{x+t} V_t dt + P dt + o(dt) \\
\frac{V_{t+dt} - V_t}{dt} &= (r + \mu_{x+t}) V_t + P + \frac{o(dt)}{dt} \\
\frac{dV_t}{dt} &= (r + \mu_{x+t}) V_t + P
\end{aligned}$$

and $V_0 = 0$.

Prospective reserve:

$$\begin{aligned}
V_t &= (1 - r dt)(1 - \mu_{x+t} dt) V_{t+dt} - P dt \\
\frac{dV_t}{dt} &= (r + \mu_{x+t}) V_t + P
\end{aligned}$$

and $V_n = 1$. Solving,

$$\begin{aligned}
V_t &= e^{-r(n-t)} {}_{n-t}p_{x+t} - P \bar{a}_{x+t:\overline{n-t}|} \\
&= \text{EPV(benefits) - EPV(premiums)}
\end{aligned}$$

2.2 Stochastic process approach

I_t: Stochastic process.

$$I_t = \begin{cases} 1 & \text{if life is alive} \\ 0 & \text{if life is dead} \end{cases}$$

N_t: No of deaths up to time t.

$$\begin{aligned}
dN_t &= N_{t+dt} - N_t = \text{ no. of deaths in } [t, t+dt) \\
I_t &= 1 - \int_0^t dN_s
\end{aligned}$$

Present value at time t of future payment:
Consider time $t \leq s \leq n$, where b_s payable if death occurs, B payable at time n if alive, and premium π_s payable when alive.
Cashflow at time s (small interval): $b_s dN_s - \pi_s I_s ds$
Discount factor: $e^{-r(s-t)} = e^{-\int_t^s r_u du}$ (if constant i.r.)
So P.V. $= e^{-r(s-t)} [b_s dN_s - \pi_s I_s ds]$.

Prospective reserve:

$$W_t = \int_t^n e^{-r(s-t)} (b_s dN_s - \pi_s I_s ds) + e^{-r(n-t)} B I_n$$

W_t is a random variable, so we take expectation.

$$\begin{aligned}
V_t &= E(W_t | \mathcal{F}_t) \\
&= E\left[\int_t^n e^{-r(s-t)} b_s dN_s - \int_t^n e^{-r(s-t)} \pi_s I_s ds + e^{-r(n-t)} B I_n | \mathcal{F}_t\right] \\
&= \int_t^n e^{-r(s-t)} b_s E(dN_s | \mathcal{F}_t) - \int_t^n e^{-r(s-t)} \pi_s E(I_s | \mathcal{F}_t) ds + e^{-r(n-t)} E(I_n | \mathcal{F}_t) B
\end{aligned}$$

Since $E(dN_s|\mathcal{F}_t) = E(N_{s+ds} - N_s|\mathcal{F}_t) = {}_{s-t}p_{x+t}\mu_{x+s}ds$ and $E(I_s|\mathcal{F}_t) = {}_{s-t}p_{x+t}$, we have

$$E(W_t|\mathcal{F}_t) = \int_t^n e^{-r(s-t)} b_s {}_{s-t}p_{x+t}\mu_{x+s}ds - \int_t^n e^{-r(s-t)} \pi_s {}_{s-t}p_{x+t}ds + e^{-r(n-t)} {}_{n-t}p_{x+t}B$$

Retrospective reserve:

$$W_t^{(R)} = \int_0^t (\pi_s I_s - b_s dN_s)\, e^{r(t-s)} ds$$

for $0 \leq s < t$. Retrospective reserve is its expectation shared among the survivors

$$V_t^{(R)} = \frac{E(W_t^{(R)})}{{}_tp_x}$$

Example: Pure endowment - Single premium

$$\begin{aligned}\text{Prospective reserve at time t} &= e^{-r(n-t)} {}_{n-t}p_{x+t} \\ &= e^{-\int_t^n (r+\mu_{x+s})ds}\end{aligned}$$

$V_n = 1$, $V_0 = e^{-rn} {}_np_x$ the single premium. And the reserve increases with time.
Or solve:

$$\frac{dV_t}{dt} = (r + \mu_{x+t})V_t$$

subject to $V_n = 1$.

$$\begin{aligned}\text{Retrospective reserve at time t} &= e^{-rn} {}_np_x e^{\int_0^t (r+\mu_{x+s})ds} \\ &= \text{Prospective reserves}\end{aligned}$$

Or solve:

$$\frac{dV_t}{dt} = (r + \mu_{x+t})V_t$$

subject to $V_0 = e^{-rn} {}_np_x$.

Example: Temporary Assurance - Continuous premium payable for n years, 1 payable on death provided it is before time n
Solve:

$$\frac{dV_t}{dt} = (r + \mu_{x+t})V_t dt + \pi - \mu_{x+t}$$

subject to the conditions $V_n = 0$ for prospective reserve, and $V_0 = 0$ for retrospective. Solving this, we have

$$\begin{aligned} V_t &= \int_0^t (\pi - \mu_{x+s}) e^{\int_s^t (r+\mu_{x+u})du} ds \\ V_n &= \int_0^n (\pi - \mu_{x+s}) e^{\int_s^n (r+\mu_{x+u})du} ds = 0 \end{aligned}$$

If μ_x is an increasing function of x, we will have $\pi - \mu_{x+s} > 0$ for some $s \leq m$ and $\pi - \mu_{x+s} < 0$ for some $s > m$, so V_t is increasing for $t < m$ and decreasing for $t > m$. On the other hand, if μ_x is decreasing in x, we have negative reserves, which is impossible.

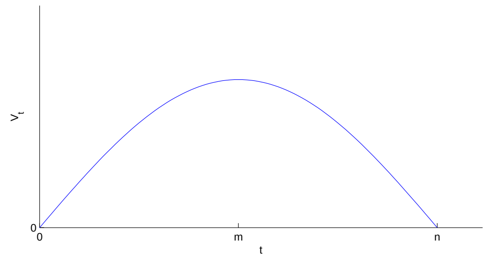

Figure 1: Positive reserves when μ_x is increasing

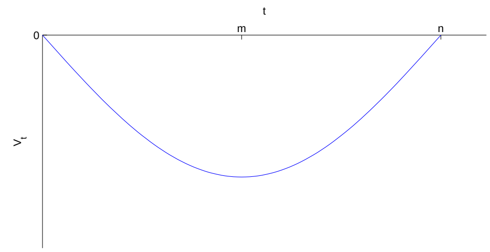

Figure 2: Negative reserves when μ_x is decreasing

Example: Variable premium - same as above but $\pi_t = \mu_{x+t}$
Thiele's equation:

$$\frac{dV_t}{dt} = (r + \mu_{x+t})V_t dt$$

subject to $V_0 = 0$. Solving this, we get $V_t = 0$ for all t. There are no reserves. Office just takes premiums and pays them out. This is impractical but very safe for the office.

Example: Endowment Assurance - 1 payable upon death or at time n, whichever comes first, continuous premium π

$$\begin{aligned} V_t &= \bar{A}_{x+t:\overline{n-t}|} - \pi \bar{a}_{x+t:\overline{n-t}|} \\ &= 1 - r\bar{a}_{x+t:\overline{n-t}|} - \frac{1 - r\bar{a}_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}} \bar{a}_{x+t:\overline{n-t}|} \\ &= 1 - \frac{\bar{a}_{x+t:\overline{n-t}|}}{\bar{a}_{x:\overline{n}|}} \end{aligned}$$

Assuming μ_x is increasing in x, we have

$$\begin{aligned} \bar{a}_{x+t:\overline{n-t}|} &= \int_t^n e^{-\int_t^s (r+\mu_{x+u})du} ds \\ &< \int_t^n e^{-\int_t^s (r+\mu_{x+t})du} \\ &= \frac{1}{r + \mu_{x+t}} - e^{-(r+\mu_{x+t})(n-t)} \\ &\leq \frac{1}{r + \mu_{x+t}} \end{aligned}$$

Now

$$\begin{aligned} \frac{d}{dt}\bar{a}_{x+t:\overline{n-t}|} &= (r + \mu_{x+t})\bar{a}_{x+t:\overline{n-t}|} - 1 \\ &< (r + \mu_{x+t})(\frac{1}{r + \mu_{x+t}}) - 1 = 0 \end{aligned}$$

Since $\bar{a}_{x+t:\overline{n-t}|}$ is decreasing, V_t is an increasing function of t.

Example: Temporary annuity - 1 payable till death or time n
Thiele's equation:

$$\frac{dV_t}{dt} = (r + \mu_{x+t})V_t + \pi - b\mu_{x+t}$$

We have $\pi = -1$, $b\mu_{x+t} = 0$, $V_n = 0$.

$$V_t = \bar{a}_{x+t:\overline{n-t}|}$$

3 Selection

(Read section 3.4 of Lifebook)
Forms of selection:

- Class selection (e.g. gender)
- Time selection (e.g. mortality improves over time)
- Self selection (e.g. antiselection)
- Temporary initial selection (e.g. time selection that wells off)

Select period: s
μ_{x+t} and ${}_tp_x$ does not lead to a proper model, because it does not take into account when the underwriting took place. So we write for a person aged x at the time of underwriting,

$$\begin{aligned} \mu_x(t) &= \mu_{x+t} \text{ for } t \geq s \\ &= \mu_{[x]+t} \end{aligned}$$

${}_\tau q_{[x]+t}$ is the probability of death within τ years for a life aged $x+t$, but went through underwriting at age x (t years ago).

$$\begin{aligned} {}_\tau q_{[x]+t} &= {}_\tau q_{x+t} \text{ for } t \geq s \\ &= P(T_x \leq t+\tau | T_x > t) \\ {}_\tau p_{[x]+t} &= P(T_x > t+\tau | T_x > t) \\ \mu_{[x]+t} &= \lim_{h \to 0} \frac{{}_h q_{[x]+t}}{h} \end{aligned}$$

Example: s=2 (selection up to time 2)

$$\begin{aligned} {}_2p_{[x]} &= {}_1p_{[x]} \times {}_1p_{[x]+1} \\ {}_3p_{[x]} &= {}_1p_{[x]} \times {}_1p_{[x]+1} \times {}_1p_{x+2} \\ \ddot{a}_{[x]} &= 1 + v \times {}_1p_{[x]} + v^2 \times {}_2p_{[x]} + v^3 \times {}_1p_{[x]} \times {}_1p_{[x]+1} \times {}_1p_{x+2} + ... \end{aligned}$$

4 Expenses

(Read Chapter 5 of Lifebook)
Types of expenses:

- Fixed: constant rate η
- Proportional to the sum assured: αb
- Proportional to premium: $\beta\pi$
- Proportional to reserve: γV_t

Expenses occur at rate $\eta + \alpha b + \beta\pi + \gamma V_t$. Thiele's equation becomes

$$\begin{aligned} V_{t+dt} &= V_t + rV_t dt + \mu_{x+t}(V_t - b)dt + \pi dt - (\eta + \alpha b + \beta\pi + \gamma V_t)dt \\ \frac{dV_t}{dt} &= (r + \mu_{x+t})V_t + \pi - \mu_{x+t}b - (\eta + \alpha b + \beta\pi + \gamma V_t) \\ &= (r - \gamma + \mu_{x+t})V_t + ((1-\beta)\pi - \eta - \alpha b) - \mu_{x+t}b \end{aligned}$$

Solve subject to $V_n = B$. We have the Thiele's equation with changed parameters:

$$V_t = \int_t^n e^{-\int_t^s (r-\gamma+\mu_{x+u})du} \left(\mu_{x+s}b - (1-\beta)\pi + \eta + \alpha b\right) ds + Be^{-\int_t^n (r-\gamma+\mu_{x+u})du}$$

We can calculate the premium π. Since $V_0 = 0$,

$$\begin{aligned} 0 &= \int_0^n e^{-\int_0^s (r-\gamma+\mu_{x+u})du} \left(\mu_{x+s}b - (1-\beta)\pi + \eta + \alpha b\right) ds + Be^{-\int_0^n (r-\gamma+\mu_{x+u})du} \\ &= \bar{A}^1_{x:\overline{n}|}b - (1-\beta)\pi\bar{a}_{x:\overline{n}|} + (\eta + \alpha b)\bar{a}_{x:\overline{n}|} + B_n p_x e^{-(r-\gamma)n} \\ (1-\beta)\pi\bar{a}_{x:\overline{n}|} &= \bar{A}^1_{x:\overline{n}|}b + (\eta + \alpha b)\bar{a}_{x:\overline{n}|} + B_n p_x e^{-(r-\gamma)n} \end{aligned}$$

where $\bar{a}_{x:\overline{n}|}$ and $\bar{A}^1_{x:\overline{n}|}$ are at force $r - \gamma$. We can get the premium from this equation.

5 Joint life

5.1 Notations

Multiple insurance
Notation:
One life: $\bar{A}_x$, $\bar{A}_{x:\overline{n}|}$, $\bar{A}^1_{x:\overline{n}|}$, $A_{x:\overset{1}{\overline{n}|}}$, $\bar{a}_x$, $\bar{a}_{x:\overline{n}|}$, $\bar{a}_{\overline{x:\overline{n}|}}$.

Now, suppose we have two lives aged x and y. Then

$$\begin{aligned} \bar{A}_{xy} &= \text{1 payable when } xy \text{ breaks on the } 1st \text{ death} \\ \bar{A}^1_{xy} &= \text{1 payable on the death of } x\text{, provided } y \text{ is alive} \\ \bar{A}_{xy} &= \bar{A}^1_{xy} + \bar{A}_{xy}{}^1 \\ \bar{A}_{\overline{xy}} &= \text{1 payable on the second death} \\ \bar{A}_{xy}{}^2 &= \bar{A}^1_{xy} \\ \bar{A}_{\overline{xy}}{}^2 &= \text{1 payable on the death of } y \text{ provided } x \text{ is already dead} \\ \bar{A}_{\overline{xy}:\overline{n}|}^{\;2} &= \text{Same as } \bar{A}_{\overline{xy}}^{\;2} \text{ before, but only if it happens within } n \text{ years} \\ \bar{A}_{\overline{xy}:\overline{n}|} &= \text{1 payable on the second death, provided it is before time } n \\ \bar{A}_{xy:\overset{1}{\overline{n}|}} &= \text{pure endowment (both alive at } n) \end{aligned}$$

Suppose lives: $1, 2, ..., r$, ages: $x_1, x_2, ..., x_r$.
Remaining lifetimes: $T_{x_1}, T_{x_2}, ..., T_{x_r}$ (Denote by: $T_1, T_2, ..., T_r$)
Joint status: $(x_1, x_2, ..., x_r)$

$$\begin{aligned} T &= \min(T_1, T_2, ..., T_r) \\ P(T > t) &= P(\min(T_1, T_2, ..., T_r) > t) \\ &= P(T_1 > t, T_2 > t, ..., T_r > t) \\ &= {}_tp_{x_1,...,x_r} \end{aligned}$$

If they are independent, then

$$
\begin{aligned}
P(T > t) &= P(T_1 > t)P(T_2 > t)...P(T_r > t) \\
&= {}_tp_{x_1} \times {}_tp_{x_2} \times ... \times {}_tp_{x_r} \\
{}_tp_{x_1...x_r} &= e^{-\int_0^t \mu_{x_1...x_r+s} ds} \\
&= e^{-\int_0^t \mu^{(1)}_{x_1+s} ds} e^{-\int_0^t \mu^{(2)}_{x_2+s} ds} ... e^{-\int_0^t \mu^{(r)}_{x_r+s} ds} \\
\mu_{x_1...x_r+t} &= \mu^{(1)}_{x_1+t} + \mu^{(2)}_{x_2+t} + ... + \mu^{(r)}_{x_r+t}
\end{aligned}
$$

GM mortality law:

$$
\begin{aligned}
\mu^{(i)}_x &= A_i + B_i e^{C_i x} \\
\mu^{(1)}_x + ... + \mu^{(r)}_x &= (A_1 + ... + A_r) + (B_1 + ... + B_r)e^{Cx} \text{ also GM law}
\end{aligned}
$$

5.2 Common joint life contracts

1. **Pure endowment -** 1 payable at time n if all are alive.

$$
\begin{aligned}
\text{PV} &= v^n \mathbf{1}_{\{T>n\}} \\
A_{x_1...x_r:\overset{1}{\overline{n|}}} &= {}_nE_{x_1...x_r} \\
&= v^n {}_np_{x_1...x_r}
\end{aligned}
$$

2. **Whole life assurance -** 1 payable at first death

$$
\begin{aligned}
\text{PV} &= v^T = v^{\min(T_1,...,T_r)} \\
\bar{A}_{x_1...x_r} &= \int_0^\infty e^{-rt} \mu_{x_1...x_r+t} {}_tp_{x_1...x_r} dt \\
&= \int_0^\infty e^{-rt} (\mu_{x_1+t} + \mu_{x_2+t} + ... + \mu_{x_r+t}) {}_tp_{x_1...x_r} dt
\end{aligned}
$$

3. **Temporary assurance -** 1 payable on first death provided it is before time n

$$
\bar{A}_{\overset{1}{\widetilde{x_1...x_r}}:\overline{n|}} = \int_0^n e^{-rt} (\mu_{x_1+t} + \mu_{x_2+t} + ... + \mu_{x_r+t}) {}_tp_{x_1...x_r} dt
$$

4. **Endowment assurance -** 1 payable on first death or at time n, whichever comes first

$$
\bar{A}_{x_1...x_r:\overline{n|}} = \bar{A}_{\overset{1}{\widetilde{x_1...x_r}}:\overline{n|}} + A_{x_1...x_r:\overset{1}{\overline{n|}}}
$$

Example: 1 payable on death of x_1 if everybody else is alive

$$
\bar{A}^{\,1}_{x_1 x_2 ... x_r} = \int_0^\infty e^{-rt} \mu_{x_1+t} {}_tp_{x_1...x_r} dt
$$

Note that $\bar{A}_{x_1...x_r} = \bar{A}^{\,1}_{x_1...x_r} + ... + \bar{A}_{x_1...\overset{1}{x_r}}$.

Example: 1 payable on death of x_1 if everybody else is alive and it happens within n years

$$
\bar{A}^{\,1}_{x_1...x_r:\overline{n|}} = \int_0^n e^{-rt} \mu_{x_1+t} {}_tp_{x_1...x_r} dt
$$

Note: $\bar{A}_{x_1...x_r:\overline{n}|} = \bar{A}^{\ \ 1}_{x_1...x_r:\overline{n}|} + \bar{A}_{x_1...x_r:\overline{n}|}^{\ \ \ \ \ \ \ \ \ 1}$.

Example: Annuity payable if everybody is alive and up to time n (n can be ∞)

$$\bar{a}_{x_1...x_r:\overline{n}|} = \int_0^n e^{-rt}\, {}_tp_{x_1...x_r} dt$$

5.3 Last survivor status

More complicated case: we are interested in the time of the last death.

$$\begin{aligned}
T &= \max(T_1, T_2, ..., T_r) \\
P(T \le t) &= P(\max(T_1, ..., T_r) \le t) \\
&= P(T_1 \le t, T_2 \le t, ..., T_r \le t) \\
&= P(T_1 \le t)P(T_2 \le t)...P(T_r \le t) \text{ if independent} \\
{}_tq_{x_1...x_r} &= {}_tq_{x_1} \times {}_tq_{x_2} \times ... \times {}_tq_{x_r} \\
&= (1 - {}_tp_{x_1})(1 - {}_tp_{x_2})...(1 - {}_tp_{x_r}) \\
&= 1 - {}_tp_{\overline{x_1...x_r}}
\end{aligned}$$

where ${}_tp_{\overline{x_1...x_r}}$ = prob. of at least one surviving.

Example: Annuity payable up to the last death

$$\begin{aligned}
\bar{a}_{\overline{x_1...x_r}} &= \int_0^\infty e^{-rt}\, {}_tp_{\overline{x_1...x_r}} dt \\
&= \int_0^\infty e^{-rt} \left(1 - (1 - {}_tp_{x_1})(1 - {}_tp_{x_2})...(1 - {}_tp_{x_r})\right) dt
\end{aligned}$$

Example: Pure endowment - 1 payable at time n if at least one life is alive

$$\begin{aligned}
\text{PV} &= v^n \mathbf{1}_{\{\max(T_1...T_r)>n\}} = v^n[1 - \mathbf{1}_{\{\max(T_1...T_r)\le n\}}] \\
\text{Expected value} &= v^n[1 - {}_tq_{x_1} \times {}_tq_{x_2} \times ... \times {}_tq_{x_r}] \\
\text{For 2 lives,} & \quad v^n[1 - {}_tq_x \times {}_tq_y]
\end{aligned}$$

Example: (Not typical) 2 lives, x and y. 1 is payable on the second death of x and y.

$$\begin{aligned}
\bar{A}_{\overline{xy}} &= E[e^{-r(T_x \vee T_y)}] \\
&= E[e^{-r(T_x \vee T_y)} \mathbf{1}_{\{T_x > T_y\}} + e^{-r(T_x \vee T_y)} \mathbf{1}_{\{T_y > T_x\}}] \\
&= E[e^{-rT_x} \mathbf{1}_{\{T_x > T_y\}}] + E[e^{-rT_y} \mathbf{1}_{\{T_y > T_x\}}] \\
&= \int_0^\infty e^{-rt}\, {}_tp_x \mu_{x+t}(1 - {}_tp_y) dt + \int_0^\infty e^{-rt}\, {}_tp_y \mu_{y+t}(1 - {}_tp_x) dt
\end{aligned}$$

Alternatively, we also have

$$\begin{aligned}
\bar{A}_{\overline{xy}} &= \bar{A}_x + \bar{A}_y - \bar{A}_{xy} \\
&= \int_0^\infty e^{-rt} \mu_{x+t}\, {}_tp_x dt + \int_0^\infty e^{-rt} \mu_{y+t}\, {}_tp_y dt - \int_0^\infty e^{-rt}\, {}_tp_x\, {}_tp_y (\mu_{x+t} + \mu_{y+t}) dt
\end{aligned}$$

Suppose we have continuous premiums as long as both are alive, then

$$\pi = \frac{\bar{A}_{\overline{xy}}}{\bar{a}_{xy}}$$

If payable as long as somebody is alive,

$$\pi = \frac{\bar{A}_{\overline{xy}}}{\bar{a}_{\overline{xy}}}$$

and we have the following relationships

$$\begin{aligned}\bar{a}_x + \bar{a}_y &= \bar{a}_{\overline{xy}} + \bar{a}_{xy} \\ \bar{a}_{\overline{xy}} &= \int_0^\infty e^{-rt}(1 - {}_tq_x {}_tq_y)dt\end{aligned}$$

If the contract finishes at time n,

$$\pi = \frac{\bar{A}_{\overline{xy}:\overline{n}|}}{\bar{a}_{xy:\overline{n}|}}$$

Example: (Most common contract) 1 payable on the first death provided it happens before time n. Premium payable as long as they are both alive up to time n.

$$\pi = \frac{\bar{A}^{\;1}_{\widetilde{xy}:\overline{n}|}}{\bar{a}_{xy:\overline{n}|}}$$

5.4 Some conditional probabilities

1. In the previous contract, the amount was paid before time n. What is the probability that y got the money (x died first)?

$$\begin{aligned}P(T_y > T_x | \min(T_x, T_y) \leq n) &= \frac{P(T_y > T_x, \min(T_x, T_y) \leq n)}{P(\min(T_x, T_y) \leq n)} \\ &= \frac{P(T_y > T_x, T_x \leq n)}{1 - {}_np_x {}_np_y} \\ &= \frac{1}{1 - {}_np_x {}_np_y} E\left[\int_0^n \mathbf{1}_{\{T_y > t\}} {}_tp_x \mu_{x+t} dt\right] \\ &= \frac{1}{1 - {}_np_x {}_np_y} \int_0^n {}_tp_x {}_tp_y \mu_{x+t} dt\end{aligned}$$

2. Somebody died at time t (exactly), and (s)he has died first. What is the probability that it was x that died?

$$\begin{aligned}\lim_{dt\to 0} \frac{P(T_x \in [t, t+dt), T_y > t)}{P(T_x \wedge T_y \in [t, t+dt))} &= \lim_{dt\to 0} \frac{P(T_x \in [t, t+dt))P(T_y > t)}{P(T_x \in [t, t+dt))P(T_y > t) + P(T_y \in [t, t+dt))P(T_x > t)} \\ &= \frac{{}_tp_x \mu_{x+t} dt {}_tp_y}{{}_tp_x \mu_{x+t} dt {}_tp_y + {}_tp_y \mu_{y+t} {}_tp_x dt} \\ &= \frac{\mu_{x+t}}{\mu_{x+t} + \mu_{y+t}}\end{aligned}$$

This extends to m lives: If we know the time of the first death is t, the probability that it is life $j, j = 1, ..., m$ is $\frac{\mu_{x_j+t}}{\sum_{i=1}^m \mu_{x_i+t}}$.

5.5 More examples

Example: Deferred annuities - On the death of x, y gets a life annuity.

$$\begin{aligned}
\text{PV} &= {}_{T_x|}\bar{a}_{\overline{T_y}|}\mathbf{1}_{\{T_y>T_x\}} \\
\text{Expected value} &= {}_{x|}\bar{a}_y \\
&= E\left[\int_{T_x}^{T_y} e^{-rt}dt\mathbf{1}_{\{T_y>T_x\}}\right] \\
&= \left[\int_0^\infty \mathbf{1}_{\{y \text{ is alive and } x \text{ is dead at time } t\}}e^{-rt}dt\right] \\
&= \int_0^\infty {}_tp_y(1-{}_tp_x)e^{-rt}dt \\
&= \bar{a}_y - \bar{a}_{xy}
\end{aligned}$$

Example: On the first death, the survivor gets a life annuity

$$\begin{aligned}
{}_{x|}\bar{a}_y + {}_{y|}\bar{a}_x &= \int_0^\infty e^{-rt}\mathbf{1}_{\{\text{exactly one is alive}\}}dt \\
&= \int_0^\infty e^{-rt}[{}_tp_y(1-{}_tp_x)+{}_tp_x(1-{}_tp_y)]dt \\
&= \int_0^\infty e^{-rt}[1-{}_tp_{x\,t}p_y - {}_tq_{x\,t}q_y]dt
\end{aligned}$$

Example: Two lives, 1 payable on second death provided this occurs before time n. There will be 3 (4) reserves.

- $V_0(t)=0$: Reserve if both dead
- $V_x(t)$: Reserve to be set up if only x is alive
- $V_y(t)$: Reserve if only y is alive
- $V_{xy}(t)$: Reserve if both are alive

Assume premium is payable continuously as long as both are alive (till time n).

$$\begin{aligned}
V_{xy}(t) &= E[\text{PV of future payments - benefits}|\text{both are alive}] \\
&= \bar{A}_{\overline{x+t,y+t}:\overline{n-t}|} - \pi\bar{a}_{x+t,y+t:\overline{n-t}|} \\
V_x(t) &= E[\text{PV of future payments - benefits}|\text{only } x \text{ is alive}] \\
&= \bar{A}_{x+t:\overline{n-t}|} \\
V_y(t) &= E[\text{PV of future payments - benefits}|\text{only } y \text{ is alive}] \\
&= \bar{A}_{y+t:\overline{n-t}|} \\
\pi &= \frac{\bar{A}_{\overline{xy}:\overline{n}|}}{\bar{a}_{xy:\overline{n}|}}
\end{aligned}$$

$V_{xy}(t)$ is thus

$$\begin{aligned}
V_{xy}(t) &= \int_t^n e^{-r(s-t)}{}_{s-t}p_{x+t}\mu_{x+s}(1-{}_{s-t}p_{y+t})ds + \int_t^n e^{-r(s-t)}{}_{s-t}p_{y+t}\mu_{y+t\,s-t}q_{x+t}ds \\
&\quad -\pi\int_t^n e^{-r(s-t)}{}_{s-t}p_{x+t\,s-t}p_{y+t}ds \\
V_{xy}(n) &= 0
\end{aligned}$$

and we can compute the derivative

$$V'_{xy}(t)|_{t=n} = 0 + 0 + \pi > 0$$

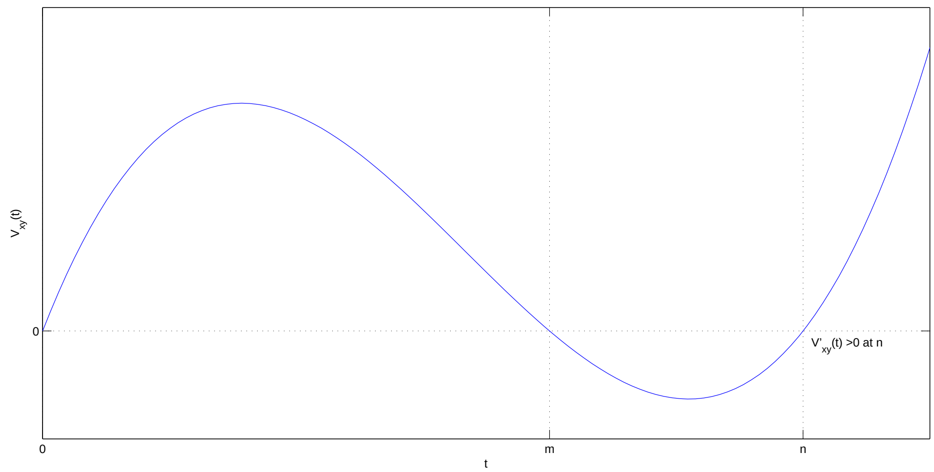

Figure 3: Graph of $V_{xy}(t)$, reserve when both are alive

Hence, $V_{xy}(t)$ is increasing for values just before n. This implies that $V_{xy}(t) < 0$ for some $m < n$. But we are not allowed to have negative reserves, so when this happens, the usual remedy is to have the premium payable up to a time $n_1 < n$ instead of being payable till time n. In which case we will have

$$V'_{xy}(t) = rV_{xy}(t) + \pi \mathbf{1}_{\{t<m\}} - \mu_{x+t}(V_y(t) - V_{xy}(t)) - \mu_{y+t}(V_x(t) - V_{xy}(t))$$

If $n = \infty$, we do not necessarily have a problem.

6 Markov Chains

Let $X(t)$, the state at time t, be a stochastic process that takes values from a finite set $\{1, 2, ..., K\}$, $t \geq 0$. $X(t)$ is a Markov process, $t_1 < t_2 < ... < t_n < t_{n+1}$. So we have

$$\begin{aligned} P(X(t_{n+1}) = j | X(t_n) = j_n, X(t_{n-1}) = j_{n-1}, ..., X(t_0) = j_0) &= P(X(t_{n+1}) = j | X(t_n) = j_n) \\ \mathcal{F}_t &= \sigma\{X(s) : 0 \leq s \leq t\} \\ P(X(t_{n+1}) = j | \mathcal{F}_{t_n}) &= P(X(t_{n+1}) = j | X_{t_n}) \end{aligned}$$

Also, conditionally on the present, the past and the future are independent:

$$P(X(t_3) = j_3, X(t_1) = j_1 | X(t_2) = j_2) = P(X(t_3) = j_3 | X(t_2) = j_2) P(X(t_1) = j_1 | X(t_2) = j_2)$$

Definitions:

- For small h,
$$P(X(t+h) = j | X(t) = i) = \mu_{ij}(t)h + o(h), \qquad i \neq j$$

- P(Two or more transitions in $[t, t+h)) = o(h)$. So

$$\begin{aligned} P(X(t+h) = i | X(t) = i) &= 1 - \sum_{j \neq i}^{K} \mu_{ij}(t)h + o(h) \\ &= 1 - \mu_{i.}(t)h + o(h) \end{aligned}$$

- $P(X(u) = j | X(t) = i) = p_{ij}(t, u)$, for $t < u$.

Aside:

$$\mu_{i.}(t) = \sum_{k=1, k \neq i}^{K} \mu_{ik}(t), \qquad \mu_{.j}(t) = \sum_{k=1, k \neq j}^{K} \mu_{kj}(t)$$

6.1 Kolmogorov Backward Equations

Consider what happens in interval $[t, t+h)$. For $t < u$,

$$\begin{aligned} p_{ij}(t, u) &= P(X(u) = j | X(t) = i) \\ &= \sum_{k=1}^{K} P(X(u) = j | X(t+h) = k) P(X(t+h) = k | X(t) = i) \\ &= \sum_{k=1, k \neq i}^{K} p_{kj}(t+h, u)[\mu_{ik}(t)h + o(h)] + p_{ij}(t+h, u)[1 - \mu_{i.}(t)h + o(h)] \end{aligned}$$

So

$$\frac{p_{ij}(t, u) - p_{ij}(t+h, u)}{h} = \frac{\sum_{k=1, k \neq i}^{K} p_{kj}(t+h, u)\mu_{ik}(t)h - \mu_{i.}(t)hp_{ij}(t+h, u) + o(h)}{h}$$

Let $h \to 0$,

$$\begin{aligned} -\frac{\delta p_{ij}(t, u)}{\delta t} &= \sum_{k=1, k \neq i}^{K} p_{kj}(t)\mu_{ik}(t) - p_{ij}(t)\mu_{i.}(t) \\ \frac{\delta p_{ij}(t, u)}{\delta t} &= p_{ij}(t)\mu_{i.}(t) - \sum_{k=1, k \neq i}^{K} p_{kj}(t)\mu_{ik}(t) \end{aligned}$$

The Kolmogorov equations for $i = 1, ..., K$ to be solved simultaneously subject to the conditions

$$\begin{aligned} p_{ij}(u, u) &= 0, \qquad \text{if } i \neq j \\ p_{jj}(u, u) &= 1 \end{aligned}$$

Backward equations: To find the probability that we are in a state j at the end given we are at various states (all possibilities) at the start.

6.2 Kolmogorov Forward Equations

Consider what happens in $[t, t+h)$. For $s < t$,

$$
\begin{aligned}
p_{ij}(s,t) &= P(X(t)=j|X(s)=i) \\
p_{ij}(s,t+h) &= \sum_{k=1}^{K} P(X(t+h)=j|X(t)=k)P(X(t)=k|X(s)=i) \\
&= \sum_{k=1,k\neq j}^{K} [\mu_{kj}(t)h+o(h)]p_{ik}(s,t) + (1-\mu_{.j}h+o(h))p_{ij}(s,t)
\end{aligned}
$$

So

$$
\frac{p_{ij}(s,t+h)-p_{ij}(s,t)}{h} = \frac{\sum_{k=1,k\neq j}^{K} \mu_{kj}(t)p_{ik}(s,t) - \mu_{.j}(t)hp_{ij}(s,t)+o(h)}{h}
$$

Letting $h \to 0$,

$$
\frac{\delta p_{ij}(s,t)}{\delta t} = \sum_{k=1,k\neq j}^{K} \mu_{kj}(t)p_{ik}(s,t) - \mu_{.j}p_{ij}(s,t)
$$

The Kolmogorov forward equations are solved for all possible $j = 1, 2, ..., K$ simultaneously, subject to

$$
\begin{aligned}
p_{ij}(s,s) &= 0 \text{ for } j \neq i \\
p_{ii}(s,s) &= 1
\end{aligned}
$$

Forward equations: We know where we are at time s and we want the probabilities of all possible outcomes at a future time. Quite often, $s = 0$ and $p_{ij}(0,t) = p_{ij}(t)$.

6.3 Occupation probabilities

Another interesting probability is the "occupation" probability $p_{\overline{ii}}(s,t)$, the probability of staying in state i from time s to t, without having left state i.

$$
\begin{aligned}
p_{\overline{ii}}(s,t) &= P(X(v)=i, \forall v \in [s,t]|X(s)=i) \\
p_{ii}(s,t) &\geq p_{\overline{ii}}(s,t) \\
p_{\overline{ii}}(s,t+h) &= p_{\overline{ii}}(s,t)p_{\overline{ii}}(t,t+h) \\
&= p_{\overline{ii}}(s,t)p_{ii}(t,t+h)+o(h) \\
&= p_{\overline{ii}}(s,t)\left(1-\mu_{i.}(t)h\right)+o(h) \\
\frac{\delta p_{\overline{ii}}(s,t)}{\delta t} &= -\mu_{i.}(t)p_{\overline{ii}}(s,t) \\
p_{\overline{ii}}(s,s) &= 1
\end{aligned}
$$

So we have

$$
p_{\overline{ii}}(s,t) = e^{-\int_s^t \mu_{i.}(u)du}
$$

Example: Model of competing risks / Multiple decrement model

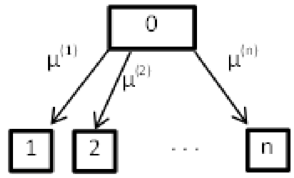

Figure 4: Model of competing risks / Multiple decrement model

Start at 0 and move to $1, ..., n$ and stay there. At time 0, we are in state 0.

$$\begin{aligned}
p_{ij}(s,t) &= 0 \text{ for } i \neq 0 \\
\mu^{(i)}(t) &= \mu^{(i)}_{x+t} \\
p_0(t) &= P(X(t)=0) \\
p_j(t) = p_{0j}(t) &= P(X(t)=j) \text{ for } j=0,1,2,...,n
\end{aligned}$$

To find $p_0(t)$, we use the forward equation

$$\begin{aligned}
p_0(t+h) &= p_0(t)\left(1-\sum_{j=1}^{n}\mu^{(j)}_{x+t}h\right)+o(h) \\
p_0(t+h)-p_0(t) &= -\sum_{j=1}^{n}\mu^{(j)}_{x+t}hp_0(t)+o(h) \\
\frac{p_0(t+h)-p_0(t)}{h} &= \frac{-\sum_{j=1}^{n}\mu^{(j)}_{x+t}hp_0(t)+o(h)}{h} \\
p_0'(t) &= -\mu_{x+t}p_0(t) \text{ (where } \mu=\sum_{j=1}^{n}\mu^{(j)})
\end{aligned}$$

subject to $p_0(0)=1$. Solving this, we get

$$\begin{aligned}
p_0(t) &= e^{-\int_0^t \mu_{x+s}ds} \\
&= e^{-\int_0^t (\mu^{(1)}_{x+s}+\mu^{(2)}_{x+s}+...+\mu^{(n)}_{x+s})ds} \\
&= e^{-\int_0^t \mu^{(1)}_{x+s}ds}e^{-\int_0^t \mu^{(2)}_{x+s}ds}...e^{-\int_0^t \mu^{(n)}_{x+s}ds}
\end{aligned}$$

For $j=1,2,...,n$, we have

$$\begin{aligned}
p_j(t+h) &= p_0(t)\mu^{(j)}_{x+t}h+p_j(t)+o(h) \\
\frac{p_j(t+h)-p_j(t)}{h} &= \frac{p_0(t)\mu^{(j)}_{x+t}h}{h}+\frac{o(h)}{h} \\
p_j'(t) &= p_0(t)\mu^{(j)}_{x+t}
\end{aligned}$$

Solving this subject to $p_j(0)=0$, we have

$$p_j(t)=\int_0^t p_0(s)\mu^{(j)}_{x+s}ds$$

Let T be the time we move out of state 0.

$$\begin{aligned} P(T > t) &= e^{-\int_0^t \mu_{x+s} ds} = p_0(t) \\ \text{Density of } T &= \mu_{x+t} p_0(t) = \sum_{i=1}^n \mu_{x+t}^{(i)} p_0(t) \\ &= \lim_{dt \to 0} \frac{P(T \in [t, t+dt))}{dt} \end{aligned}$$

We also have

$$\lim_{dt \to 0} \frac{P(T \in [t, t+dt), X(T) = j)}{dt} = \mu_{x+t}^{(j)} p_0(t)$$

So

$$P(X(T) = i | T = t) = \frac{\mu_{x+t}^{(i)}}{\mu_{x+t}}$$

Suppose there is benefit b_j, payable on transition to state j.

$$\text{EPV(benefits)} = \sum_{j=1}^n b_j \int_0^\infty e^{-rt} \mu_{x+t}^{(j)} p_0(t) dt$$

Example: James Bond insurance

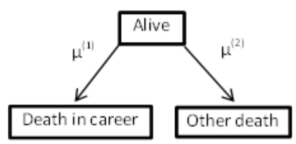

Figure 5: James Bond insurance

Benefit b for other death, $2b$ for death in service.

$$\begin{aligned} \text{EPV(benefits)} &= b \int_0^\infty e^{-rs} p_0(s) \mu_{x+s}^{(2)} ds + 2b \int_0^\infty e^{-rs} p_0(s) \mu_{x+s}^{(1)} ds \\ p_0(t) &= e^{-\int_0^t (\mu_{x+s}^{(1)} + \mu_{x+s}^{(2)}) ds} \\ p_j(t) &= \int_0^t p_0(s) \mu_{x+s}^{(j)} ds \end{aligned}$$

Example: Two lives x, y

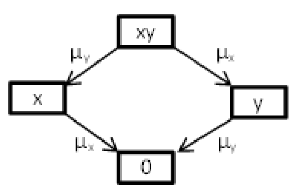

Figure 6: Two lives x, y

We use forward equations to find $p_{xy}(t)$.

$$\begin{aligned} p_{xy}(t+dt) &= p_{xy}(t)[1-(\mu_{x+t}+\mu_{y+t})dt]+o(dt) \\ \frac{dp_{xy}(t)}{dt} &= -(\mu_{x+t}+\mu_{y+t})p_{xy}(t) \\ p_{xy}(t) &= e^{-\int_0^t(\mu_{x+s}+\mu_{y+s})ds} = {}_tp_{xt}p_y \\ p_{xy}(0) &= 1 \end{aligned}$$

For $p_x(t)$, we have

$$\begin{aligned} p_x(t+dt) &= p_x(t)(1-\mu_{x+t}dt)+p_{xy}(t)\mu_{y+t}dt+o(dt) \\ \frac{p_x(t+dt)-p_x(t)}{dt} &= -\frac{p_x(t)\mu_{x+t}dt}{dt}+\frac{p_{xy}(t)\mu_{y+t}dt}{dt}+\frac{o(dt)}{dt} \\ p'_x(t) &= -\mu_{x+t}p_x(t)+\mu_{y+t}p_{xy}(t) \end{aligned}$$

We need to solve this subject to $p_x(0)=0$. After some calculations, we will obtain

$$\begin{aligned} p_x(t) &= e^{-\int_0^t \mu_{x+u}du}\int_0^t e^{\int_0^s \mu_{x+u}du}\mu_{y+s}p_{xy}(s)ds \\ &= {}_tp_{xt}q_y \end{aligned}$$

Similarly, $p_y(t) = {}_tp_{yt}q_x$. And $p_0(t)=1-p_{xy}(t)-p_x(t)-p_y(t)$. Alternatively,

$$\begin{aligned} p_0(t+dt) &= p_{xy}(t)o(dt)+p_x(t)\mu_{x+t}dt+p_y(t)\mu_{y+t}dt+p_0(t)+o(dt) \\ p'_0(t) &= \mu_{x+t}p_x(t)+\mu_{y+t}p_y(t) \end{aligned}$$

Solve subject to $p_0(0)=0$ to get

$$\begin{aligned} p_0(t) &= \int_0^t(\mu_{x+s}p_x(s)+\mu_{y+s}p_y(s))ds \\ &= \int_0^t {}_sp_x\mu_{x+s\,s}q_y ds+\int_0^t {}_sp_y\mu_{y+s\,s}q_y ds \end{aligned}$$

Example: Health-Sickness Model

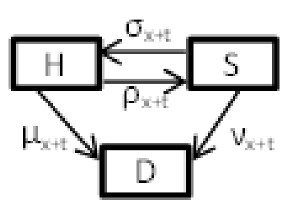

Figure 7: Health-Sickness Model

At time 0, we are in state H (healthy). $X(0)=H$ and $p_H(t)=P(X(t)=H|X(0)=H)$. (We use $x+t$ here, but in practice it can depend on t in other ways). The forward equations are:

$$\begin{aligned} p_H(t+dt) &= p_H(t)(1-\sigma_{x+t}dt-\mu_{x+t}dt)+p_S(t)\rho_{x+t}dt+o(dt) \\ p'_H(t) &= -(\sigma_{x+t}+\mu_{x+t})p_H(t)+p_S(t)\rho_{x+t} \end{aligned}$$

with $p_H(0) = 1$, and

$$p'_S(t) = -(\rho_{x+t} + \nu_{x+t})p_S(t) + p_H(t)\sigma_{x+t}$$

with $p_S(0) = 0$. Solve the two equations simultaneously. In special cases, e.g. if $\sigma_{x+t} = \sigma$, $\rho_{x+t} = \rho$, $\mu_{x+t} = \mu$, $\nu_{x+t} = \nu$, we can solve them explicitly.

We denote the backward probabilities (for $t < u$):

$$\begin{aligned}
p_H^B(t) &= P(X(u) = S|X(t) = H) \\
p_S^B(t) &= P(X(u) = S|X(t) = S) \\
p_D^B(t) &= P(X(u) = S|X(t) = D) = 0
\end{aligned}$$

The backward equations are:

$$\begin{aligned}
p_H^B(t) &= P(X(u) = S|X(t) = H) \\
&= P(X(u) = S|X(t+dt) = H)(1 - \sigma_{x+t}dt - \mu_{x+t}dt) \\
&\quad + P(X(u) = S|X(t+dt) = S)\sigma_{x+t}dt \\
&= p_H^B(t+dt)(1 - \sigma_{x+t}dt - \mu_{x+t}dt) + p_S^B(t+dt)\sigma_{x+t}dt + o(dt) \\
\frac{p_H^B(t) - p_H^B(t+dt)}{dt} &= -(\sigma_{x+t} + \mu_{x+t})p_H^B(t+dt) + p_S^B(t+dt)\sigma_{x+t} + \frac{o(dt)}{dt} \\
{p'}_H^B(t) &= (\sigma_{x+t} + \mu_{x+t})p_H^B(t) - \sigma_{x+t}p_S^B(t)
\end{aligned}$$

and $p_H^B(u) = 0$. Similarly, we have for $p_S^B(t)$

$${p'}_S^B(t) = (\rho_{x+t} + \nu_{x+t})p_S^B(t) - \rho_{x+t}p_H^B(t)$$

and $p_S^B(u) = 1$. Solve simultaneously. If we want to find $P(X(u) = H|X(t) = H)$ and $P(X(u) = H|X(t) = S)$ we can use the same equations but with terminal conditions $p_H^B(u) = 1$ and $p_S^B(u) = 0$.

Clearly, the value of an annuity payable as long as I am alive is $\int_0^\infty e^{-rt}(p_H(t) + p_S(t))dt$. The value of the sickness benefit, a continuous annuity payable while sick is $\int_0^\infty e^{-rt}p_S(t)dt$.

Criticisms of this model:

- Recovery does not depend on how long sickness has lasted.
- Recovery does not depend on how many times you have been sick.

6.4 Stochastic Process Approach

We define the following stochastic processes.

$$\begin{aligned}
X(t) &= \{1, 2, ..., K\} \\
I_j(t) &= \begin{cases} 1 & \text{if } X(t) = j \\ 0 & \text{if } X(t) \neq j \end{cases} \\
N_{jk}(t) &= \text{No. of transitions from state } j \text{ to state } k \text{ up to time } t \\
dN_{jk}(t) &= N_{jk}(t+dt) - N_{jk}(t) = \begin{cases} 1 & \text{if } X(t+dt) = k, X(t) = j \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

We have the following expectations.

$$\begin{aligned}
E(I_j(t)) &= P(X(t)=j) \\
E(dN_{jk}(t)) &= E(N_{jk}(t+dt)-N_{jk}(t)) \\
&= P(X(t)=j)P(X(t+dt)=k|X(t)=j) \\
&= P(X(t)=j)\mu_{jk}(t)dt+o(dt) \\
E(N_{jk}(t)) &= \int_0^t P(X(s)=j)\mu_{jk}(s)ds
\end{aligned}$$

Example: Alive-dead model

$$E(N_{AD}(t)) = \int_0^t {}_sp_x\mu_{x+s}ds = {}_tq_x$$

Example: States $\{1,...,K\}$. Suppose we have continuous payments with rate $b_j(t)$ made while at state j at time t. Discrete payment $B_{jk}(t)$ on transition from state j to state k at time t. Final payment at time n: A_j if then at state j. Continuous premium $\pi_j(t)$ in state j at time t.

$$\begin{aligned}
\text{PV of benefits} &= \int_0^n \left(\sum_{j=1}^K b_j(t)e^{-\int_0^t r(s)ds}I_j(t)\right)dt \\
&\quad+\sum_{j=1}^K\sum_{k=1,k\neq j}^K\left(\int_0^n B_{jk}(t)dN_{jk}(t)e^{-\int_0^t r(s)ds}\right) \\
&\quad+\sum_{j=1}^K A_jI_j(n)e^{-\int_0^n r(s)ds} \\
\text{PV of premiums} &= \sum_{j=1}^K\left(\int_0^n \pi_j(t)e^{-\int_0^t r(s)ds}I_j(t)dt\right)
\end{aligned}$$

Actuarial Principle: EPV(benefits)=EPV(premiums).

$$\begin{aligned}
\text{EPV(benefits)} &= \sum_{j=1}^K\left(\int_0^n b_j(t)e^{-\int_0^t r(s)ds}P(X(t)=j)dt\right) \\
&\quad+\sum_{j=1}^K\sum_{k=1,k\neq j}^K\left(\int_0^n B_{jk}(t)e^{-\int_0^t r(s)ds}P(X(t)=j)\mu_{jk}(t)dt\right) \\
&\quad+\sum_{j=1}^K A_jP(X(n)=j)e^{-\int_0^n r(s)ds} \\
\text{EPV(premiums)} &= \sum_{j=1}^K\left(\int_0^n \pi_j(t)e^{-\int_0^t r(s)ds}P(X(t)=j)dt\right)
\end{aligned}$$

6.5 Reserves in the multi-state model

$$\begin{aligned} W(t) &= \sum_{j=1}^{K}\int_t^n e^{-r(s-t)}\left(b_j(s)-\pi_j(s)\right)I_j(s)ds+\sum_{j=1}^{K}\sum_{k=1,k\neq j}^{K}\int_t^n e^{-r(s-t)}B_{jk}dN_{jk} \\ &+\sum_{j=1}^{K}A_jI_j(n)e^{-r(n-t)} \\ V_i(t) &= E(W(t)|\mathcal{F}_t)=E(W(t)|X(t)=i) \end{aligned}$$

Reserves at time u given we are in state i (for simplicity, $r(t)=r$):

$$\begin{aligned} V_i(u) &= \sum_{j=1}^{K}\int_u^n b_j(t)e^{-r(t-u)}P(X(t)=j|X(u)=i)dt \\ &+\sum_{j=1}^{K}\sum_{k=1,k\neq j}^{K}\int_u^n B_{jk}(t)P(X(t)=j|X(u)=i)\mu_{jk}(u)e^{-r(t-u)}dt \\ &+\sum_{j=1}^{K}A_jP(X(n)=j|X(u)=i)e^{-r(n-u)} \\ &-\sum_{j=1}^{K}\int_u^n \pi_j(t)e^{-r(t-u)}P(X(t)=j|X(u)=i)dt \end{aligned}$$

6.6 Thiele's equations for the multi-state model

$$\begin{aligned} V_i(t+dt) &= V_i(t)(1+rdt)-(b_i(t)-\pi_i(t))dt-\sum_{k=1,k\neq i}^{K}\mu_{ik}dtB_{ik} \\ &-\sum_{k=1,k\neq i}^{K}\mu_{ik}dtV_k(t)+\sum_{k=1,k\neq i}^{K}\mu_{ik}dtV_i(t)+o(dt) \\ \frac{V_i(t+dt)-V_i(t)}{dt} &= (r+\mu_{i.})V_i(t)-\sum_{k=1}^{K}\mu_{ik}(B_{ik}(t)+V_k(t))-b_i(t)+\pi_i(t) \\ \frac{dV_i(t)}{dt} &= (r+\mu_{i.})V_i(t)-\sum_{k=1,k\neq i}^{K}\mu_{ik}(B_{ik}(t)+V_k(t))-b_i(t)+\pi_i(t) \\ \frac{dV_i(t)}{dt} &= rV_i(t)-\sum_{k=1,k\neq i}^{K}\mu_{ik}B_{ik}(t)-b_i(t)+\pi_i(t)-\sum_{k=1,k\neq i}^{K}\mu_{ik}(t)(V_k(t)-V_i(t)) \end{aligned}$$

for $i=1,...,K$ such that $V_i(n-)=A_i$. If there is no premium upfront, $V_i(0)=0$, $\forall i$.

Example: 2 lives aged x, y. Premium payable in advance. Benefit of 1 payable on second death.

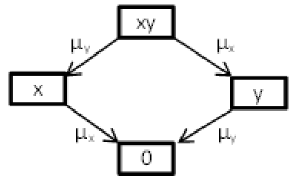

Figure 8: Two lives x, y

Thiele's equations for the reserves:

$$\begin{aligned}
\frac{dV_{xy}(t)}{dt} &= (r+\mu_{x+t}+\mu_{y+t})V_{xy}(t)-\mu_{x+t}V_y(t)-\mu_{y+t}V_x(t)\\
\frac{dV_x(t)}{dt} &= (r+\mu_{x+t})V_x(t)-\mu_{x+t}\\
\frac{dV_y(t)}{dt} &= (r+\mu_{y+t})V_y(t)-\mu_{y+t}
\end{aligned}$$

and the conditions

$$V_{xy}(n)=V_x(n)=V_y(n)=0, \qquad V_{xy}(0)=\pi$$

Suppose there is no premium upfront, but premium is payable continuously while both lives are alive. Then

$$\begin{aligned}
\frac{dV_{xy}(t)}{dt} &= (r+\mu_{x+t}+\mu_{y+t})V_{xy}(t)-\mu_{x+t}V_y(t)-\mu_{y+t}V_x(t)+\pi\\
\pi &= \frac{\bar{A}_{\overline{xy}}}{\bar{a}_{xy}}
\end{aligned}$$

Example: Health-Sickness model. A continuous benefit b is payable as long as the life is sick. Single premium payable upfront. The contract lasts for n years.

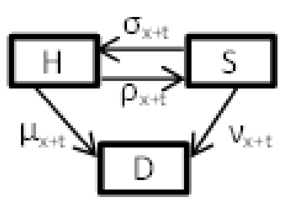

Figure 9: Health-Sickness Model

$$\begin{aligned}
V_H'(t) &= rV_H(t)+\mu_{x+t}V_H(t)-\sigma_{x+t}(V_S(t)-V_H(t))\\
V_S'(t) &= rV_S(t)+\nu_{x+t}V_s(t)-\rho_{x+t}(V_H(t)-V_S(t))-b
\end{aligned}$$

subject to the conditions

$$V_H(n)=0, \qquad V_S(n)=0$$

Solve and we have $V_H(0)=\pi$.

Example: Health-Sickness model. Same as above, but instead of a single premium we have a continuous premium π payable while healthy, then

$$\begin{aligned} V_H'(t) &= rV_H(t) + \mu_{x+t}V_H(t) - \sigma_{x+t}(V_S(t) - V_H(t)) + \pi \\ V_S'(t) &= rV_S(t) + \nu_{x+t}V_S(t) - \rho_{x+t}(V_H(t) - V_S(t)) - b \end{aligned}$$

and the conditions

$$V_H(n) = 0, \qquad V_S(n) = 0$$

π should be such that $V_H(0) = 0$. But how do we find π?

First, we want to find the value of a continuous annuity payable as long as a life is healthy and up to time n (this can be viewed as a continuous benefit of 1 payable as long as healthy). In order to find that, we solve:

$$\begin{aligned} W_H'(t) &= rW_H(t) + \mu_{x+t}W_H(t) - \sigma_{x+t}(W_S(t) - W_H(t)) - 1 \\ W_S'(t) &= rW_S(t) + \nu_{x+t}W_S(t) - \rho_{x+t}(W_H(t) - W_H(t)) \end{aligned}$$

subject to the conditions

$$W_H(n) = 0, \qquad W_S(n) = 0$$

$W_H(0)$ will be the value of this annuity (by the Principle of equivalence).

Now, consider the single premium model as in the previous example:

$$\begin{aligned} U_H'(t) &= rU_H(t) + \mu_{x+t}U_H(t) - \sigma_{x+t}(U_S(t) - U_H(t)) \\ U_S'(t) &= rU_S(t) + \nu_{x+t}U_S(t) - \rho_{x+t}(U_H(t) - U_S(t)) - b \end{aligned}$$

subject to the conditions

$$U_H(n) = 0, \qquad U_S(n) = 0$$

By the Principle of equivalence,

$$\begin{aligned} \pi W_H(0) &= U_H(0) \\ \pi &= \frac{U_H(0)}{W_H(0)} \end{aligned}$$

Then we substitute π into the differential equations for $V_H(t)$, $V_S(t)$ and solve to find the reserves. Check to see if $V_H(0) = 0$.

7 Higher Moments

We have

- $W(t)$ is a stochastic process.
- $V_i(t) = E(W(t)|\mathcal{F}_t) = E(W(t)|X(t) = i)$.
- $V_i^{(2)}(t) = E(W^2(t)|X(t) = i)$, which will lead to $Var(W(t)|X(t) = i)$.

Example: Life-Death Model - Continuous benefit b payable as long as alive until time n, one off payment B at time of death, and C at time n if alive.

$$W(t) = \int_t^n e^{-r(s-t)} b I_s ds + \int_t^n e^{-r(s-t)} B dN_s + Ce^{-r(n-t)} I_n$$

Let T_{x+t} be the remaining lifetime. So

$$\begin{aligned}
W(t) &= b\bar{a}_{\overline{T_{x+t}\wedge n|}} + Be^{-rT_{x+t}}\mathbf{1}_{\{T_{x+t}<n\}} + Ce^{-r(n-t)}\mathbf{1}_{\{T_{x+t}>n\}} \\
\text{Second moment} &= \int_0^\infty \left(b\frac{1-e^{-r(s\wedge n)}}{r} + e^{-rs}B\mathbf{1}_{\{s<n\}} + Ce^{-r(n-t)}\mathbf{1}_{\{s>n\}} \right)^2 {}_sp_{x+t}\mu_{x+t+s}ds
\end{aligned}$$

Alternatively, we can use differential equations.
The first moment:

$$\begin{aligned}
W(t) &= e^{-rdt}W(t+dt) + b(t)dt + (B(t) - W(t+dt))dN_t \\
V(t) &= E(W(t)|\mathcal{F}_t) \\
&= (1-rdt)V(t+dt) + b(t)dt + (B(t) - V(t+dt))\mu_{x+t}dt \\
\frac{dV^{(1)}(t)}{dt} &= rV^{(1)}(t) - b(t) - \left(B(t) - V^{(1)}(t)\right)\mu_{x+t}
\end{aligned}$$

such that $V^{(1)}(n) = C$.
The second moment:

$$\begin{aligned}
W^2(t) &= [W(t+dt) - rdtW(t+dt) + b(t)dt + (B(t) - W(t+dt))dN_t]^2 \\
&= W^2(t+dt) - 2rdtW^2(t+dt) + 2b(t)dtW(t+dt) + 2W(t+dt)(B(t) - W(t+dt))dN_t \\
&\quad +(B(t) - W(t+dt))^2dN_t + o(dt) \\
V^{(2)}(t) &= E(W^2(t)|\mathcal{F}_t) \\
&= V^{(2)}(t+dt) - 2rV^{(2)}dt + 2b(t)V^{(1)}(t+dt) + \left(2B(t)V^{(1)}(t+dt) - 2V^{(2)}(t+dt)\right)\mu_{x+t}dt \\
&\quad + \left(B(t)^2 - 2B(t)V^{(1)}(t+dt) + V^{(2)}(t+dt)\right)\mu_{x+t}dt + o(dt) \\
&= V^{(2)}(t+dt) - 2rdtV^{(2)}(t+dt)dt + 2b(t)V^{(1)}(t+dt) + \left(B(t)^2 - V^{(2)}(t+dt)\right)\mu_{x+t}dt \\
\frac{dV^{(2)}(t)}{dt} &= 2rV^{(2)}(t) - 2b(t)V^{(1)}(t) - \left(B^2(t) - V^{(2)}(t)\right)\mu_{x+t}
\end{aligned}$$

such that $V^{(2)}(n) = C^2$. Note that we have to solve $V^{(1)}(t)$ first. In MAPLE, we solve simultaneously (1) and (2). $V^{(2)}(0) - \left(V^{(1)}(0)\right)^2$ is the variance of the contract.

Example: A more general case

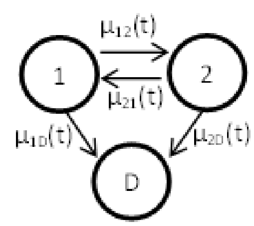

Figure 10: 3 state model

$$\begin{aligned}
W(t) &= \int_t^n \Big(e^{-r(s-t)}b_1I_1(s)ds + e^{-r(s-t)}b_2I_2(s)ds + B_{12}e^{-r(s-t)}dN_{12}(s) + B_{21}e^{-r(s-t)}dN_{21}(s) \\
&\quad +B_{1D}e^{-r(s-t)}dN_{1D}(s) + B_{2D}e^{-r(s-t)}dN_{2D}(s)\Big) + C_1e^{-r(n-s)}I_1(n) + C_2e^{-r(n-t)}I_2(n)
\end{aligned}$$

We denote $W_1(t) = W(t)|$ state 1 at t, $W_2(t) = W(t)|$ state 2 at t.
Thiele's equations for the first moments:

$$\begin{aligned}
W_1(t) &= (1-rdt)W_1(t+dt) + b_1 dt + (B_{12} + W_2(t+dt) - W_1(t+dt))\, dN_{12}(t) \\
&\quad + (B_{1D} - W_1(t+dt))\, dN_{1D}(t) \\
V_1^{(1)}(t) &= (1-rdt)V_1^{(1)}(t+dt) + b_1 dt + \left(B_{12} + V_2^{(1)}(t+dt) - V_1^{(1)}(t+dt)\right)\mu_{12}(t)dt \\
&\quad + \left(B_{1D} - V_1^{(1)}(t+dt)\right)\mu_{1D}(t)dt \\
\frac{dV_1^{(1)}(t)}{dt} &= rV_1^{(1)}(t) - b_1 - \mu_{12}(t)\left(B_{12} + V_2^{(1)}(t) - V_1^{(1)}(t)\right) - \mu_{1D}(t)\left(B_{1D} - V_1^{(1)}(t)\right)
\end{aligned}$$

Similarly,

$$\frac{dV_2^{(1)}(t)}{dt} = rV_2^{(1)}(t) - b_2 - \mu_{21}(t)\left(B_{21} + V_1^{(1)}(t) - V_2^{(1)}(t)\right) - \mu_{2D}(t)\left(B_{2D} - V_2^{(1)}(t)\right)$$

Solve subject to $V_1(n) = C_1$, $V_2(n) = C_2$.

To get the second moments,

$$\begin{aligned}
W_1^2(t) &= (1-rdt)^2 W_1^2(t+dt) + (B_{12} + W_2(t+dt) - W_1(t+dt))^2\, dN_{12}(t) + 2b_1 dt W_1(t+dt) \\
&\quad + (B_{1D} - W_1(t+dt)dt)^2\, dN_{1D}(t) + 2\,(B_{12} + W_2(t+dt) - W_1(t+dt))\, W_1(t+dt) dN_{12}(t) \\
&\quad +2\,(B_{1D} - W_1(t+dt))\, W_1(t+dt) dN_{1D} + o(dt) \\
V_1^{(2)}(t) &= V_1^{(2)}(t+dt) - 2rV_1^{(2)}(t+dt)dt + \left(B_{1D}^2 - V_1^{(2)}(t+dt)\right)\mu_{1D}(t)dt \\
&\quad + \left(B_{12}^2 + 2B_{12}V_2^{(1)}(t+dt) + V_2^{(2)}(t+dt) - V_1^{(2)}(t+dt)\right)\mu_{12}(t)dt \\
&\quad +2b_1 V_1^{(1)}(t+dt)dt \\
\frac{dV_1^{(2)}}{dt} &= 2rV_1^{(2)}(t) - \left(B_{1D}^2 - V_1^{(2)}(t)\right)\mu_{1D}(t) \\
&\quad - \left(B_{12}^2 + 2B_{12}V_2^{(1)}(t) + V_2^{(2)}(t) - V_1^{(2)}(t)\right)\mu_{12}(t) - 2b_1 V_1^{(1)}(t)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{dV_2^{(2)}}{dt} &= 2rV_2^{(2)}(t) - \left(B_{2D}^2 - V_2^{(2)}(t)\right)\mu_{2D}(t) \\
&\quad - \left(B_{21}^2 + 2B_{21}V_1^{(1)}(t) + V_1^{(2)}(t) - V_2^{(2)}(t)\right)\mu_{21}(t) - 2b_2 V_2^{(1)}(t)
\end{aligned}$$

Solve subject to $V_1^{(1)}(n) = C_1$, $V_1^{(2)}(n) = C_1^2$, $V_2^{(1)}(n) = C_2$, $V_2^{(2)} = C_2^2$.

8 Safety Margin

Principle of Equivalence: EPV(benefits) = EPV(premiums).

However, to make a profit, a safety margin is included so that premiums charged are higher than that implied by the principle of equivalence.

Example: Consider a one year temporary assurance, where benefit b is payable at the end of 1 year if life is dead.

$$EPV(benefit) = e^{-r} b q_x$$

and the single premium is $e^{-r} b q_x (1+\theta)$ where $\theta > 0$ is the loading factor. Suppose we have 10000 such policies, independent of each other.

$$\begin{aligned}
E(profits) &= 10000\left(e^{-r} b q_x (1+\theta) - e^{-r} b q_x\right) \\
&\quad 10000\theta e^{-r} b q_x \\
E(profits^2) &= 10000 b^2 \theta^2 e^{-2r} q_x \\
Var(profits) &= 10000^2 b^2 \theta^2 e^{-2r} q_x p_x
\end{aligned}$$

By the Central limit theorem, the profits

$$Z \sim \mathcal{N}\left(10000\theta e^{-r} b q_x, 10000 b^2 \theta^2 e^{-2r} q_x p_x\right)$$

So we have

$$P(loss) = \Psi\left(\frac{-10000\theta e^{-r} b q_x}{100 b\theta e^{-r}\sqrt{q_x p_x}}\right) \approx 0$$

This is especially so if there is a large number of policies.

9 Stochastic Interest Rates

We have previously been working with constant interest rate. In practice, interest rates do not stay constant. Now, we consider the case when the force of interest r_t is a stochastic process. Amount S invested at time 0 will grow to $Se^{\int_0^t r_s ds}$ at time t. We might want to calculate $E(e^{\int_0^t r_s ds})$.

Suppose a markov chain model with states $\{1, 2, ..., K\}$. At state i, the force of interest is $r^{(i)}$. If $X(t) = i$, $r_t = r^{(i)}$. Let λ_{ij} be the force of transition from i to j. Then

$$\begin{aligned}
E\left(e^{\int_0^n r_s ds}|\mathcal{F}_t\right) &= E\left(e^{\int_0^t r_s ds} e^{\int_t^n r_s ds}|\mathcal{F}_t\right) \\
&= e^{\int_0^t r_s ds} E\left(e^{\int_t^n r_s ds}|\mathcal{F}_t\right) \\
&= e^{\int_0^t r_s ds} E\left(e^{\int_t^n r_s ds}|X(t) = i\right) \\
&= e^{\int_0^t r_s ds} W_i(t)
\end{aligned}$$

where we have denoted $W_i(t) = E\left(e^{\int_t^n r_s ds}|X(t) = i\right)$.

Now we obtain the differential equations:

$$\begin{aligned}
W_i(t-dt) &= (1 + r^{(i)} dt)\left(W_i(t)\left(1 - \sum_{j=1, j\neq i}^{K} \lambda_{ij} dt\right) + \sum_{j=1, j\neq i}^{K} \lambda_{ij} W_j(t)\right) \\
&= W_i(t) + r^{(i)} W_i(t) dt - W_i(t)\sum_{j=1, j\neq i}^{K} \lambda_{ij} dt + \sum_{j=1, j\neq i}^{K} \lambda_{ij} W_j(t) dt + o(dt) \\
\frac{W_i(t-dt) - W_i(t)}{dt} &= r^{(i)} W_i(t) + \sum_{j=1, j\neq i}^{K} \lambda_{ij}\left(W_j(t) - W_i(t)\right) + \frac{o(dt)}{dt}
\end{aligned}$$

So

$$\frac{d}{dt}W_i(t) = -r^{(i)}W_i(t) - \sum_{j=1,j\neq i}^{K} \lambda_{ij}\left(W_j(t) - W_i(t)\right)$$

Solve simultaneously such that $W_i(n) = 1$ for all $i = 1, ..., K$. I solve all my predicted accumulation at time t. If I know $X(t) = i$, then accumulation is $E\left(e^{\int_0^t r(s)ds}\right) = W_i(t)$. In particular, $E\left(e^{\int_0^n r(s)ds}\right) = W_{i_0}(t)$ if we know $X(0) = i_0$.

For variances calculate second moment $E\left(e^{2\int_0^n r_s ds}|\mathcal{F}_t\right)$. So replace $r^{(i)}$ by $2r^{(i)}$ everywhere and we get the second moment. Solve

$$\frac{dW_i^{(2)}(t)}{dt} = -2r_iW_i^{(2)}(t) + \sum \lambda_{ij}\left(W_i^{(2)}(t) - W_j^{(2)}(t)\right)$$

such that $W_i^{(2)}(n) = 1$. Then

$$E\left(e^{\int_0^n 2r(s)ds}\right) = W_{i_0}^{(2)}(0)$$

and variance is $W_{i_0}^{(2)}(0) - \left(W_{i_0}^{(2)}(0)\right)^2$.

9.1 Financial risk vs Mortality risk

With stochastic interest rate, what is the price of 1 payable at time n? Or equivalently, the single premium at time 0. Here, the principle of equivalence does not work. Mortality risk is diversifiable, i.e. in a large portfolio of people aged x, q_x will die on average, hence we can simply take expected values. However, interest rate risks are non-diversifiable no matter how large the size of the portfolio is.

Example: Consider one year term insurance policies with a sum assured of 1, paid out at the end of the year if dead. The probability of death $q_x = 0.01$. The assumed force of interest is $r = 0.05$.

$$EPV(payments) = e^{-0.05} \times 0.01 = 0.0095$$

Premium will be slightly higher, suppose 0.0097. If there are 1000000 policies, in one year's time, the 9700 premium will accumulate to $9700e^{0.05} = 10197$. The expected loss and variance are

$$\begin{aligned} E(loss) &= 1000000 \times 0.01 = 10000 \\ Var(loss) &= 1000000 \times 0.01 \times 0.99 \approx 100 \end{aligned}$$

Hence, the probability of a loss for the office is

$$P(loss) = \Psi\left(\frac{10000 - 10197}{100}\right) = \Psi(-1.97) \approx 0.024$$

Now suppose r is not 0.05, but it is 0 with probability $\frac{1}{3}$ and 0.1 with probability $\frac{2}{3}$. Then

$$\begin{aligned} P(loss) &= \frac{2}{3}\Psi\left(\frac{10000 - 9700e^{0.1}}{100}\right) + \frac{1}{3}\Psi\left(\frac{10000 - 9700e^{0}}{100}\right) \\ &= \frac{2}{3} \times 0 + \frac{1}{3} \times 0.99 \\ &= 0.33 \end{aligned}$$

As we can see, financial risk cannot be hedged out, it applies to all policies. There is still a bit of mortality risk, if we did not get 0.01 right. In order to calculate price in this case, we can use the equivalent martingale measure so that we work under the risk neutral measure. Typically in our calculation we will be using a risk-neutral measure for interest rates and the physical measure for mortality.
Quite often, the structure of the model is the same under the EMM except with changed parameters. So to find $E\left(e^{-\int_0^n r_s ds}\right)$, we denote $\tilde{W}_i(t)$ as

$$\begin{aligned}\tilde{W}_i(t) &= E\left(e^{-\int_t^n r_s ds}|\mathcal{F}_t\right)\\ &= E\left(e^{-\int_t^n r_s ds}|X(t)=i\right)\end{aligned}$$

This replaces $r(t)$ in $W_i(t)$ with $-r(t)$. Hence the equation becomes

$$\begin{aligned}\frac{d\tilde{W}_i(t)}{dt} &= r_i\tilde{W}_i(t) - \sum_{i=1,i\neq j}^{k}\lambda_{ij}\left(\tilde{W}_j(t)-\tilde{W}_i(t)\right)\\ \tilde{W}_i(n) &= 1\end{aligned}$$

So we have $E\left(e^{-\int_0^n r(s)ds}\right) = \tilde{W}_{i_0}(0)$ if $X(0)=i_0$.
Boredom: calculation from scratch

$$\begin{aligned}\tilde{W}_i(t-dt) &= e^{-r^{(i)}dt}\left(\tilde{W}_i(t)\left(1-\sum_{j=1,j\neq i}^{K}\lambda_{ij}dt\right)+\sum_{j=1,j\neq i}^{K}\lambda_{ij}\tilde{W}_j(t)dt\right)\\ &= (1-r^{(i)}dt)\left(\tilde{W}_i(t)\left(1-\sum_{j=1,j\neq i}^{K}\lambda_{ij}dt\right)+\sum_{j=1,j\neq i}^{K}\lambda_{ij}\tilde{W}_j(t)dt\right)\\ \frac{\tilde{W}_i(t-dt)-\tilde{W}_i(t)}{dt} &= -r^{(i)}\tilde{W}_i(t)+\sum_{j=1,j\neq i}^{K}\lambda_{ij}(\tilde{W}_j(t)-\tilde{W}_i(t))+\frac{o(dt)}{dt}\end{aligned}$$

So we have

$$\frac{d\tilde{W}_i(t)}{dt} = r^{(i)}\tilde{W}_i(t) - \sum_{j=1,j\neq i}^{K}\lambda_{ij}(\tilde{W}_j(t)-\tilde{W}_i(t))$$

such that $\tilde{W}_i(n)=1$ for all $i=1,...,K$. $\tilde{W}_i(0)$ is the premium we want.

One can become more sophisticated and try to price derivatives. For example, suppose one promised the accumulation or a guarantee g whichever is larger. The expected value of the accumulation is $E\left(e^{\int_0^n r(s)ds}\vee g\right)$, $g>1$.

10 With-profits policies

One way to safeguard against non-diversifiable risks is to assume a "basis" (a set of interest and mortality rates), and calculate premiums and prospective reserves based on a "safe" or "conservative" basis which we denote $(r*, \mu*)$. Premiums and reserves calculated based on the safe basis will be more prudent, hence generating a surplus. With-profits policies are such that policyholders get a share of the surplus. We calculate the premium using the safe basis (also called "first order" basis), and calculate technical "safe" reserve using the equation:

$$\begin{aligned} \frac{dV_t^*}{dt} &= (r^* + \mu_{x+t}^*)V_t^* - \mu_{x+t}^* b_t + \pi \\ V_n^* &= B \end{aligned}$$

Also $V_0^* = 0$ by our choice of π. As we will see, the retrospective accumulation will in general exceed V_t^* producing a surplus.

Discounted new surplus at time 0:

$$\begin{aligned} S_t &= \int_0^t e^{-\int_0^s (r(u)+\mu_{x+u})du}(\pi - \mu_{x+s}b_s)ds - e^{-\int_0^t (r(u)+\mu_{x+u})du}V_t^* \\ \frac{dS_t}{dt} &= e^{-\int_0^t (r(u)+\mu_{x+u})du}(\pi - \mu_{x+t}b_t) + (r(t)+\mu_{x+t})e^{-\int_0^t (r(u)+\mu_{x+t})}V_t^* - e^{-\int_0^t (r(u)+\mu_{x+u})du}\frac{dV_t^*}{dt} \\ &= e^{-\int_0^t (r(u)+\mu_{x+u})du}\left(\pi - \mu_{x+t}b_t + (r(t)+\mu_{x+t})V_t^* - (r^* + \mu_{x+t}^*)V_t^* - \pi + \mu_{x+t}^* b_t\right) \\ &= e^{-\int_0^t (r(u)+\mu_{x+u})du}\left((r(t) - r^*)V_t^* + (\mu_{x+t}^* - \mu_{x+t})(b_t - V_t^*)\right) \\ &= e^{-\int_0^t (r(u)+\mu_{x+u})du}c(t) \end{aligned}$$

where

$$c(t) = (r(t) - r^*)V_t^* + (\mu_{x+t}^* - \mu_{x+t})(b_t - V_t^*)$$

This is the emerging surplus value (per survivor). It is the sum of the interest rate profit and mortality surplus.

Remark: If $r(t) = r^*$, $\mu_{x+t} = \mu_{x+t}^*$, then $c(t) = 0$, $\frac{dS_t}{dt} = 0$, $S_t = 0$.
A safe basis is one where r^* is as low as possible. The choice of μ^* depends on the products, for example:

1. Temporary assurance. $V_t^* < b$ the death benefit. A high μ^* is safe.
2. Pure endowment. $b_t = 0$, we choose a low μ^* to be safe.
3. Endowment assurance. $b_t = B = b_n$ (survival benefit). Recall that V_t^* increases to B, so $V_t^* \le B$. We would choose a high μ^*, but it does not matter very much.

Distribution of surplus (Bonus):

1. Cash bonus: Pass it on as it emerges. Pay $c(t)$ at time t to each surviving member.
2. Pay a terminal bonus: Accumulate all surpluses and pay it as a terminal bonus at maturity.

$$L_n = \int_0^n e^{\int_s^n (r_u + \mu_{x+u})du} c(s)ds$$

   Sometimes there is a guarantee so the bonus will be $L_n \vee g$; actually there is always a guarantee because in reality you pay $L_n \vee 0$.

3. Combination of the above: Pay a bit of a cash bonus (but not the whole surplus), accumulate the rest to a terminal bonus. Note that the equation of value will have to hold. Let $l(t)$ be the cash bonus value and let L be the terminal bonus.

$$\int_0^n l(s)e^{-\int_0^s (r(u)+\mu_{x+u})du}ds + Le^{-\int_0^n (r(u)+\mu_{x+u})du} = \int_0^n c(s)e^{-\int_0^s (r(u)+\mu_{x+u})du}$$

4. Purchase additional insurance: We use surplus to purchase some units of the product; the emerging surplus is used as a single premium to buy more units. This is a little complicated because extra units generate more surplus to buy extra units and so on.

Purchase additional insurance

A surplus $c(t)dt$ emerges at time t. This will buy $q(t)dt$ extra units. $c(t)dt$ is used as a single premium, so calculate first how much is the single premium for an extra unit. We need to solve

$$\begin{aligned}\frac{dV_t^{**}}{dt} &= (r^* + \mu^*_{x+t})V_t^{**} - \mu^*_{x+t}b_t \\ V_n^{**} &= B\end{aligned}$$

V_t^{**} represents the single premium to be paid at time t to buy an extra unit.

$$\begin{aligned}c(t) &= q(t)V_t^{**} \\ q(t) &= \frac{c(t)}{V_t^{**}}\end{aligned}$$

We want to calculate $Q(t) = \int_0^t q(s)ds$, the accumulated extra units. The benefits will then be $b_t(1 + Q(t))$ and survival benefit $B(1 + Q(n))$. The surplus is:

$$c(t) = (r(t) - r^*)(V_t^* + Q(t)V_t^{**}) + (\mu^*_{x+t} - \mu_{x+t})(b_t(1 + Q(t)) - V_t^* - Q(t)V_t^{**})$$

So we have

$$\begin{aligned}\frac{dQ(t)}{dt} &= \frac{(r(t) - r^*)(V_t^* + Q(t)V_t^{**}) + (\mu^*_{x+t} - \mu_{x+t})(b_t(1 + Q(t)) - V_t^* - Q(t)V_t^{**})}{V_t^{**}} \\ \frac{dV_t^*}{dt} &= (r^* + \mu^*_{x+t})V_t^* + (\pi - \mu^*_{x+t}b_t) \\ \frac{dV_t^{**}}{dt} &= (r^* + \mu^*_{x+t})V_t^{**} - \mu^{**}_{x+t}b_t\end{aligned}$$

subject to $Q(0) = 0$, $V_n^* = B$ or $V_0^* = 0$, and $V_n^{**} = B$.

We do not have V_0^{**}, so in order to solve this we need two steps:

1. Solve

$$\begin{aligned}\frac{dV_t^{**}}{dt} &= (r^* + \mu^*_{x+t})V_t^* - \mu^*_{x+t}b_t \\ V_n^{**} &= B\end{aligned}$$

   and extract V_0^{**}.

2. Solve 3 differential equations for $Q(t)$ (replacing V_t^{**} with W_t^{**}), V_t^*, and W_t^{**} where

$$\frac{dW^{**}}{dt} = (r^* + \mu^*_{x+t})W_t^{**} - \mu^*_{x+t}b_t$$

   together subject to the initial conditions

$$\begin{aligned}Q(0) &= 0 \\ V_0^* &= 0 \\ W_0^{**} &= V_0^{**}\end{aligned}$$

   and we have $Q(t)$ for all values of t.

If we use the basis $r(t)$, μ_{x+t} for predictions then we can represent that the predicted survival benefit is $[1+Q(n)]B$ and the predicted death benefit if it occurs at time t is $b_t(1+Q(t))$.

Variations: Extra units do not generate extra units. The extra surplus generated by new units is given out as cash bonus.

Suppose $c(t)dt$ buys $q(t)dt$ units. $c(t) = q(t)V_t^{**}$. Since extra units do not generate extra units, $q(t) = \frac{c(t)}{V_t^{**}}$. $Q(t) = \int_0^t q(s)dt$. So

$$c(t) = (r(t) - r^*)V_t^* + (\mu^*_{x+t} - \mu_{x+t})(b_t - V_t^*)$$

and we solve the three differential equations below

$$\begin{aligned}
\frac{dQ(t)}{dt} &= \frac{(r(t) - r^*)V_t^* + (\mu^*_{x+t} - \mu_{x+t})(b_t - V_t^*)}{W_t^{**}} \\
\frac{dV_t^*}{dt} &= (r^* + \mu^*_{x+t})V_t^* + \pi - \mu^*_{x+t}b_t \\
\frac{dW_t^{**}}{dt} &= (r^* + \mu^*_{x+t})W_t^{**} - \mu^*_{x+t}b_t
\end{aligned}$$

subject to

$$\begin{aligned}
Q(0) &= 0 \\
V_0^* &= 0 \\
W_0^{**} &= V_0^{**}
\end{aligned}$$

The extra surplus to be given out as cash bonus will be

$$Q(t)\left((r(t) - r^*)V_t^{**} + (\mu^*_{x+t} - \mu_{x+t})(b_t - V_t^{**})\right)$$

If you want it to be terminal bonus, this will be

$$\int_0^n Q(s)\left((r(s) - r^*)V_s^{**} + (\mu^*_{x+s} - \mu_{x+s})(b_s - V_s^{**})\right) e^{\int_s^n (r(u)+\mu_{x+u})du} ds$$

We can also summarise this into four equations. We have

$$\begin{aligned}
L(t) &= \int_0^t Q(s)\left((r(s) - r^*)V_s^{**} + (\mu^*_{x+s} - \mu_{x+s})(b_s - V_s^{**})\right) e^{\int_s^t (r(u)+\mu_{x+u})du} ds \\
\frac{dL(t)}{dt} &= (r(t) + \mu_{x+t})L(t) + Q(t)\left((r(t) - r^*)W_t^{**} + (\mu^*_{x+t} - \mu_{x+t})(b_t - W_t^{**})\right)
\end{aligned}$$

Solve this and the above three differential equations subject to $L(0) = 0$, $Q(0) = 0$, $V_0^* = 0$, $W_0^{**} = V_0^{**}$. $L(n)$ is the terminal bonus prediction.

Predicting cash bonus

We will use the Markov model for interest rate introduced three weeks ago. States $1, 2, ..., k$. While at state i, interest $r(t) = r_i$ if $X(t) = i$. λ_{ij} are the discount rate. We will also assume for simplicity that $\mu_{x+t} = \mu^*_{x+t}$. If we were not to do that we could also assume a model for stochastic mortality rates so for example when $X(t) = i$, $r(t) = r_i$, $\mu_{x+t} = \mu^{(i)}_{x+t}$,

$$c_t = (r(t) - r^*)V_t^*$$

So we need to predict $c_u = (r(u) - r^*)V_u^* = f(r(u))$. We need to calculate

$$E\left[f(r_u)|\mathcal{F}_t\right] = E\left[f(r_u)|r_t = r^{(i)}\right] = W_i(t)$$

Because $X(t)$ and $r(t)$ is a Markov process, we just need to calculate

$$E[c_u|X(t) = i] = E[(r(u) - r^*)V_u^*|X(t) = i] = V_u^* E[r(u)|X(t) = i] - r^* V_u^*$$

So we define $W_i(t) = E[r(u)|X(t) = i]$. Then

$$\begin{aligned}
W_i(t - dt) &= \left(1 - \sum_{j \neq i} \lambda_{ij} dt\right) W_i(t) + \sum_{j \neq i} \lambda_{ij} dt W_j(t) + o(dt) \\
\frac{W_i(t - dt) - W_i(t)}{dt} &= -\sum_{j \neq i} \lambda_{ij} W_i(t) + \sum_{j \neq i} \lambda_{ij} W_j(t) + \frac{o(dt)}{dt} \\
\frac{dW_i(t)}{dt} &= \sum_{j=1, j \neq i}^{k} \lambda_{ij}(W_i(t) - W_j(t))
\end{aligned}$$

subject to $W_i(u) = r_i$. We can predict the cash bonus by $W_i(t)V_u^* - r^* V_u^*$ if $X(t) = i$.

Alternatively, we can define $W_i(t) = E[(r(u) - r^*)V_u^*|X(t) = i]$, and solve the equation

$$\begin{aligned}
\frac{dW_i(t)}{dt} &= \sum_{j \neq i} \lambda_{ij}(W_i(t) - W_j(t)) \\
W_i(t) &= (r_i - r^*)V_u^*
\end{aligned}$$

V_u^* is to be calculated from Thiele's equation. **Predicting Terminal Bonus**

$$\begin{aligned}
W &= \int_0^n e^{\int_t^n (r(u) + \mu_{x+u}) du} (r(t) - r^*) V_t^* dt \\
&= \int_0^t e^{\int_s^n (r(u) + \mu_{x+u}) du} (r(s) - r^*) V_s^* ds + \int_t^n e^{\int_s^n (r(u) + \mu_{x+u}) du} (r(s) - r^*) V_s^* ds \\
&= e^{\int_t^n (r(u) + \mu_{x+u}) du} \int_0^t e^{\int_s^t (r(u) + \mu_{x+u}) du} (r(s) - r^*) V_s^* ds + \int_t^n e^{\int_s^n (r(u) + \mu_{x+u}) du} (r(s) - r^*) V_s^* ds
\end{aligned}$$

Define the following:

$$\begin{aligned}
U(t) &= e^{\int_t^n (r(u) + \mu_{x+u}) du} && \text{(Accumulation of past surplus)} \\
\tilde{W}(t) &= \int_0^t e^{\int_s^t (r(u) + \mu_{x+u}) du} (r(s) - r^*) V_s^* ds && \text{(Past surplus)} \\
\tilde{\tilde{W}}(t) &= \int_t^n e^{\int_s^n (r(u) + \mu_{x+u}) du} (r(s) - r^*) V_s^* ds && \text{(Future surplus)}
\end{aligned}$$

Then we have

$$E(U(t)|\mathcal{F}_t) = E\left(e^{\int_t^n (r(u) + \mu_{x+u}) du} | \mathcal{F}_t\right)$$

Define

$$
\begin{aligned}
V_i(t) &= E\left(e^{\int_t^n (r(u)+\mu_{x+u})du} | X(t) = i\right) \\
V_i(t-dt) &= \left(1 - \sum_{j\neq i} \lambda_{ij} dt\right) (1 + (r_i + \mu_{x+t} dt))\, V_i(t) + \sum_{j\neq i} \lambda_{ij} V_j(t) dt + o(dt) \\
\frac{V_i(t-dt) - V_i(t)}{dt} &= (r_i + \mu_{x+t}) V_i(t) dt - \sum_{j\neq i} \lambda_{ij} (V_i(t) - V_j(t)) + \frac{o(dt)}{dt} \\
\frac{dV_i(t)}{dt} &= -(r_i + \mu_{x+t}) V_i(t) + \sum_{j\neq i} \lambda_{ij} (V_i(t) - V_j(t))
\end{aligned}
$$

for $i = 1, 2, ..., k$ subject to $V_i(n) = 1$ for all i. Next define

$$
\tilde{V}_i(t) = E(\tilde{\tilde{W}} | \mathcal{F}_t)
$$

If no change occurs,

$$
\begin{aligned}
\tilde{\tilde{W}}_i(t-dt) &= \tilde{\tilde{W}}_i(t) + (r_i - r^*) V_t^* e^{\int_t^n (r(u)+\mu_{x+u})du} + o(dt) \\
E(\tilde{\tilde{W}}_i(t-dt) | \text{no change}) &= E(\tilde{\tilde{W}}_i(t) | \text{no change}) + (r_i - r^*) V_t^* E(e^{\int_t^n (r(u)+\mu_{x+u})du} | X(t) = i) \\
&= E(\tilde{\tilde{W}}_i(t) | \text{no change}) + (r_i - r^*) V_t^* V_i(t)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{V}_i(t-dt) &= \left(1 - \sum_{j\neq i} \lambda_{ij} dt\right) \left(\tilde{V}_i(t) + (r_i - r^*) V_t^* V_i(t) dt\right) + \sum_{j\neq i} \lambda_{ij} dt \tilde{V}_j(t) + o(dt) \\
\frac{\tilde{V}_i(t-dt) - \tilde{V}_i(t)}{dt} &= (r_i - r^*) V_t^* V_i(t) - \sum_{j\neq i} \lambda_{ij} (\tilde{V}_i(t) - \tilde{V}_j(t)) + \frac{o(dt)}{dt} \\
\frac{d\tilde{V}_i(t)}{dt} &= -(r_i - r^*) V_t^* V_i(t) + \sum_{j\neq i} \lambda_{ij} (\tilde{V}_i(t) - \tilde{V}_j(t))
\end{aligned}
$$

with $\tilde{V}_i(n) = 0$ for all i. We need to solve the following simultaneously.

$$
\begin{aligned}
\frac{d\tilde{V}_i(t)}{dt} &= -(r_i - r^*) V_i(t) V_t^* + \sum_{j\neq i} \lambda_{ij} (\tilde{V}_i(t) - \tilde{V}_j(t)) \\
\frac{dV_i(t)}{dt} &= -(r_i + \mu_{x+t}) V_i(t) + \sum_{j\neq i} \lambda_{ij} (V_i(t) - V_j(t)) \\
\frac{dV_t^*}{dt} &= (r^* + \mu_{x+t}) V_t^* + \pi - \mu_{x+t} b_t
\end{aligned}
$$

subject to the conditions $\tilde{V}_i(n) = 0$, $V_i(n) = 1$, $V_n^* = B$ for all i. And

$$
\tilde{W}(t) = \int_0^t e^{\int_0^t (r(u)+\mu_{x+u})du} (r(s) - r^*) V_s^* ds
$$

has been recorded or calculated from past value. If we are at time t and the past accumulated surplus is $\tilde{W}(t)$ and $X(t) = i$, our prediction of the terminal bonus is $\tilde{W}(t) V_i(t) + \tilde{V}_i(t)$.

Cash bonus with guarantee

With guarantee that we will never pay less than gV_t^*. Typically $g=0$. We pay bonuses $max(r(t)-r^*,0)V_t^*$. In general $max(r(t)-r^*,g)V_t^*$. We need to calculate the cost of the guarantee to the office and change the policyholder (either by a premium upfront or an extra surcharge on the premium). Expenses of the office is $(r^*-r(t))_+V_t^*$.

$$\begin{aligned} \text{Cost of the guarantee} &= E\left(\int_0^n e^{-\int_0^s (r(u)+\mu_{x+u})du}(r^*-r(s))_+V_s^*ds\right) \\ &= E\left(\int_0^n e^{-\int_0^s r(u)du}(r^*-r(s))_+V_s^*{}_sp_xds\right) \end{aligned}$$

Define

$$\begin{aligned} W_i(t) &= E\left(\int_t^n e^{-\int_t^s r(u)du}(r^*-r(s))_+V_s^*{}_sp_xds|X(t)=i\right) \\ W_i(t-dt) &= \left(1-\sum_{j\neq i}\lambda_{ij}dt\right)((r^*-r_i)_+V_t^*{}_tp_xdt+W_i(t)(1-r_idt))+\sum_{j\neq i}\lambda_{ij}dtW_j(t)+o(dt) \\ \frac{dW_i(t)}{dt} &= r_iW_i(t)-(r^*-r_i)_+V_t^*{}_tp_x+\sum_{j\neq i}\lambda_{ij}(W_i(t)-W_j(t)) \end{aligned}$$

such that $W_i(n)=0$ for all $i=1,...,k$. If $X(0)=i$, $W_i(0)$ is the cost of the guarantee.

A slightly different way:
Define

$$\begin{aligned} \tilde{W}_i(t) &= E\left(\int_t^n e^{-\int_t^s (r(u)+\mu_{x+u})du}(r^*-r(s))_+V_s^*ds|X(t)=i\right) \\ \tilde{W}_i(t-dt) &= \left(1-\sum_{j\neq i}\lambda_{ij}dt\right)\left((r^*-r_i)_+V_t^*+\tilde{W}_i(t)\left(1-r_idt-\mu_{x+t}dt\right)\right)+\sum_{j\neq i}\lambda_{ij}dt\tilde{W}_j(t) \\ \frac{d\tilde{W}_i(t)}{dt} &= (r_i+\mu_{x+t})\tilde{W}_i(t)-(r^*-r_i)_+V_t^*+\sum_{j\neq i}\lambda_{ij}(\tilde{W}_i(t)-\tilde{W}_j(t)) \end{aligned}$$

such that $\tilde{W}_i(n)=0$ for all i. If $X(0)=i$, $\tilde{W}_i(0)$ is the cost. Observe that ${}_tp_x\tilde{W}_i(t)=W_i(t)$.

Remarks:

1. The calculated cost of the guarantee should be relatively small.
2. A large value could indicate that the safe basis was too aggressive (not conservative enough). Maybe r^* should have been lower.
3. This measure can be used as an indicator of how "safe" the basis is.

10.1 Unit-linked Insurance

Managed Unit Link

r_t = interest rate at which the fund grows (unknown and variable). We set up two funds:

1. Unit fund (grows at rate r_t): The money there belongs to the insured and they will get it back at maturity or earlier death.
2. Cash fund (grows with a "safe" rate r_t^*): This money will be used to cover any shortfall between the guarantee and the unit fund.

Split the premium between the unit fund and the cash fund (in such a way that there is no shortfall). $\gamma_t \pi_t$ goes to the cash fund. $(1-\gamma_t)\pi_t$ goes to the unit fund.

$$\begin{aligned}\text{Rate of growth of the unit fund is } \frac{dU_t}{dt} &= r_t U_t + (1-\gamma_t)\pi_t \\ \text{Rate of growth of the cash fund is } \frac{dV_t}{dt} &= (r_t^* + \mu_{x+t})V_t + \gamma_t\pi_t - \mu_{x+t}(g - U_t)_+\end{aligned}$$

If there is a survival benefit: $V_n = (g - U_n)_+$. We would like this to be positive.

Example: Endowment assurance without reserves

Let π be the premium, r is the interest, b is the sum assured (guaranteed). Choose γ_t such that

$$\begin{aligned}\gamma_t \pi &= \mu_{x+t}(b - U_t) \\ (1-\gamma_t)\pi &= \pi - \mu_{x+t}(b - U_t)\end{aligned}$$

Then

$$\frac{dV_t}{dt} = (r^* + \mu_{x+t})V_t$$

$V_0 = 0$, so $V_t = 0$ for all t. The cash fund is 0 at all times. Since $U_0 = 0$, we have $b - U(0) > 0$ so for small t we will have $b - U(t) \geq 0$.

$$\begin{aligned}\frac{dU_t}{dt} &= rU_t + \pi - \mu_{x+t}(b - U_t)_+ \\ &= rU_t + \pi - \mu_{x+t}(b - U_t) \\ &= (r + \mu_{x+t})U_t + \pi - \mu_{x+t}b\end{aligned}$$

We can compute

$$U_t = e^{\int_0^t (r+\mu_{x+u})du} \int_0^t (\pi - \mu_{x+s}b)e^{-\int_0^s (r+\mu_{x+u})du} ds$$

Suppose we choose $\pi = \frac{\bar{A}_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}} b$. Then the accumulation of the fund at time n,

$$\begin{aligned}U_n &= e^{\int_0^n (r+\mu_{x+u})du} \int_0^n \left((\pi - \mu_{x+s}b)e^{-\int_0^s (r+\mu_{x+u})du} \right) ds \\ &= e^{\int_0^n (r+\mu_{x+u})du} \left(\pi \bar{a}_{x:\overline{n}|} - b\bar{A}^1_{x:\overline{n}|} \right) \\ &= e^{\int_0^n (r+\mu_{x+u})du} \left(b\bar{A}^1_{x:\overline{n}|} + be^{-\int_0^n (r_u+\mu_{x+u})du} - b\bar{A}^1_{x:\overline{n}|} \right) \\ &= e^{\int_0^n (r+\mu_{x+u})du} \left(be^{-\int_0^n (r+\mu_{x+u})du} \right) \\ &= b\end{aligned}$$

Since $U(t)$ is an increasing function, we will have that $b - U(t) \geq 0$ for all $t \leq n$. So $(b - U(t))_+ = b - U(t)$ for all $t \leq n$.

If the premium is larger than the actuarial premium, $\pi > b\frac{\bar{A}_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}}$, then

$$U(n) \quad = \quad e^{\int_0^n (r+\mu_{x+u})du} \left(b\frac{\bar{A}_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}} \bar{a}_{x:\overline{n}|} - b\bar{A}_{x:\overline{n}|} \right) = b$$

So if $U(n) > b$, there exists an m such that $U(m) = b$, and for $t > m$, $U(t) > b$. So $(b - U(t))_+ = 0$. In this case, the unit fund will work as follows.

$$\begin{aligned} \text{For } t \le m, \ \frac{dU(t)}{dt} &= rU(t) + \pi - \mu_{x+t}(b - U(t)) \\ \text{For } m < t \le n, \ \frac{dU(t)}{dt} &= rU(t) + \pi \qquad \text{such that } U(m) = b \end{aligned}$$

In general,

$$U(t) = be^{r(t-m)} + \pi \int_m^t e^{\int_s^t r du} ds$$

Now, if $\pi < b\frac{\bar{A}_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}}$, then $U(n) < b$. So $U(t) < b$ for all $t \le n$. So $(b - U(t))_+ = (b - U(t))$ for all $t \le n$. But in this case, we did not make the sum assured and we do not know where to find it from. This is not good, so we have to charge enough premium, or there will be a loss.

However, $r(t)$ can follow a stochastic model. If $r(t)$ is stochastic, then there is no guarantee that $U(t) \ge b$. If $U(n) < b$, then there will be no way to pay the survival benefit. By the principle of equivalence,

$$E\left[\int_0^n \left(\gamma_t \pi - \mu_{x+t}(b - U(t))_+\right) e^{-r^* t} {}_t p_x dt - (b - U(n))_+ e^{-r^* n} {}_n p_x \right] = 0$$

If we set $\gamma_t \pi = \mu_{x+t}(b - U(t))_+$, the LHS becomes:

$$-E\left[(b - U(n))_+ e^{-r^* n} {}_n p_x \right] < 0$$

We did not put in enough in the cash fund. Hence, we should choose γ_t such that $\gamma_t \pi > \mu_{x+t}(b - U(t))_+$ so net money will be going into the cash fund. We have

$$\gamma_t \pi = \mu_{x+t}(b - U(t))_+ + \alpha_t \pi$$

How do we choose α_t? It should be such that

$$E\left[\int_t^n \alpha_s \pi e^{-r^* s} {}_s p_x - (b - U(n))_+ e^{-r^* n} {}_n p_x + V_t | \mathcal{F}_t \right] = 0$$

Mark to market:

Observe every day V_t and U_t. Calculate $E[(b - U(n))_+ | \mathcal{F}_t]$ using α_t that has been used the previous day. Choose new α_t such that the above equation is satisfied. Continue and repeat every day. Note of course that if at any point, $U(t) = b$, scrap the whole thing and stop putting money into the cash fund because we have made the guarantee.

Very important: How do we choose the premium?

Assume a "safe" interest rate r for the growth of the unit fund $r > r^*$ but still low enough. Then assume (from experience and relevant simulation) a future α.

$$\begin{aligned} \frac{dV_t}{dt} &= (r^* + \mu_{x+t})V_t + \alpha_t \pi \\ \frac{dU_t}{dt} &= rU(t) + \pi - \mu_{x+t}(b - U(t))_+ - \alpha_t \pi \end{aligned}$$

where $V_0 = 0$ and $U_0 = 0$. Solve and find $U(n)$ and $V(n)$ for a given premium. Calculate $U(n)+V(n)$ and choose π such that $U(n) + V(n) = b$.

11 Pensions

1. Defined contributions
   (a) Contributions defined in terms of salary.
   (b) Financial risk is with the member.
   (c) The accumulated value at the time of retirement is random.
   (d) There is uncertainty about interest rate at retirement.
2. Defined benefits

Profit reporting should be as follows:

1. Predict the accumulated value of the contributions of retirement (A).
2. For various interest rates calculate A and $\bar{a}_y$.

We will use a force of interest $r(t)$ and a force of mortality μ_{x+t}. Also, we use a salary scale which will be such that the salary at time t is

$$S_t = S_0 e^{\int_0^t a(u)du}$$

where $a(t)$ is the force of salary inflation. When we need a stochastic model, we will use a larger Markov chain model, where $X(t)$ takes values from $1, 2, ..., k$ and when $X(t) = i$, $r(t) = r_i$ and $a(t) = a_i$. Transition rates λ_{ij} as before.

11.1 Defined Contributions

The contribution is usually a proportion γ of the salary at any time. Contribution rate is γS_t. If γ depends on time, then contribution rate is $\gamma_t S_t$. The accumulation at time n is

$$\int_0^n \gamma S_s e^{\int_s^n (r(u)+\mu_{x+u})du} ds$$

Assume there are no death benefits. (If contributions are returned on death, then the accumulation will be $\int_0^n \gamma S_s e^{\int_s^n r(u)du} ds$.) Since $S_s = S_0 e^{\int_0^s a(u)du}$, we have the accumulation at time n

$$\gamma S_0 \int_0^n e^{\int_0^s a(u)du} e^{\int_s^n (r(u)+\mu_{x+u})du} ds$$

If we use a Markov chain model, we will have to predict:

$$
\begin{aligned}
& \gamma S_0 E\left(\int_0^n e^{\int_0^s a(u)du} e^{\int_s^n (r(u)+\mu_{x+u})du} ds|\mathcal{F}_t\right) \\
= \;& \gamma S_0 E\left(\int_0^t e^{\int_0^s a(u)du} e^{\int_s^n (r(u)+\mu_{x+u})du} ds|\mathcal{F}_t\right) + \gamma S_0 E\left(\int_t^n e^{\int_0^s a(u)du} e^{\int_s^n (r(u)+\mu_{x+u})du} ds|\mathcal{F}_t\right) \\
= \;& \gamma S_0 \int_0^t e^{\int_0^s a(u)du} e^{\int_s^t (r(u)+\mu_{x+u})du} ds E\left(e^{\int_t^n (r(u)+\mu_{x+u})du}|\mathcal{F}_t\right) \\
& +\gamma S_0 e^{\int_0^t a(u)du} E\left(e^{\int_t^s a(u)du} e^{\int_s^n (r(u)+\mu_{x+u})du} ds|\mathcal{F}_t\right)
\end{aligned}
$$

Define

$$
\begin{aligned}
V_i(t) &= E\left(e^{\int_t^n (r(u)+\mu_{x+u})du}|X(t)=i\right) \\
W_i(t) &= E\left(\int_t^n e^{\int_t^s a(u)du} e^{\int_s^n (r(u)+\mu_{x+u})du} ds|X(t)=i\right)
\end{aligned}
$$

Then we have the differential equations

$$
\begin{aligned}
V_i(t-dt) &= (1+(r_i+\mu_{x+t})dt)V_i(t)\left(1-\sum_{j\neq i}\lambda_{ij}dt\right)+\sum_{j\neq i}\lambda_{ij}V_j(t)dt+o(dt) \\
-\frac{dV_i(t)}{dt} &= (r_i+\mu_{x+t})V_i(t)-\sum_{j\neq i}\lambda_{ij}(V_i(t)-V_j(t)) \\
\frac{dV_i(t)}{dt} &= -(r_i+\mu_{x+t})V_i(t)+\sum_{j\neq i}\lambda_{ij}(V_i(t)-V_j(t))
\end{aligned}
$$

For $W_i(t)$, we have

$$
\begin{aligned}
W_i(t-dt) &= \left(1-\sum_{j\neq i}\lambda_{ij}dt\right)\left((1+a_i dt)W_i(t)+E\left(e^{\int_t^n (r(u)+\mu_{x+u})du}|X(t)=i\right)dt\right)+\sum_{j\neq i}\lambda_{ij}dtW_j(t) \\
&= \left(1-\sum_{j\neq i}\lambda_{ij}dt\right)(1+a_i dtW_i(t)+V_i(t)dt)+\sum_{j\neq i}\lambda_{ij}dtW_j(t)+o(dt) \\
\frac{W_i(t-dt)-W_i(t)}{dt} &= a_iW_i(t)+V_i(t)-\sum_{j\neq i}\lambda_{ij}(W_i(t)-W_j(t))+\frac{o(dt)}{dt} \\
\frac{dW_i(t)}{dt} &= -a_iW_i(t)-V_i(t)+\sum_{j\neq i}\lambda_{ij}(W_i(t)-W_j(t))
\end{aligned}
$$

Solve both sets of equations together such that $V_i(n)=1$ and $W_i(n)=0$ for all i. Our prediction will be (if $X(t)=i$)

$$U(t)V_i(t)+\gamma S_t W_i(t)$$

where

$$U(t)=\gamma S_0\int_0^t e^{\int_0^s a(u)du} e^{\int_s^t (r(u)+\mu_{x+u})du} ds$$

11.2 Defined Benefits

Use the principle of equivalence to calculate γ, the proportion of salary to contribute.

$$\text{EPV(contributions)} = \text{EPV(benefits)} + \text{EPV(expenses)}$$

The EPV of future contributions is

$$\begin{aligned}\int_0^n \gamma S_s e^{-\int_0^s (r(u)+\mu_{x+u})du} ds &= \gamma S_0 \int_0^n e^{\int_0^s a(u)du} e^{-\int_0^s (r(u)+\mu_{x+u})du} ds \\ &= \gamma S_0 \int_0^n e^{-\int_0^s (r(u)+\mu_{x+u}-a(u))du} ds\end{aligned}$$

If stochastic, we use the Markov chain model. We need to calculate

$$\begin{aligned}& E\left(\gamma S_0 \int_0^n e^{-\int_0^s (r(u)+\mu_{x+u}-a(u))du} ds | \mathcal{F}_t\right) \\ = \;& \gamma S_0 \int_0^t e^{-\int_0^s (r(u)+\mu_{x+u})du} ds + \gamma S_0 e^{-\int_0^t (r(u)+\mu_{x+u}-a(u))du} E\left(\int_t^n e^{-\int_t^s (r(u)+\mu_{x+u})du} ds | \mathcal{F}_t\right)\end{aligned}$$

Now we define

$$W_i(t) = E\left(\int_t^n e^{-\int_t^s (r(u)+\mu_{x+u})du} ds | X(t) = i\right)$$

Then

$$\begin{aligned}W_i(t-dt) &= \left(1 - \sum_{j\neq i} \lambda_{ij} dt\right) ((1 - (r_i + \mu_{x+t} - a_i)dt)\, W_i(t) + 1dt) + \sum_{j\neq i} \lambda_{ij} W_j(t) dt + o(dt) \\ \frac{W_i(t-dt) - W_i(t)}{dt} &= -(r_i + \mu_{x+t} - a_i) W_i(t) + 1 - \sum_{j\neq i} \lambda_{ij}(W_i(t) - W_j(t)) + \frac{o(dt)}{dt} \\ \frac{dW_i(t)}{dt} &= (r_i + \mu_{x+t} - a_i) W_i(t) - 1 + \sum_{j\neq i} \lambda_{ij}(W_i(t) - W_j(t))\end{aligned}$$

such that $W_i(n) = 0$ for all i. The EPV at time t of future contributions is $\gamma S_0 W_i(t)$ if $X(t) = i$.

Expenses

Suppose they occur at a rate C_t. EPV of expenses is

$$\int_0^\infty C_s e^{-\int_0^s (r(u)+\mu_{x+u})du} ds$$

If we assume a stochastic model, $E\left(\int_0^\infty C_s e^{-\int_0^s (r(u)+\mu_{x+u})du} ds\right)$ can be calculated by solving

$$\frac{dW_i^{(c)}(t)}{dt} = (r_i + \mu_{x+t}) W_i^{(c)}(t) - C_t + \sum_{j\neq i} \lambda_{ij}(W_i^{(c)}(t) - W_j^{(c)}(t))$$

such that $W_i^{(c)}(\infty) = 0$. EPV of expense is thus $W_i^{(c)}(0)$ if $X(0) = i$.

Benefits: Pensions

Final salary scheme: A continuous pension payable at rate αS_n where S_n is the salary at retirement. Usually $\alpha = \frac{n}{80}$ or $\frac{n}{60}$ where n is the number of years of service. If aged x at time 0 and age of retirement is 65, then $\alpha = \frac{65-x}{80}$. EPV of pension is

$$\begin{aligned}
& \int_n^\infty \alpha S_n e^{-\int_0^s (r(u)+\mu_{x+u})du} ds \\
= \quad & \alpha S_0 e^{\int_0^n a(u)du} \int_n^\infty e^{-\int_0^s (r(u)+\mu_{x+u})du} ds \\
= \quad & \alpha S_0 e^{\int_0^n a(u)du} e^{-\int_0^n (r(u)+\mu_{x+u})du} \int_n^\infty e^{-\int_n^s (r(u)+\mu_{x+u})du} ds
\end{aligned}$$

Suppose we want to calculate

$$\begin{aligned}
& E\left(\alpha S_0 e^{-\int_0^n (r(u)+\mu_{x+u}-a(u))du} \int_n^\infty e^{-\int_n^s (r(u)+\mu_{x+u})du} ds\right) \\
= \quad & E\left(E\left(\alpha S_0 e^{-\int_0^n (r(u)+\mu_{x+u}-a(u))du} \int_n^\infty e^{-\int_n^s (r(u)+\mu_{x+u})du} ds|\mathcal{F}_n\right)\right) \\
= \quad & \alpha S_0 E\left(e^{-\int_0^n (r(u)+\mu_{x+u}-a(u))du} E\left(\int_n^\infty e^{-\int_n^s (r(u)+\mu_{x+u})du} ds|\mathcal{F}_n\right)\right)
\end{aligned}$$

Define

$$W_i(t) = E\left(\int_t^\infty e^{-\int_t^s (r(u)+\mu_{x+u})du} ds|X(t)=i\right)$$

for $t > n$. From before,

$$\frac{dW_i(t)}{dt} = (r_i + \mu_{x+t})W_i(t) - 1 + \sum_{j\neq i} \lambda_{ij}(W_i(t) - W_j(t))$$

and $W_i(\infty) = 0$ for all i. Then

$$W_i(n) = E\left(\int_n^\infty e^{-\int_n^s (r(u)+\mu_{x+u})du} |X(n)=i\right)$$

Now, define

$$\begin{aligned}
V_i(t) &= E\left(e^{-\int_t^n (r(u)+\mu_{x+u}-a(u))du} W_{X(n)}(n)|X(t)=i\right) \\
V_i(t-dt) &= \left(1-(r_i+\mu_{x+t}-a_i)dt\right) V_i(t) \left(1 - \sum_{j\neq i}\lambda_{ij}dt\right) + \sum_{j\neq i}\lambda_{ij}V_j(t) \\
\frac{V_i(t-dt)-V_i(t)}{dt} &= -(r_i+\mu_{x+t}-a_i)V_i(t) - \sum_{j\neq i}\lambda_{ij}(V_i(t)-V_j(t)) + \frac{o(dt)}{dt} \\
\frac{dV_i(t)}{dt} &= (r_i+\mu_{x+t}-a_i)V_i(t) + \sum_{j\neq i}\lambda_{ij}(V_i(t)-V_j(t))
\end{aligned}$$

such that $V_i(n) = W_i(n)$. $W_i(n)$ has been calculated from the previous set of equations. Hence, the procedure would be:

1. Solve the set of differential equations for $W_i(t)$ such that $W_i(\infty) = 0$ for all i.

2. Extract all $W_i(n)$.
3. Solve the set of differential equations for $V_i(t)$ such that $V_i(n) = W_i(n)$ for all i.
4. EPV of benefits is $\alpha S_0 V_i(0)$ for $X(0) = i$.

Varying retirement time: Suppose the member can choose to retire at any time between n_1 and n_2. We assume the deterministic case but if we want to use stochastic just need to calculate expectation of everything.

It is safe to assume the member will try to maximise the EPV of the pension:

$$\frac{n}{80}\int_n^\infty S_n e^{-\int_n^s (r(u)+\mu_{x+u})du} ds$$

Find which n in $[n_1, n_2]$ maximise the quantity and do the calculation for this n. Usually but not always it is n_2. In practise, however, people do not always retire at the optimal time. Suppose for example the time of retirement is distributed between n_1 and n_2, with density $f(n)$ (for survivors). Then

$$\begin{aligned}
\text{EPV of pension} &= \int_{n_1}^{n_2} \frac{n}{80}\int_n^\infty S_n e^{-\int_0^s (r(u)+\mu_{x+u})du} ds f(n) dn \\
\text{EPV of contribution} &= \int_{n_1}^{n_2}\int_0^n \gamma S_t e^{-\int_0^t (r(u)+\mu_{x+u}-a(u))du} dt f(n) dn
\end{aligned}$$

Other pension schemes:

1. Final salary but it is the average over the last j years.

$$\text{EPV benefit} = \frac{n}{80}\int_n^\infty \frac{\int_{n-j}^n S_t dt}{j} e^{-\int_0^s (r(u)+\mu_{x+u})du} ds$$

2. Career average

$$\text{EPV benefit} = \frac{n}{80}\int_n^\infty \frac{\int_0^n S_t dt}{n} e^{-\int_0^s (r(u)+\mu_{x+u})du} ds$$

3. Maximum over the last j years.

$$\text{EPV benefit} = \frac{n}{80}\int_n^\infty \left(\max_{n-j\le t\le n} S_t\right) e^{-\int_0^s (r(u)+\mu_{x+u})du} ds$$

Other benefits

Lump sum payable on retirement of 1 or 2 times final salary βS_n.

$$\begin{aligned}
\text{EPV benefit} &= \beta S_n e^{-\int_0^n (r(u)+\mu_{x+u})du} \\
&= \beta S_0 e^{\int_0^n a(u)du} e^{-\int_0^n (r(u)+\mu_{x+u})du}
\end{aligned}$$

If we assume a stochastic model, we already have calculated

$$\beta S_0 E\left(e^{-\int_0^n (r(u)+\mu_{x+u}-a(u))du}\right)$$

Death benefits

On death before time n, a lump sum βS_t is paid if death occurs at time t, $t < n$. Then

$$\begin{aligned}\text{EPV benefit} &= \int_0^n \beta S_0 e^{\int_0^t a(u)du} e^{-\int_0^t (r(u)+\mu_{x+u})du} \mu_{x+t} dt \\ &= \beta S_0 \int_0^n e^{-\int_0^s (r(u)+\mu_{x+u}-a(u))du} \mu_{x+s} ds\end{aligned}$$

With a stochastic model, we want to calculate the EPV

$$\beta S_0 E\left(\int_0^n e^{-\int_0^s (r(u)+\mu_{x+u}-a(u))du} \mu_{x+s} ds\right)$$

Define

$$\begin{aligned}W_i(t) &= E\left(\int_t^n e^{-\int_t^s (r(u)+\mu_{x+u}-a(u))du} \mu_{x+s} ds | X(t)=i\right) \\ \frac{dW_i(t)}{dt} &= (r_i + \mu_{x+t} - a_i) W_i(t) - \mu_{x+t} + \sum_{j \neq i} \lambda_{ij}(W_i(t) - W_j(t))\end{aligned}$$

Solve such that $W_i(n) = 0$. The EPV of benefit is $\beta S_0 W_i(0)$ if $X(0) = i$.

12 Factors Affecting Mortality and Mortality Indices

1. Nutritions and Lifestyle
2. Climate
3. Health care services
4. Housing
5. Genetics
6. Marital status
7. Occupation
8. Education

This leads to selection.

12.1 Selection

Selection in life insurance

1. Temporary initial selection (due to underwriting).
2. Class selection (gender, occupation, education etc).
3. Time selection. In general mortality improves with time but this improvement can vary according to age and sex.
4. Adverse selection. Ill people have more incentive to buy life insurance.

5. Spurious selection. It manifests itself as temporary initial selection that does not wear off.

We should produce as many separate tables as possible. Failure to differentiate will mean that we discourage lives with low mortality and encourage lives with high mortality and leading to insolvency. However, we should not over-classify because small sets of data is unreliable. Also, classification can be costly, e.g. DNA testing.

Selection in pensions

All of the above are important especially so adverse selection. The office should be possible to produce separate tables for

1. Active lives
2. Normal retirement
3. Ill health retirement
4. Early retirement
5. Deferred retirement (indicate good health)

12.2 Mortality Indices

To find a single figure to describe the mortality of a region, a country or a group of people etc.

Suppose we are given either

E_x : initial exposed to risk for age group x in an area (population at the start of the year of measurement)
q_x : probability of death of the age group in the area
${}^sE_x$: some quantities for a standard population (eg the country as a whole)
${}^sq_x$: probability of death for a standard population

or

$E^c_{x,t}$: central exposed-to-risk over the period $(x, x+t)$
$M^c_{x,t}$: central death rates approximate to the force of mortality
${}^sE_{x,t}$: quantities for a standard population
${}^sM_{x,t}$: death rates for a standard population

Indices:

1. Crude death rate

$$\begin{aligned} \text{CDR} &= \frac{\sum_x E_x q_x}{\sum_x E_x} = \frac{\sum_x E_{x,t} M_{x,t}}{\sum_x E^c_{x,t}} \\ &= \frac{\#\text{ of death}}{\text{population}} \end{aligned}$$

   Obviously this can be very misleading. For example if area A has lower mortality rate than area B but much older population it will probably have a higher CDR.

2. Standardised mortality rate (SMRate): instead of using the population of the area we use the standard population (direct standardisation).

$$\text{SMRate} = \frac{\sum_x^s E_x q_x}{\sum_x^s E_x} = \frac{\sum_x^s E_{x,t}^c M_{x,t}}{\sum_x^s E_{x,t}^c}$$

   A criticism of this is that we might not have age specific rates for the area.

3. Standardised mortality ratio (SMRatio): compare the actual mortality experience of the area, with what it would have been if we had the standard mortality experience instead (indirect standardisation).

$$\begin{aligned}\text{SMRatio} &= \frac{\text{Actual deaths}}{\text{"Expected" deaths}} \\ &= \frac{\sum_x E_x q_x}{\sum_x E_x^s q_x} = \frac{\sum_x E_{x,t}^c M_{x,t}}{\sum_x E_{x,t}^c {}^s M_{x,t}}\end{aligned}$$

We can even produce an indirectly standardised mortality rate: $SMRatio \times CDR_{\text{standard}}$. $SMRatio > 1$ could mean higher mortality than "standard". $SMRatio < 1$ could mean lower mortality than "standard". We can even perform a standardised test. If the "standard" mortality is applicable, the expected death is $E = \sum_x E_x^s q_x$. The distribution of this is binomial and also approximately Poisson with parameter E so the variance is approximately E. And so it can be approximated by a normal distribution $\mathcal{N}(E, E)$. Thus, we can look at the test statistic $\frac{A-E}{\sqrt{E}}$ and perform a 2-sided or 1-sided test depending on the circumstances. For example, at the 5% (2-sided) level, there is a significant difference if $\left|\frac{A-E}{\sqrt{E}}\right| > 1.96$ (one-sided this would be 2.5%). All the above indices use the number of deaths, this is heavily geared towards old ages which will be more influential on your conclusions.

Example of WHO standard:

$$\begin{aligned}CDR_{US} &= \frac{0.068 \times 1.7 + 0.142 \times 0.2 + ... + 0.016 \times 148.3}{1} \\ &= 8.5 \qquad \text{(per 1000)} \\ \text{SMRate for US} &= \frac{0.089 \times 1.7 + 0.173 \times 0.2 + ... + 0.006 \times 148.3}{1} \\ &= 5.5 \\ \text{SMRate for Venezuela} &= 5.8 \\ \text{SMRatio for US} &= \frac{8.5}{0.068 \times 10 + 0.142 \times 1 + ... + 0.016 \times 100} \\ &= 1.25 \\ \text{SMRatio for Venezuela} &= \frac{4.2}{0.111 \times 10 + 0.216 \times 1 + ... + 0.003 \times 100} \\ &= 1.08\end{aligned}$$

If you are young, it is better to be in US since lower mortality rate. But after 55, the mortality rate is higher in US. Both countries have low mortality rate for young and high mortality rate for old people.