ST305 - Actuarial Mathematics: Life

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1 Introduction

1.1 Actuarial Notations

Consider one individual. Define the following:

$$\begin{array}{rcl} T_x & : & \text{remaining lifetime of a life aged } x \\ \mu_{x+t} & : & \text{force of mortality} \\ {}_tp_x & : & \text{survival probability} \\ {}_tq_x & : & \text{probability of death within time } [0,t] \\ f(t) & : & \text{density of } T_x \end{array}$$

We have the following expressions:

$$\begin{aligned} {}_tp_x &= e^{-\int_0^t \mu_{x+s}ds} \\ {}_tq_x &= 1 - {}_tp_x = P(T_x \le t) \\ f(t) &= \mu_{x+t}e^{-\int_0^t \mu_{x+s}ds} = \mu_{x+t}\,{}_tp_x \\ E(T_x) &= \int_0^\infty tf(t)dt = \int_0^\infty t\,{}_tp_x\mu_{x+t}dt \\ &= \int_0^\infty P(T_x > t)dt = \int_0^\infty {}_tp_x dt \end{aligned}$$

1.2 Some insurance contracts

1.2.1 Actuarial principle - Expected present value

A payment of 1 at time t is worth $e^{-rt} = v^t$.
A payment of 1 made at time T_x is worth $e^{-rT_x} = v^{T_x}$ now. (A random quantity)
The value of a contract is the expected present value of its payments, in this case $E(e^{-rT_x})$. Below are some examples of insurance contracts and their expected present value:

1. **Whole life assurance**
   A payment of 1 made at time T_x. Payment is worth e^{-rT_x} now.

$$E[e^{-rT_x}] = \int_0^\infty e^{-rt}\,{}_tp_x\mu_{x+t}dt = \bar{A}_x$$

2. **Pure endowment**
   A payment of 1 is made at time n if life is alive. Payment is worth $e^{-rn}\mathbf{1}_{\{T_x>n\}}$ now.

$$\begin{aligned} E\left[e^{-rn}\mathbf{1}_{\{T_x>n\}}\right] &= e^{-rn}P(T_x>n) \\ &= e^{-rn}{}_np_x \\ &= e^{-rn}e^{-\int_0^n \mu_{x+t}dt} \\ &= e^{-\int_0^n (r+\mu_{x+t})dt} = A_{x:\overset{1}{\overline{n}|}} = {}_nE_x \end{aligned}$$

3. **Temporary Life Assurance**
   A payment of 1 immediately on death if that occurs before time n. Present value is $e^{-rT_x}\mathbf{1}_{\{T_x\leq n\}}$.

$$\begin{aligned} E\left[e^{-rT_x}\mathbf{1}_{\{T_x\leq n\}}\right] &= \int_0^n e^{-rt}f(t)dt = \int_0^n e^{-rt}\mu_{x+t\,t}p_x dt \\ &= \int_0^n e^{-rt}\mu_{x+t}e^{-\int_0^t \mu_{x+s}ds}dt = \bar{A}^1_{x:\overline{n}|} \end{aligned}$$

4. **Endowment Assurance**
   A payment of 1 either immediately upon death or at time n, whichever comes first. Present value is $e^{-rn}\mathbf{1}_{\{T_x>n\}} + e^{-rT_x}\mathbf{1}_{\{T_x\leq n\}} = e^{-r(T_x\wedge n)}$.

$$\begin{aligned} E\left[e^{-r(T_x\wedge n)}\right] &= A_{x:\overset{1}{\overline{n}|}} + \bar{A}^1_{x:\overline{n}|} \\ &= \bar{A}_{x:\overline{n}|} \end{aligned}$$

1.2.2 Variances

For the above examples, we have

1. **Whole life assurance**

$$\begin{aligned} \text{Present value} &= e^{-rT_x} \\ 2^{nd}\text{ moment} &= E\left[e^{-2rT_x}\right] \\ &= \bar{A}_x \text{ at force of interest } 2r \\ \text{Variance} &= \bar{A}_x \text{ at } 2r - \left(\bar{A}_x\right)^2 \text{ at } r \end{aligned}$$

2. **Pure endowment**

$$\begin{aligned} \text{Present value} &= e^{-rn}\mathbf{1}_{\{T_x>n\}} \\ 2^{nd}\text{ moment} &= E\left[e^{-2rn}\mathbf{1}_{\{T_x>n\}}\right] \\ &= e^{-2rn}{}_np_x \\ &= A_{x:\overset{1}{\overline{n}|}} \text{ at } 2r \\ \text{Variance} &= A_{x:\overset{1}{\overline{n}|}} \text{ at } 2r - \left(A_{x:\overset{1}{\overline{n}|}}\right)^2 \text{ at } r \end{aligned}$$

3. **Temporary Life Assurance**

$$\begin{aligned} \text{Present value:} &= e^{-rT_x}\mathbf{1}_{\{T_x\leq n\}} \\ 2^{nd}\text{ moment:} &= E\left[e^{-2rT_x}\mathbf{1}_{\{T_x\leq n\}}\right] \\ &= \bar{A}^1_{x:\overline{n}|} \text{ at } 2r \\ \text{Variance} &= \bar{A}^1_{x:\overline{n}|} \text{ at 2r} - \left(\bar{A}^1_{x:\overline{n}|}\right)^2 \text{ at r} \end{aligned}$$

4. **Endowment Assurance**

$$\begin{aligned}
\text{Present value} &= e^{-r(T_x \wedge n)} \\
2^{nd} \text{ moment} &= E\left[e^{-2r(T_x \wedge n)}\right] \\
&= \bar{A}_{x:\overline{n}|} \text{ at } 2r \\
\text{Variance} &= \bar{A}_{x:\overline{n}|} \text{ at } 2r - \left(\bar{A}_{x:\overline{n}|}\right)^2 \text{ at } r
\end{aligned}$$

1.3 Annuities

Annuity certain: $a_{\overline{n}|}$, $\ddot{a}_{\overline{n}|}$, $\bar{a}_{\overline{n}|}$

1. **Whole Life annuity (Continuous):**
   1 per annum payable continuously till death.

$$\begin{aligned}
\text{Present value} &= \bar{a}_{\overline{T_x}|} = \frac{1 - e^{-rT_x}}{r} \\
\text{Expected value} &= \bar{a}_x = E\left[\bar{a}_{\overline{T_x}|}\right] = \frac{1 - \bar{A}_x}{r} \Rightarrow \bar{A}_x + r\bar{a}_x = 1 \\
\text{Variance} &= Var\left(\frac{1 - e^{-rT_x}}{r}\right) \\
&= \frac{1}{r^2} Var(e^{-rT_x}) \\
&= \frac{1}{r^2}\left(\bar{A}_x \text{ at } 2r - \left(\bar{A}_x\right)^2 \text{ at } r\right)
\end{aligned}$$

   Alternatively, we also have

$$\begin{aligned}
\bar{a}_{\overline{T_x}|} &= \int_0^\infty \mathbf{1}_{\{T_x > t\}} e^{-rt} dt \\
\bar{a}_x &= E\left[\int_0^\infty \mathbf{1}_{\{T_x > t\}} e^{-rt} dt\right] \\
&= \int_0^\infty P(T_x > t) e^{-rt} dt \\
&= \int_0^\infty {}_t p_x e^{-rt} dt
\end{aligned}$$

2. **Whole Life annuity (Discrete):**

$$\begin{aligned}
\ddot{a}_{\overline{T_x}|} &= \sum_{j=0}^\infty e^{-rj} \mathbf{1}_{\{T_x > j\}} \\
\ddot{a}_x &= \sum_{j=0}^\infty e^{-rj} {}_j p_x \\
a_x &= \sum_{j=1}^\infty e^{-rj} {}_j p_x \\
\ddot{a}_x &\approx \bar{a}_x + \frac{1}{2} \text{ for whole life annuities}
\end{aligned}$$

3. **Temporary annuity (Continuous):**
   1 per annum payable continuously till death or time n, whichever comes first.

$$\begin{aligned}
\text{Present value} &= \bar{a}_{\overline{T_x \wedge n}|} = \frac{1 - e^{-r(T_x \wedge n)}}{r} \\
\text{Expected value} &= \bar{a}_{x:\overline{n}|} = \frac{1 - \bar{A}_{x:\overline{n}|}}{r} \Rightarrow \bar{A}_{x:\overline{n}|} + r\bar{a}_{x:\overline{n}|} = 1 \\
\text{Variance} &= Var\left(\frac{1 - e^{-r(T_x \wedge n)}}{r}\right) \\
&= \frac{1}{r^2} Var\left(e^{-r(T_x \wedge n)}\right) \\
&= \frac{1}{r^2}\left(\bar{A}_{x:\overline{n}|} \text{ at 2r} - \left(\bar{A}_{x:\overline{n}|}\right)^2 \text{ at } r\right)
\end{aligned}$$

   We also have

$$\begin{aligned}
\bar{a}_{\overline{T_x \wedge n}|} &= \int_0^n \mathbf{1}_{\{T_x > t\}} e^{-rt} dt \\
\bar{a}_{x:\overline{n}|} &= \int_0^n {}_t p_x e^{-rt} dt
\end{aligned}$$

4. **Temporary annuity (Discrete):**

$$\begin{aligned}
\ddot{a}_{x:\overline{n}|} &= \sum_{j=0}^{n-1} e^{-rj} {}_j p_x \\
a_{x:\overline{n}|} &= \sum_{j=1}^{n} e^{-rj} {}_j p_x
\end{aligned}$$

1.4 Principle of Equivalence

Expected P.V. of premiums = EPV(benefits) + EPV(expenses)
Ignoring expenses for the time being, EPV(premiums) = EPV(benefits)

Example: Suppose we have a whole life assurance financed by a life annuity of P per annum, then

$$\begin{aligned}
P\bar{a}_x &= \bar{A}_x \\
P &= \frac{\bar{A}_x}{\bar{a}_x}
\end{aligned}$$

If the premium is only payable till time n, then

$$\begin{aligned}
P\bar{a}_{x:\overline{n}|} &= \bar{A}_x \\
P &= \frac{\bar{A}_x}{\bar{a}_{x:\overline{n}|}}
\end{aligned}$$

Example: For a temporary life assurance, we have

$$\begin{aligned}
P\bar{a}_{x:\overline{n}|} &= \bar{A}^1_{x:\overline{n}|} \\
P &= \frac{\bar{A}^1_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}}
\end{aligned}$$

If premium is payable annually in advance,

$$P = \frac{\bar{A}^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$$

Example: Deferred annuity (pensions):
A whole life (or temporary) annuity commencing at a fixed time m provided the policyholder is alive then.

$$\begin{aligned}\text{Expected present value} = {}_{m|}\bar{a}_{x:\overline{n}|} &= e^{-rm}{}_m p_x \bar{a}_{x+m:\overline{n}|} \\ &= \bar{a}_{x:\overline{m+n}|} - \bar{a}_{x:\overline{m}|} \\ &= \bar{a}_x - \bar{a}_{x:\overline{m}|} \text{ (for } n = \infty)\end{aligned}$$

For a simple pension contract where premium is paid till time m and life annuity starts thereafter, we have

$$P = \frac{{}_{m|}\bar{a}_x}{\bar{a}_{x:\overline{m}|}}$$

2 Reserves

Money set aside to finance the rest of the contract. Two kinds of reserves:

1. **Prospective Reserves**: Value at time t of future liability minus future income = EPV(future benefits) - EPV(future premiums)

2. **Retrospective Reserves**: Net accumulation of money already received.
   If interest rates remain the same, prospective and retrospective reserves should equal.

Example: Consider a pure endowment, financed by a single premium P payable in advance.

$$P = v^n {}_n p_x = A_{x:\overline{n}|}^{\;\;1} = {}_nE_x$$

At time t,

$$\begin{aligned}\text{Prospective reserve, } V_t &= v^{n-t}{}_{n-t}p_{x+t} \\ \text{Retrospective reserve, } V_t^{(R)} &= Pe^{rt} \times \frac{1}{{}_tp_x} \\ &= Pe^{\int_0^t (r+\mu_{x+s})ds} \\ &= e^{-\int_0^n (r+\mu_{x+s})ds} e^{\int_0^t (r+\mu_{x+s})ds} \\ &= e^{-\int_t^n (r+\mu_{x+s})ds} \\ &= e^{-\int_0^{n-t} (r+\mu_{x+t+s})ds} \\ &= V_t\end{aligned}$$

The "accumulation" of P at time t is $Pe^{\int_0^t (r+\mu_{x+s})ds}$.

2.1 Thiele's Differential Equation

At any time, a premium π_t is being paid continuously and there is a death benefit b_t. Consider a small time interval $(t, t + dt)$.

Retrospective argument

$$\begin{aligned} V_{t+dt} &= V_t + r_t V_t dt + \pi_t dt + \mu_{x+t} dt (V_t - b_t) + o(dt) \\ V_{t+dt} - V_t &= (r + \mu_{x+t}) V_t dt + \pi_t dt - \mu_{x+t} b_t dt + o(dt) \\ \frac{dV_t}{dt} &= (r + \mu_{x+t}) V_t + \pi_t - \mu_{x+t} b_t \end{aligned}$$

Solve subject to $V_0 = \pi_0$, the premium upfront.

Prospective argument

$$\begin{aligned} V_t &= (1 - r_t dt)(1 - \mu_{x+t} dt) V_{t+dt} - \pi_t dt + \mu_{x+t} dt b_t + o(dt) \\ \frac{dV_t}{dt} &= (r_t + \mu_{x+t}) V_t + \pi_t - \mu_{x+t} b_t \end{aligned}$$

subject to $V_{n-} = B$, the terminal benefit. In the case of whole life assurance, the condition is $\lim_{t \to \infty} V_t = 0$.

Solutions:
Retrospective

$$V_t = \int_0^t e^{\int_s^t (r_u + \mu_{x+u}) du} \left(\pi_s - \mu_{x+s} b_s \right) ds + \pi_0 e^{\int_0^t (r_u + \mu_{x+u}) du}$$

Prospective

$$V_t = \int_t^n e^{-\int_t^s (r + \mu_{x+u}) du} \left(\mu_{x+s} b_s - \pi_s \right) ds + B e^{-\int_t^n (r_s + \mu_{x+s}) ds}$$

Example: Pure endowment - single premium

Retrospective reserve:

$$\begin{aligned} V_{t+dt} &= V_t + V_t r dt + V_t \mu_{x+t} dt \\ \frac{V_{t+dt} - V_t}{dt} &= \frac{V_t r dt + V_t \mu_{x+t} dt}{dt} + \frac{o(dt)}{dt} \end{aligned}$$

As $dt \to 0$, $\frac{dV_t}{dt} = (r + \mu_{x+t}) V_t$, $V_0 = P$. Solving, $V_t = P e^{\int_0^t (r + \mu_{x+s}) ds}$.

Prospective reserve:

$$\begin{aligned} V_t &= (1 - r dt)(1 - \mu_{x+t} dt) V_{t+dt} \\ e^{-r dt} &\approx 1 - r dt + o(dt) \\ e^{-\int_t^{t+dt} \mu_{x+s} ds} &\approx 1 - \mu_{x+t} dt + o(dt) \\ \frac{V_t - V_{t+dt}}{dt} &= \frac{-(r + \mu_{x+t}) dt}{dt} V_{t+dt} + \frac{o(dt)}{dt} \end{aligned}$$

Letting $dt \to 0$, $\frac{dV_t}{dt} = (r + \mu_{x+t}) V_t$, $V_{n-} = 1$. Solving, $V_t = e^{-\int_0^{n-t} (r + \mu_{x+t+s}) ds}$ = retrospective.

Example: Pure endowment with premium P payable continuously.

$$P = \frac{v^n {}_n p_x}{\bar{a}_{x:\overline{n}|}}$$

Retrospective reserve:

$$\begin{aligned}
V_{t+dt} &= V_t + V_t r dt + \mu_{x+t} V_t dt + P dt + o(dt) \\
\frac{V_{t+dt} - V_t}{dt} &= (r + \mu_{x+t}) V_t + P + \frac{o(dt)}{dt} \\
\frac{dV_t}{dt} &= (r + \mu_{x+t}) V_t + P
\end{aligned}$$

and $V_0 = 0$.

Prospective reserve:

$$\begin{aligned}
V_t &= (1 - r dt)(1 - \mu_{x+t} dt) V_{t+dt} - P dt \\
\frac{dV_t}{dt} &= (r + \mu_{x+t}) V_t + P
\end{aligned}$$

and $V_n = 1$. Solving,

$$\begin{aligned}
V_t &= e^{-r(n-t)}{}_{n-t}p_{x+t} - P \bar{a}_{x+t:\overline{n-t}|} \\
&= \text{EPV(benefits) - EPV(premiums)}
\end{aligned}$$

2.2 Stochastic process approach

I_t: Stochastic process.

$$I_t = \begin{cases} 1 & \text{if life is alive} \\ 0 & \text{if life is dead} \end{cases}$$

N_t: No of deaths up to time t.

$$\begin{aligned}
dN_t &= N_{t+dt} - N_t = \text{ no. of deaths in } [t, t+dt) \\
I_t &= 1 - \int_0^t dN_s
\end{aligned}$$

Present value at time t of future payment:
Consider time $t \leq s \leq n$, where b_s payable if death occurs, B payable at time n if alive, and premium π_s payable when alive.
Cashflow at time s (small interval): $b_s dN_s - \pi_s I_s ds$
Discount factor: $e^{-r(s-t)} = e^{-\int_t^s r_u du}$ (if constant i.r.)
So P.V. $= e^{-r(s-t)} \left[b_s dN_s - \pi_s I_s ds \right]$.

Prospective reserve:

$$W_t = \int_t^n e^{-r(s-t)} \left(b_s dN_s - \pi_s I_s ds \right) + e^{-r(n-t)} B I_n$$

W_t is a random variable, so we take expectation.

$$\begin{aligned}
V_t &= E(W_t | \mathcal{F}_t) \\
&= E\left[\int_t^n e^{-r(s-t)} b_s dN_s - \int_t^n e^{-r(s-t)} \pi_s I_s ds + e^{-r(n-t)} B I_n | \mathcal{F}_t \right] \\
&= \int_t^n e^{-r(s-t)} b_s E(dN_s | \mathcal{F}_t) - \int_t^n e^{-r(s-t)} \pi_s E(I_s | \mathcal{F}_t) ds + e^{-r(n-t)} E(I_n | \mathcal{F}_t) B
\end{aligned}$$

Since $E(dN_s|\mathcal{F}_t) = E(N_{s+ds} - N_s|\mathcal{F}_t) = {}_{s-t}p_{x+t}\mu_{x+s}ds$ and $E(I_s|\mathcal{F}_t) = {}_{s-t}p_{x+t}$, we have

$$E(W_t|\mathcal{F}_t) = \int_t^n e^{-r(s-t)} b_s {}_{s-t}p_{x+t}\mu_{x+s}ds - \int_t^n e^{-r(s-t)}\pi_s {}_{s-t}p_{x+t}ds + e^{-r(n-t)}{}_{n-t}p_{x+t}B$$

Retrospective reserve:

$$W_t^{(R)} = \int_0^t (\pi_s I_s - b_s dN_s)\, e^{r(t-s)}ds$$

for $0 \le s < t$. Retrospective reserve is its expectation shared among the survivors

$$V_t^{(R)} = \frac{E(W_t^{(R)})}{{}_tp_x}$$

Example: Pure endowment - Single premium

$$\begin{aligned}\text{Prospective reserve at time t} &= e^{-r(n-t)}{}_{n-t}p_{x+t} \\ &= e^{-\int_t^n (r+\mu_{x+s})ds}\end{aligned}$$

$V_n = 1$, $V_0 = e^{-rn}{}_np_x$ the single premium. And the reserve increases with time.
Or solve:

$$\frac{dV_t}{dt} = (r + \mu_{x+t})V_t$$

subject to $V_n = 1$.

$$\begin{aligned}\text{Retrospective reserve at time t} &= e^{-rn}{}_np_x e^{\int_0^t (r+\mu_{x+s})ds} \\ &= \text{Prospective reserves}\end{aligned}$$

Or solve:

$$\frac{dV_t}{dt} = (r + \mu_{x+t})V_t$$

subject to $V_0 = e^{-rn}{}_np_x$.

Example: Temporary Assurance - Continuous premium payable for n years, 1 payable on death provided it is before time n
Solve:

$$\frac{dV_t}{dt} = (r + \mu_{x+t})V_t dt + \pi - \mu_{x+t}$$

subject to the conditions $V_n = 0$ for prospective reserve, and $V_0 = 0$ for retrospective. Solving this, we have

$$\begin{aligned}V_t &= \int_0^t (\pi - \mu_{x+s})e^{\int_s^t (r+\mu_{x+u})du}ds \\ V_n &= \int_0^n (\pi - \mu_{x+s})e^{\int_s^n (r+\mu_{x+u})du}ds = 0\end{aligned}$$

If μ_x is an increasing function of x, we will have $\pi - \mu_{x+s} > 0$ for some $s \le m$ and $\pi - \mu_{x+s} < 0$ for some $s > m$, so V_t is increasing for $t < m$ and decreasing for $t > m$. On the other hand, if μ_x is decreasing in x, we have negative reserves, which is impossible.

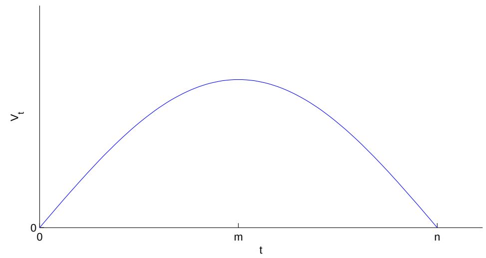

Figure 1: Positive reserves when μ_x is increasing

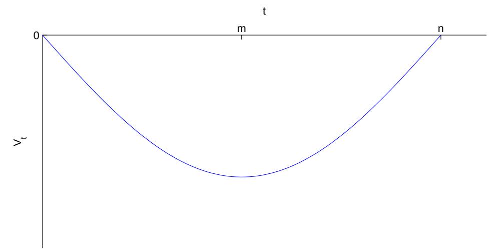

Figure 2: Negative reserves when μ_x is decreasing

Example: Variable premium - same as above but $\pi_t = \mu_{x+t}$
Thiele's equation:

$$\frac{dV_t}{dt} = (r + \mu_{x+t})V_t dt$$

subject to $V_0 = 0$. Solving this, we get $V_t = 0$ for all t. There are no reserves. Office just takes premiums and pays them out. This is impractical but very safe for the office.

Example: Endowment Assurance - 1 payable upon death or at time n, whichever comes first, continuous premium π

$$\begin{aligned} V_t &= \bar{A}_{x+t:\overline{n-t}|} - \pi \bar{a}_{x+t:\overline{n-t}|} \\ &= 1 - r\bar{a}_{x+t:\overline{n-t}|} - \frac{1 - r\bar{a}_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}} \bar{a}_{x+t:\overline{n-t}|} \\ &= 1 - \frac{\bar{a}_{x+t:\overline{n-t}|}}{\bar{a}_{x:\overline{n}|}} \end{aligned}$$

Assuming μ_x is increasing in x, we have

$$\begin{aligned} \bar{a}_{x+t:\overline{n-t}|} &= \int_t^n e^{-\int_t^s (r+\mu_{x+u})du} ds \\ &< \int_t^n e^{-\int_t^s (r+\mu_{x+t})du} \\ &= \frac{1}{r + \mu_{x+t}} - e^{-(r+\mu_{x+t})(n-t)} \\ &\leq \frac{1}{r + \mu_{x+t}} \end{aligned}$$

Now

$$\begin{aligned} \frac{d}{dt}\bar{a}_{x+t:\overline{n-t}|} &= (r + \mu_{x+t})\bar{a}_{x+t:\overline{n-t}|} - 1 \\ &< (r + \mu_{x+t})(\frac{1}{r + \mu_{x+t}}) - 1 = 0 \end{aligned}$$

Since $\bar{a}_{x+t:\overline{n-t}|}$ is decreasing, V_t is an increasing function of t.

Example: Temporary annuity - 1 payable till death or time n
Thiele's equation:

$$\frac{dV_t}{dt} = (r + \mu_{x+t})V_t + \pi - b\mu_{x+t}$$

We have $\pi = -1$, $b\mu_{x+t} = 0$, $V_n = 0$.

$$V_t = \bar{a}_{x+t:\overline{n-t|}}$$

3 Selection

(Read section 3.4 of Lifebook)
Forms of selection:

- Class selection (e.g. gender)
- Time selection (e.g. mortality improves over time)
- Self selection (e.g. antiselection)
- Temporary initial selection (e.g. time selection that wells off)

Select period: s
μ_{x+t} and ${}_tp_x$ does not lead to a proper model, because it does not take into account when the underwriting took place. So we write for a person aged x at the time of underwriting,

$$\begin{aligned} \mu_x(t) &= \mu_{x+t} \text{ for } t \geq s \\ &= \mu_{[x]+t} \end{aligned}$$

${}_\tau q_{[x]+t}$ is the probability of death within τ years for a life aged $x+t$, but went through underwriting at age x (t years ago).

$$\begin{aligned} {}_\tau q_{[x]+t} &= {}_\tau q_{x+t} \text{ for } t \geq s \\ &= P(T_x \leq t+\tau | T_x > t) \\ {}_\tau p_{[x]+t} &= P(T_x > t+\tau | T_x > t) \\ \mu_{[x]+t} &= \lim_{h\to 0} \frac{{}_hq_{[x]+t}}{h} \end{aligned}$$

Example: s=2 (selection up to time 2)

$$\begin{aligned} {}_2p_{[x]} &= {}_1p_{[x]} \times {}_1p_{[x]+1} \\ {}_3p_{[x]} &= {}_1p_{[x]} \times {}_1p_{[x]+1} \times {}_1p_{x+2} \\ \ddot{a}_{[x]} &= 1 + v \times {}_1p_{[x]} + v^2 \times {}_2p_{[x]} + v^3 \times {}_1p_{[x]} \times {}_1p_{[x]+1} \times {}_1p_{x+2} + ... \end{aligned}$$

4 Expenses

(Read Chapter 5 of Lifebook)
Types of expenses:

- Fixed: constant rate η
- Proportional to the sum assured: αb
- Proportional to premium: $\beta\pi$
- Proportional to reserve: γV_t

Expenses occur at rate $\eta + \alpha b + \beta\pi + \gamma V_t$. Thiele's equation becomes

$$\begin{aligned} V_{t+dt} &= V_t + rV_t dt + \mu_{x+t}(V_t - b)dt + \pi dt - (\eta + \alpha b + \beta\pi + \gamma V_t)dt \\ \frac{dV_t}{dt} &= (r + \mu_{x+t})V_t + \pi - \mu_{x+t}b - (\eta + \alpha b + \beta\pi + \gamma V_t) \\ &= (r - \gamma + \mu_{x+t})V_t + ((1-\beta)\pi - \eta - \alpha b) - \mu_{x+t}b \end{aligned}$$

Solve subject to $V_n = B$. We have the Thiele's equation with changed parameters:

$$V_t = \int_t^n e^{-\int_t^s (r-\gamma+\mu_{x+u})du} \left(\mu_{x+s}b - (1-\beta)\pi + \eta + \alpha b\right) ds + Be^{-\int_t^n (r-\gamma+\mu_{x+u})du}$$

We can calculate the premium π. Since $V_0 = 0$,

$$\begin{aligned} 0 &= \int_0^n e^{-\int_0^s (r-\gamma+\mu_{x+u})du} \left(\mu_{x+s}b - (1-\beta)\pi + \eta + \alpha b\right) ds + Be^{-\int_0^n (r-\gamma+\mu_{x+u})du} \\ &= \bar{A}^1_{x:\overline{n}|}b - (1-\beta)\pi\bar{a}_{x:\overline{n}|} + (\eta + \alpha b)\bar{a}_{x:\overline{n}|} + B_n p_x e^{-(r-\gamma)n} \\ (1-\beta)\pi\bar{a}_{x:\overline{n}|} &= \bar{A}^1_{x:\overline{n}|}b + (\eta + \alpha b)\bar{a}_{x:\overline{n}|} + B_n p_x e^{-(r-\gamma)n} \end{aligned}$$

where $\bar{a}_{x:\overline{n}|}$ and $\bar{A}^1_{x:\overline{n}|}$ are at force $r - \gamma$. We can get the premium from this equation.

5 Joint life

5.1 Notations

Multiple insurance
Notation:
One life: $\bar{A}_x$, $\bar{A}_{x:\overline{n}|}$, $\bar{A}^1_{x:\overline{n}|}$, $A_{x:\overset{1}{\overline{n}|}}$, $\bar{a}_x$, $\bar{a}_{x:\overline{n}|}$, $\bar{a}_{\overline{x:\overline{n}|}}$.

Now, suppose we have two lives aged x and y. Then

$$\begin{aligned} \bar{A}_{xy} &= \text{1 payable when } xy \text{ breaks on the } 1st \text{ death} \\ \bar{A}^1_{xy} &= \text{1 payable on the death of } x\text{, provided } y \text{ is alive} \\ \bar{A}_{xy} &= \bar{A}^1_{xy} + \bar{A}_{xy}{}^1 \\ \bar{A}_{\overline{xy}} &= \text{1 payable on the second death} \\ \bar{A}_{xy}{}^2 &= \bar{A}^1_{xy} \\ \bar{A}_{\overline{xy}}{}^2 &= \text{1 payable on the death of } y \text{ provided } x \text{ is already dead} \\ \bar{A}_{\overline{xy}:\overline{n}|}{}^2 &= \text{Same as } \bar{A}_{\overline{xy}}{}^2 \text{ before, but only if it happens within } n \text{ years} \\ \bar{A}_{\overline{xy}:\overline{n}|} &= \text{1 payable on the second death, provided it is before time } n \\ \bar{A}_{xy:\overset{1}{\overline{n}|}} &= \text{pure endowment (both alive at } n) \end{aligned}$$

Suppose lives: $1, 2, ..., r$, ages: $x_1, x_2, ..., x_r$.
Remaining lifetimes: $T_{x_1}, T_{x_2}, ..., T_{x_r}$ (Denote by: $T_1, T_2, ..., T_r$)
Joint status: $(x_1, x_2, ..., x_r)$

$$\begin{aligned} T &= \min(T_1, T_2, ..., T_r) \\ P(T > t) &= P(\min(T_1, T_2, ..., T_r) > t) \\ &= P(T_1 > t, T_2 > t, ..., T_r > t) \\ &= {}_t p_{x_1,...,x_r} \end{aligned}$$

If they are independent, then

$$
\begin{aligned}
P(T>t) &= P(T_1>t)P(T_2>t)\ldots P(T_r>t) \\
&= {}_tp_{x_1} \times {}_tp_{x_2} \times \ldots \times {}_tp_{x_r} \\
{}_tp_{x_1\ldots x_r} &= e^{-\int_0^t \mu_{x_1\ldots x_r+s}ds} \\
&= e^{-\int_0^t \mu^{(1)}_{x_1+s}ds} e^{-\int_0^t \mu^{(2)}_{x_2+s}ds} \ldots e^{-\int_0^t \mu^{(r)}_{x_r+s}ds} \\
\mu_{x_1\ldots x_r+t} &= \mu^{(1)}_{x_1+t} + \mu^{(2)}_{x_2+t} + \ldots + \mu^{(r)}_{x_r+t}
\end{aligned}
$$

GM mortality law:

$$
\begin{aligned}
\mu^{(i)}_x &= A_i + B_i e^{C_i x} \\
\mu^{(1)}_x + \ldots + \mu^{(r)}_x &= (A_1+\ldots+A_r) + (B_1+\ldots+B_r)e^{Cx} \text{ also GM law}
\end{aligned}
$$

5.2 Common joint life contracts

1. **Pure endowment -** 1 payable at time n if all are alive.

$$
\begin{aligned}
\text{PV} &= v^n \mathbf{1}_{\{T>n\}} \\
A_{x_1\ldots x_r:\overset{1}{\overline{n|}}} &= {}_nE_{x_1\ldots x_r} \\
&= v^n {}_np_{x_1\ldots x_r}
\end{aligned}
$$

2. **Whole life assurance -** 1 payable at first death

$$
\begin{aligned}
\text{PV} &= v^T = v^{\min(T_1,\ldots,T_r)} \\
\bar{A}_{x_1\ldots x_r} &= \int_0^\infty e^{-rt}\mu_{x_1\ldots x_r+t}\,{}_tp_{x_1\ldots x_r}dt \\
&= \int_0^\infty e^{-rt}(\mu_{x_1+t}+\mu_{x_2+t}+\ldots+\mu_{x_r+t})\,{}_tp_{x_1\ldots x_r}dt
\end{aligned}
$$

3. **Temporary assurance -** 1 payable on first death provided it is before time n

$$
\bar{A}_{\overset{1}{\widetilde{x_1\ldots x_r}}:\overline{n|}} = \int_0^n e^{-rt}(\mu_{x_1+t}+\mu_{x_2+t}+\ldots+\mu_{x_r+t})\,{}_tp_{x_1\ldots x_r}dt
$$

4. **Endowment assurance -** 1 payable on first death or at time n, whichever comes first

$$
\bar{A}_{x_1\ldots x_r:\overline{n|}} = \bar{A}_{\overset{1}{\widetilde{x_1\ldots x_r}}:\overline{n|}} + A_{x_1\ldots x_r:\overset{1}{\overline{n|}}}
$$

Example: 1 payable on death of x_1 if everybody else is alive

$$
\bar{A}_{\overset{1}{x_1}x_2\ldots x_r} = \int_0^\infty e^{-rt}\mu_{x_1+t}\,{}_tp_{x_1\ldots x_r}dt
$$

Note that $\bar{A}_{x_1\ldots x_r} = \bar{A}_{\overset{1}{x_1}\ldots x_r} + \ldots + \bar{A}_{x_1\ldots \overset{1}{x_r}}$.

Example: 1 payable on death of x_1 if everybody else is alive and it happens within n years

$$
\bar{A}_{\overset{1}{x_1}\ldots x_r:\overline{n|}} = \int_0^n e^{-rt}\mu_{x_1+t}\,{}_tp_{x_1\ldots x_r}dt
$$

Note: $\bar{A}_{x_1...x_r:\overline{n|}} = \bar{A}^{\ 1}_{x_1...x_r:\overline{n|}} + \bar{A}_{x_1...x_r:\overline{n|}}^{\quad\ \ 1}$.

Example: Annuity payable if everybody is alive and up to time n (n can be ∞)

$$\bar{a}_{x_1...x_r:\overline{n|}} = \int_0^n e^{-rt}{}_tp_{x_1...x_r}dt$$

5.3 Last survivor status

More complicated case: we are interested in the time of the last death.

$$\begin{aligned}
T &= \max(T_1, T_2, ..., T_r) \\
P(T \le t) &= P(\max(T_1, ..., T_r) \le t) \\
&= P(T_1 \le t, T_2 \le t, ..., T_r \le t) \\
&= P(T_1 \le t)P(T_2 \le t)...P(T_r \le t) \text{ if independent} \\
{}_tq_{x_1...x_r} &= {}_tq_{x_1} \times {}_tq_{x_2} \times ... \times {}_tq_{x_r} \\
&= (1 - {}_tp_{x_1})(1 - {}_tp_{x_2})...(1 - {}_tp_{x_r}) \\
&= 1 - {}_tp_{\overline{x_1...x_r}}
\end{aligned}$$

where ${}_tp_{\overline{x_1...x_r}}$ = prob. of at least one surviving.

Example: Annuity payable up to the last death

$$\begin{aligned}
\bar{a}_{\overline{x_1...x_r}} &= \int_0^\infty e^{-rt}{}_tp_{\overline{x_1...x_r}}dt \\
&= \int_0^\infty e^{-rt}\left(1 - (1 - {}_tp_{x_1})(1 - {}_tp_{x_2})...(1 - {}_tp_{x_r})\right)dt
\end{aligned}$$

Example: Pure endowment - 1 payable at time n if at least one life is alive

$$\begin{aligned}
\text{PV} &= v^n \mathbf{1}_{\{\max(T_1...T_r)>n\}} = v^n[1 - \mathbf{1}_{\{\max(T_1...T_r)\le n\}}] \\
\text{Expected value} &= v^n[1 - {}_tq_{x_1} \times {}_tq_{x_2} \times ... \times {}_tq_{x_r}] \\
\text{For 2 lives,} & \quad v^n[1 - {}_tq_x \times {}_tq_y]
\end{aligned}$$

Example: (Not typical) 2 lives, x and y. 1 is payable on the second death of x and y.

$$\begin{aligned}
\bar{A}_{\overline{xy}} &= E[e^{-r(T_x \vee T_y)}] \\
&= E[e^{-r(T_x \vee T_y)}\mathbf{1}_{\{T_x>T_y\}} + e^{-r(T_x \vee T_y)}\mathbf{1}_{\{T_y>T_x\}}] \\
&= E[e^{-rT_x}\mathbf{1}_{\{T_x>T_y\}}] + E[e^{-rT_y}\mathbf{1}_{\{T_y>T_x\}}] \\
&= \int_0^\infty e^{-rt}{}_tp_x\mu_{x+t}(1 - {}_tp_y)dt + \int_0^\infty e^{-rt}{}_tp_y\mu_{y+t}(1 - {}_tp_x)dt
\end{aligned}$$

Alternatively, we also have

$$\begin{aligned}
\bar{A}_{\overline{xy}} &= \bar{A}_x + \bar{A}_y - \bar{A}_{xy} \\
&= \int_0^\infty e^{-rt}\mu_{x+t}{}_tp_xdt + \int_0^\infty e^{-rt}\mu_{y+t}{}_tp_ydt - \int_0^\infty e^{-rt}{}_tp_x{}_tp_y(\mu_{x+t} + \mu_{y+t})dt
\end{aligned}$$

Suppose we have continuous premiums as long as both are alive, then

$$\pi = \frac{\bar{A}_{\overline{xy}}}{\bar{a}_{xy}}$$

If payable as long as somebody is alive,

$$\pi = \frac{\bar{A}_{\overline{xy}}}{\bar{a}_{\overline{xy}}}$$

and we have the following relationships

$$\begin{aligned}
\bar{a}_x + \bar{a}_y &= \bar{a}_{\overline{xy}} + \bar{a}_{xy} \\
\bar{a}_{\overline{xy}} &= \int_0^\infty e^{-rt}(1 - {}_tq_{x\,t}q_y)dt
\end{aligned}$$

If the contract finishes at time n,

$$\pi = \frac{\bar{A}_{\overline{xy}:\overline{n}|}}{\bar{a}_{xy:\overline{n}|}}$$

Example: (Most common contract) 1 payable on the first death provided it happens before time n. Premium payable as long as they are both alive up to time n.

$$\pi = \frac{\bar{A}^{\,1}_{\widetilde{xy}:\overline{n}|}}{\bar{a}_{xy:\overline{n}|}}$$

5.4 Some conditional probabilities

1. In the previous contract, the amount was paid before time n. What is the probability that y got the money (x died first)?

$$\begin{aligned}
P(T_y > T_x | \min(T_x, T_y) \leq n) &= \frac{P(T_y > T_x, \min(T_x, T_y) \leq n)}{P(\min(T_x, T_y) \leq n)} \\
&= \frac{P(T_y > T_x, T_x \leq n)}{1 - {}_np_{x\,n}p_y} \\
&= \frac{1}{1 - {}_np_{x\,n}p_y} E\left[\int_0^n \mathbf{1}_{\{T_y > t\}}\, {}_tp_x\mu_{x+t}dt\right] \\
&= \frac{1}{1 - {}_np_{x\,n}p_y}\int_0^n {}_tp_{x\,t}p_y\mu_{x+t}dt
\end{aligned}$$

2. Somebody died at time t (exactly), and (s)he has died first. What is the probability that it was x that died?

$$\begin{aligned}
\lim_{dt\to 0}\frac{P(T_x \in [t, t+dt), T_y > t)}{P(T_x \wedge T_y \in [t, t+dt))} &= \lim_{dt\to 0}\frac{P(T_x \in [t, t+dt))P(T_y > t)}{P(T_x \in [t, t+dt))P(T_y > t) + P(T_y \in [t, t+dt))P(T_x > t)} \\
&= \frac{{}_tp_x\mu_{x+t}dt\,{}_tp_y}{{}_tp_x\mu_{x+t}dt\,{}_tp_y + {}_tp_y\mu_{y+t}\,{}_tp_xdt} \\
&= \frac{\mu_{x+t}}{\mu_{x+t} + \mu_{y+t}}
\end{aligned}$$

This extends to m lives: If we know the time of the first death is t, the probability that it is life $j, j = 1, ..., m$ is $\frac{\mu_{x_j+t}}{\sum_{i=1}^m \mu_{x_i+t}}$.

5.5 More examples

Example: Deferred annuities - On the death of x, y gets a life annuity.

$$\begin{aligned}
\text{PV} &= {}_{T_x|}\bar{a}_{\overline{T_y}|}\mathbf{1}_{\{T_y > T_x\}} \\
\text{Expected value} &= {}_{x|}\bar{a}_y \\
&= E\left[\int_{T_x}^{T_y} e^{-rt}dt\mathbf{1}_{\{T_y>T_x\}}\right] \\
&= \left[\int_0^\infty \mathbf{1}_{\{y \text{ is alive and } x \text{ is dead at time } t\}} e^{-rt}dt\right] \\
&= \int_0^\infty {}_tp_y(1-{}_tp_x)e^{-rt}dt \\
&= \bar{a}_y - \bar{a}_{xy}
\end{aligned}$$

Example: On the first death, the survivor gets a life annuity

$$\begin{aligned}
{}_{x|}\bar{a}_y + {}_{y|}\bar{a}_x &= \int_0^\infty e^{-rt}\mathbf{1}_{\{\text{exactly one is alive}\}}dt \\
&= \int_0^\infty e^{-rt}[{}_tp_y(1-{}_tp_x) + {}_tp_x(1-{}_tp_y)]dt \\
&= \int_0^\infty e^{-rt}[1 - {}_tp_{xt}p_y - {}_tq_{xt}q_y]dt
\end{aligned}$$

Example: Two lives, 1 payable on second death provided this occurs before time n. There will be 3 (4) reserves.

- $V_0(t) = 0$: Reserve if both dead
- $V_x(t)$: Reserve to be set up if only x is alive
- $V_y(t)$: Reserve if only y is alive
- $V_{xy}(t)$: Reserve if both are alive

Assume premium is payable continuously as long as both are alive (till time n).

$$\begin{aligned}
V_{xy}(t) &= E[\text{PV of future payments - benefits}|\text{both are alive}] \\
&= \bar{A}_{\overline{x+t,y+t}:\overline{n-t}|} - \pi\bar{a}_{x+t,y+t:\overline{n-t}|} \\
V_x(t) &= E[\text{PV of future payments - benefits}|\text{only } x \text{ is alive}] \\
&= \bar{A}_{x+t:\overline{n-t}|} \\
V_y(t) &= E[\text{PV of future payments - benefits}|\text{only } y \text{ is alive}] \\
&= \bar{A}_{y+t:\overline{n-t}|} \\
\pi &= \frac{\bar{A}_{\overline{xy}:\overline{n}|}}{\bar{a}_{xy:\overline{n}|}}
\end{aligned}$$

$V_{xy}(t)$ is thus

$$\begin{aligned}
V_{xy}(t) &= \int_t^n e^{-r(s-t)}{}_{s-t}p_{x+t}\mu_{x+s}(1-{}_{s-t}p_{y+t})ds + \int_t^n e^{-r(s-t)}{}_{s-t}p_{y+t}\mu_{y+t}{}_{s-t}q_{x+t}ds \\
&\quad -\pi\int_t^n e^{-r(s-t)}{}_{s-t}p_{x+t}{}_{s-t}p_{y+t}ds \\
V_{xy}(n) &= 0
\end{aligned}$$

and we can compute the derivative

$$V'_{xy}(t)|_{t=n} = 0 + 0 + \pi > 0$$

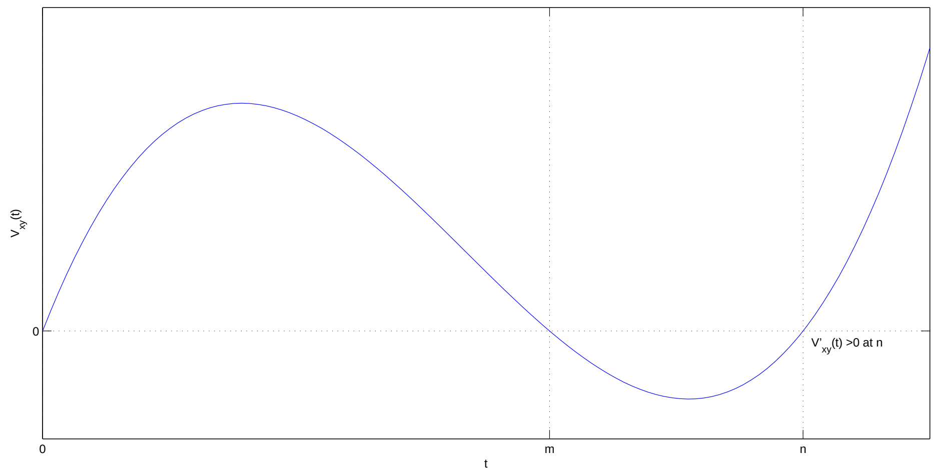

Figure 3: Graph of $V_{xy}(t)$, reserve when both are alive

Hence, $V_{xy}(t)$ is increasing for values just before n. This implies that $V_{xy}(t) < 0$ for some $m < n$. But we are not allowed to have negative reserves, so when this happens, the usual remedy is to have the premium payable up to a time $n_1 < n$ instead of being payable till time n. In which case we will have

$$V'_{xy}(t) = rV_{xy}(t) + \pi \mathbf{1}_{\{t<m\}} - \mu_{x+t}(V_y(t) - V_{xy}(t)) - \mu_{y+t}(V_x(t) - V_{xy}(t))$$

If $n = \infty$, we do not necessarily have a problem.

6 Markov Chains

Let $X(t)$, the state at time t, be a stochastic process that takes values from a finite set $\{1, 2, ..., K\}$, $t \geq 0$. $X(t)$ is a Markov process, $t_1 < t_2 < ... < t_n < t_{n+1}$. So we have

$$\begin{aligned} P(X(t_{n+1}) = j|X(t_n) = j_n, X(t_{n-1}) = j_{n-1}, ..., X(t_0) = j_0) &= P(X(t_{n+1}) = j|X(t_n) = j_n) \\ \mathcal{F}_t &= \sigma\{X(s) : 0 \leq s \leq t\} \\ P(X(t_{n+1}) = j|\mathcal{F}_{t_n}) &= P(X(t_{n+1}) = j|X_{t_n}) \end{aligned}$$

Also, conditionally on the present, the past and the future are independent:

$$P(X(t_3) = j_3, X(t_1) = j_1|X(t_2) = j_2) = P(X(t_3) = j_3|X(t_2) = j_2)P(X(t_1) = j_1|X(t_2) = j_2)$$

Definitions:

- For small h,
$$P(X(t+h) = j|X(t) = i) = \mu_{ij}(t)h + o(h), \qquad i \neq j$$

- P(Two or more transitions in $[t, t+h)) = o(h)$. So

$$
\begin{aligned}
P(X(t+h) = i | X(t) = i) &= 1 - \sum_{j \neq i}^{K} \mu_{ij}(t)h + o(h) \\
&= 1 - \mu_{i.}(t)h + o(h)
\end{aligned}
$$

- $P(X(u) = j | X(t) = i) = p_{ij}(t, u)$, for $t < u$.

Aside:

$$
\mu_{i.}(t) = \sum_{k=1, k \neq i}^{K} \mu_{ik}(t), \qquad \mu_{.j}(t) = \sum_{k=1, k \neq j}^{K} \mu_{kj}(t)
$$

6.1 Kolmogorov Backward Equations

Consider what happens in interval $[t, t+h)$. For $t < u$,

$$
\begin{aligned}
p_{ij}(t, u) &= P(X(u) = j | X(t) = i) \\
&= \sum_{k=1}^{K} P(X(u) = j | X(t+h) = k) P(X(t+h) = k | X(t) = i) \\
&= \sum_{k=1, k \neq i}^{K} p_{kj}(t+h, u)[\mu_{ik}(t)h + o(h)] + p_{ij}(t+h, u)[1 - \mu_{i.}(t)h + o(h)]
\end{aligned}
$$

So

$$
\frac{p_{ij}(t, u) - p_{ij}(t+h, u)}{h} = \frac{\sum_{k=1, k \neq i}^{K} p_{kj}(t+h, u)\mu_{ik}(t)h - \mu_{i.}(t)hp_{ij}(t+h, u) + o(h)}{h}
$$

Let $h \to 0$,

$$
\begin{aligned}
-\frac{\delta p_{ij}(t, u)}{\delta t} &= \sum_{k=1, k \neq i}^{K} p_{kj}(t)\mu_{ik}(t) - p_{ij}(t)\mu_{i.}(t) \\
\frac{\delta p_{ij}(t, u)}{\delta t} &= p_{ij}(t)\mu_{i.}(t) - \sum_{k=1, k \neq i}^{K} p_{kj}(t)\mu_{ik}(t)
\end{aligned}
$$

The Kolmogorov equations for $i = 1, ..., K$ to be solved simultaneously subject to the conditions

$$
\begin{aligned}
p_{ij}(u, u) &= 0, \qquad \text{if } i \neq j \\
p_{jj}(u, u) &= 1
\end{aligned}
$$

Backward equations: To find the probability that we are in a state j at the end given we are at various states (all possibilities) at the start.

6.2 Kolmogorov Forward Equations

Consider what happens in $[t, t+h)$. For $s < t$,

$$
\begin{aligned}
p_{ij}(s,t) &= P(X(t) = j|X(s) = i) \\
p_{ij}(s,t+h) &= \sum_{k=1}^{K} P(X(t+h) = j|X(t) = k)P(X(t) = k|X(s) = i) \\
&= \sum_{k=1,k\neq j}^{K} [\mu_{kj}(t)h + o(h)]p_{ik}(s,t) + (1 - \mu_{.j}h + o(h))p_{ij}(s,t)
\end{aligned}
$$

So

$$\frac{p_{ij}(s,t+h) - p_{ij}(s,t)}{h} = \frac{\sum_{k=1,k\neq j}^{K} \mu_{kj}(t)p_{ik}(s,t) - \mu_{.j}(t)hp_{ij}(s,t) + o(h)}{h}$$

Letting $h \to 0$,

$$\frac{\delta p_{ij}(s,t)}{\delta t} = \sum_{k=1,k\neq j}^{K} \mu_{kj}(t)p_{ik}(s,t) - \mu_{.j}p_{ij}(s,t)$$

The Kolmogorov forward equations are solved for all possible $j = 1, 2, ..., K$ simultaneously, subject to

$$
\begin{aligned}
p_{ij}(s,s) &= 0 \text{ for } j \neq i \\
p_{ii}(s,s) &= 1
\end{aligned}
$$

Forward equations: We know where we are at time s and we want the probabilities of all possible outcomes at a future time. Quite often, $s = 0$ and $p_{ij}(0,t) = p_{ij}(t)$.

6.3 Occupation probabilities

Another interesting probability is the "occupation" probability $p_{\overline{ii}}(s,t)$, the probability of staying in state i from time s to t, without having left state i.

$$
\begin{aligned}
p_{\overline{ii}}(s,t) &= P(X(v) = i, \forall v \in [s,t]|X(s) = i) \\
p_{ii}(s,t) &\geq p_{\overline{ii}}(s,t) \\
p_{\overline{ii}}(s,t+h) &= p_{\overline{ii}}(s,t)p_{\overline{ii}}(t,t+h) \\
&= p_{\overline{ii}}(s,t)p_{ii}(t,t+h) + o(h) \\
&= p_{\overline{ii}}(s,t)\,(1 - \mu_{i.}(t)h) + o(h) \\
\frac{\delta p_{\overline{ii}}(s,t)}{\delta t} &= -\mu_{i.}(t)p_{\overline{ii}}(s,t) \\
p_{\overline{ii}}(s,s) &= 1
\end{aligned}
$$

So we have

$$p_{\overline{ii}}(s,t) = e^{-\int_s^t \mu_{i.}(u)du}$$

Example: Model of competing risks / Multiple decrement model

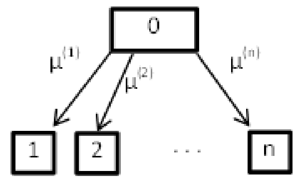

Figure 4: Model of competing risks / Multiple decrement model

Start at 0 and move to $1, ..., n$ and stay there. At time 0, we are in state 0.

$$\begin{aligned} p_{ij}(s,t) &= 0 \text{ for } i \neq 0 \\ \mu^{(i)}(t) &= \mu^{(i)}_{x+t} \\ p_0(t) &= P(X(t)=0) \\ p_j(t) = p_{0j}(t) &= P(X(t)=j) \text{ for } j = 0,1,2,...,n \end{aligned}$$

To find $p_0(t)$, we use the forward equation

$$\begin{aligned} p_0(t+h) &= p_0(t)\left(1 - \sum_{j=1}^{n} \mu^{(j)}_{x+t} h\right) + o(h) \\ p_0(t+h) - p_0(t) &= -\sum_{j=1}^{n} \mu^{(j)}_{x+t} h p_0(t) + o(h) \\ \frac{p_0(t+h)-p_0(t)}{h} &= \frac{-\sum_{j=1}^{n} \mu^{(j)}_{x+t} h p_0(t) + o(h)}{h} \\ p_0'(t) &= -\mu_{x+t} p_0(t) \text{ (where } \mu = \sum_{j=1}^{n} \mu^{(j)}) \end{aligned}$$

subject to $p_0(0) = 1$. Solving this, we get

$$\begin{aligned} p_0(t) &= e^{-\int_0^t \mu_{x+s} ds} \\ &= e^{-\int_0^t (\mu^{(1)}_{x+s} + \mu^{(2)}_{x+s} + ... + \mu^{(n)}_{x+s}) ds} \\ &= e^{-\int_0^t \mu^{(1)}_{x+s} ds} e^{-\int_0^t \mu^{(2)}_{x+s} ds} ... e^{-\int_0^t \mu^{(n)}_{x+s} ds} \end{aligned}$$

For $j = 1, 2, ..., n$, we have

$$\begin{aligned} p_j(t+h) &= p_0(t)\mu^{(j)}_{x+t} h + p_j(t) + o(h) \\ \frac{p_j(t+h)-p_j(t)}{h} &= \frac{p_0(t)\mu^{(j)}_{x+t} h}{h} + \frac{o(h)}{h} \\ p_j'(t) &= p_0(t)\mu^{(j)}_{x+t} \end{aligned}$$

Solving this subject to $p_j(0) = 0$, we have

$$p_j(t) = \int_0^t p_0(s)\mu^{(j)}_{x+s} ds$$

Let T be the time we move out of state 0.

$$\begin{aligned} P(T > t) &= e^{-\int_0^t \mu_{x+s} ds} = p_0(t) \\ \text{Density of } T &= \mu_{x+t} p_0(t) = \sum_{i=1}^{n} \mu_{x+t}^{(i)} p_0(t) \\ &= \lim_{dt \to 0} \frac{P(T \in [t, t+dt))}{dt} \end{aligned}$$

We also have

$$\lim_{dt \to 0} \frac{P(T \in [t, t+dt), X(T) = j)}{dt} = \mu_{x+t}^{(j)} p_0(t)$$

So

$$P(X(T) = i | T = t) = \frac{\mu_{x+t}^{(i)}}{\mu_{x+t}}$$

Suppose there is benefit b_j, payable on transition to state j.

$$\text{EPV(benefits)} = \sum_{j=1}^{n} b_j \int_0^{\infty} e^{-rt} \mu_{x+t}^{(j)} p_0(t) dt$$

Example: James Bond insurance

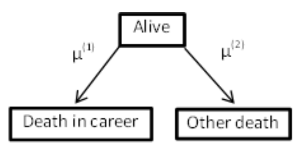

Figure 5: James Bond insurance

Benefit b for other death, $2b$ for death in service.

$$\begin{aligned} \text{EPV(benefits)} &= b \int_0^{\infty} e^{-rs} p_0(s) \mu_{x+s}^{(2)} ds + 2b \int_0^{\infty} e^{-rs} p_0(s) \mu_{x+s}^{(1)} ds \\ p_0(t) &= e^{-\int_0^t (\mu_{x+s}^{(1)} + \mu_{x+s}^{(2)}) ds} \\ p_j(t) &= \int_0^t p_0(s) \mu_{x+s}^{(j)} ds \end{aligned}$$

Example: Two lives x, y

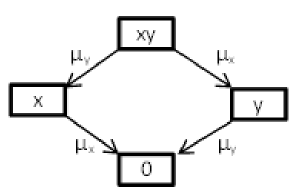

Figure 6: Two lives x, y

We use forward equations to find $p_{xy}(t)$.

$$\begin{aligned}
p_{xy}(t+dt) &= p_{xy}(t)[1-(\mu_{x+t}+\mu_{y+t})dt]+o(dt) \\
\frac{dp_{xy}(t)}{dt} &= -(\mu_{x+t}+\mu_{y+t})p_{xy}(t) \\
p_{xy}(t) &= e^{-\int_0^t(\mu_{x+s}+\mu_{y+s})ds} = {}_tp_{xt}p_y \\
p_{xy}(0) &= 1
\end{aligned}$$

For $p_x(t)$, we have

$$\begin{aligned}
p_x(t+dt) &= p_x(t)(1-\mu_{x+t}dt)+p_{xy}(t)\mu_{y+t}dt+o(dt) \\
\frac{p_x(t+dt)-p_x(t)}{dt} &= -\frac{p_x(t)\mu_{x+t}dt}{dt}+\frac{p_{xy}(t)\mu_{y+t}dt}{dt}+\frac{o(dt)}{dt} \\
p_x'(t) &= -\mu_{x+t}p_x(t)+\mu_{y+t}p_{xy}(t)
\end{aligned}$$

We need to solve this subject to $p_x(0)=0$. After some calculations, we will obtain

$$\begin{aligned}
p_x(t) &= e^{-\int_0^t\mu_{x+u}du}\int_0^t e^{\int_0^s\mu_{x+u}du}\mu_{y+s}p_{xy}(s)ds \\
&= {}_tp_{xt}q_y
\end{aligned}$$

Similarly, $p_y(t)={}_tp_{yt}q_x$. And $p_0(t)=1-p_{xy}(t)-p_x(t)-p_y(t)$. Alternatively,

$$\begin{aligned}
p_0(t+dt) &= p_{xy}(t)o(dt)+p_x(t)\mu_{x+t}dt+p_y(t)\mu_{y+t}dt+p_0(t)+o(dt) \\
p_0'(t) &= \mu_{x+t}p_x(t)+\mu_{y+t}p_y(t)
\end{aligned}$$

Solve subject to $p_0(0)=0$ to get

$$\begin{aligned}
p_0(t) &= \int_0^t(\mu_{x+s}p_x(s)+\mu_{y+s}p_y(s))ds \\
&= \int_0^t {}_sp_x\mu_{x+s\,s}q_yds+\int_0^t {}_sp_y\mu_{y+s\,s}q_yds
\end{aligned}$$

Example: Health-Sickness Model

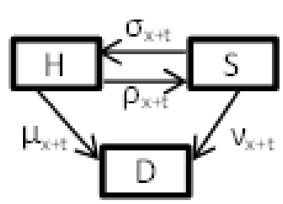

Figure 7: Health-Sickness Model

At time 0, we are in state H (healthy). $X(0)=H$ and $p_H(t)=P(X(t)=H|X(0)=H)$. (We use $x+t$ here, but in practice it can depend on t in other ways). The forward equations are:

$$\begin{aligned}
p_H(t+dt) &= p_H(t)(1-\sigma_{x+t}dt-\mu_{x+t}dt)+p_S(t)\rho_{x+t}dt+o(dt) \\
p_H'(t) &= -(\sigma_{x+t}+\mu_{x+t})p_H(t)+p_S(t)\rho_{x+t}
\end{aligned}$$

with $p_H(0) = 1$, and

$$p'_S(t) = -(\rho_{x+t} + \nu_{x+t})p_S(t) + p_H(t)\sigma_{x+t}$$

with $p_S(0) = 0$. Solve the two equations simultaneously. In special cases, e.g. if $\sigma_{x+t} = \sigma$, $\rho_{x+t} = \rho$, $\mu_{x+t} = \mu$, $\nu_{x+t} = \nu$, we can solve them explicitly.

We denote the backward probabilities (for $t < u$):

$$\begin{aligned}
p_H^B(t) &= P(X(u) = S|X(t) = H) \\
p_S^B(t) &= P(X(u) = S|X(t) = S) \\
p_D^B(t) &= P(X(u) = S|X(t) = D) = 0
\end{aligned}$$

The backward equations are:

$$\begin{aligned}
p_H^B(t) &= P(X(u) = S|X(t) = H) \\
&= P(X(u) = S|X(t + dt) = H)(1 - \sigma_{x+t}dt - \mu_{x+t}dt) \\
&\quad +P(X(u) = S|X(t + dt) = S)\sigma_{x+t}dt \\
&= p_H^B(t + dt)(1 - \sigma_{x+t}dt - \mu_{x+t}dt) + p_S^B(t + dt)\sigma_{x+t}dt + o(dt) \\
\frac{p_H^B(t) - p_H^B(t + dt)}{dt} &= -(\sigma_{x+t} + \mu_{x+t})p_H^B(t + dt) + p_S^B(t + dt)\sigma_{x+t} + \frac{o(dt)}{dt} \\
{p'}_H^B(t) &= (\sigma_{x+t} + \mu_{x+t})p_H^B(t) - \sigma_{x+t}p_S^B(t)
\end{aligned}$$

and $p_H^B(u) = 0$. Similarly, we have for $p_S^B(t)$

$${p'}_S^B(t) = (\rho_{x+t} + \nu_{x+t})p_S^B(t) - \rho_{x+t}p_H^B(t)$$

and $p_S^B(u) = 1$. Solve simultaneously. If we want to find $P(X(u) = H|X(t) = H)$ and $P(X(u) = H|X(t) = S)$ we can use the same equations but with terminal conditions $p_H^B(u) = 1$ and $p_S^B(u) = 0$.

Clearly, the value of an annuity payable as long as I am alive is $\int_0^\infty e^{-rt}(p_H(t) + p_S(t))dt$. The value of the sickness benefit, a continuous annuity payable while sick is $\int_0^\infty e^{-rt}p_S(t)dt$.

Criticisms of this model:

- Recovery does not depend on how long sickness has lasted.
- Recovery does not depend on how many times you have been sick.

6.4 Stochastic Process Approach

We define the following stochastic processes.

$$\begin{aligned}
X(t) &= \{1, 2, ..., K\} \\
I_j(t) &= \begin{cases} 1 & \text{if } X(t) = j \\ 0 & \text{if } X(t) \neq j \end{cases} \\
N_{jk}(t) &= \text{No. of transitions from state } j \text{ to state } k \text{ up to time } t \\
dN_{jk}(t) &= N_{jk}(t + dt) - N_{jk}(t) = \begin{cases} 1 & \text{if } X(t + dt) = k, X(t) = j \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

We have the following expectations.

$$\begin{aligned}
E(I_j(t)) &= P(X(t)=j) \\
E(dN_{jk}(t)) &= E(N_{jk}(t+dt) - N_{jk}(t)) \\
&= P(X(t)=j)P(X(t+dt)=k|X(t)=j) \\
&= P(X(t)=j)\mu_{jk}(t)dt + o(dt) \\
E(N_{jk}(t)) &= \int_0^t P(X(s)=j)\mu_{jk}(s)ds
\end{aligned}$$

Example: Alive-dead model

$$E(N_{AD}(t)) = \int_0^t {}_s p_x \mu_{x+s} ds = {}_t q_x$$

Example: States $\{1, ..., K\}$. Suppose we have continuous payments with rate $b_j(t)$ made while at state j at time t. Discrete payment $B_{jk}(t)$ on transition from state j to state k at time t. Final payment at time n: A_j if then at state j. Continuous premium $\pi_j(t)$ in state j at time t.

$$\begin{aligned}
\text{PV of benefits} &= \int_0^n \left(\sum_{j=1}^K b_j(t) e^{-\int_0^t r(s)ds} I_j(t) \right) dt \\
&\quad + \sum_{j=1}^K \sum_{k=1, k\neq j}^K \left(\int_0^n B_{jk}(t) dN_{jk}(t) e^{-\int_0^t r(s)ds} \right) \\
&\quad + \sum_{j=1}^K A_j I_j(n) e^{-\int_0^n r(s)ds} \\
\text{PV of premiums} &= \sum_{j=1}^K \left(\int_0^n \pi_j(t) e^{-\int_0^t r(s)ds} I_j(t) dt \right)
\end{aligned}$$

Actuarial Principle: EPV(benefits)=EPV(premiums).

$$\begin{aligned}
\text{EPV(benefits)} &= \sum_{j=1}^K \left(\int_0^n b_j(t) e^{-\int_0^t r(s)ds} P(X(t)=j) dt \right) \\
&\quad + \sum_{j=1}^K \sum_{k=1, k\neq j}^K \left(\int_0^n B_{jk}(t) e^{-\int_0^t r(s)ds} P(X(t)=j)\mu_{jk}(t) dt \right) \\
&\quad + \sum_{j=1}^K A_j P(X(n)=j) e^{-\int_0^n r(s)ds} \\
\text{EPV(premiums)} &= \sum_{j=1}^K \left(\int_0^n \pi_j(t) e^{-\int_0^t r(s)ds} P(X(t)=j) dt \right)
\end{aligned}$$

6.5 Reserves in the multi-state model

$$
\begin{aligned}
W(t) &= \sum_{j=1}^{K} \int_t^n e^{-r(s-t)} \left(b_j(s) - \pi_j(s)\right) I_j(s) ds + \sum_{j=1}^{K} \sum_{k=1,k\neq j}^{K} \int_t^n e^{-r(s-t)} B_{jk} dN_{jk} \\
&+ \sum_{j=1}^{K} A_j I_j(n) e^{-r(n-t)} \\
V_i(t) &= E(W(t)|\mathcal{F}_t) = E(W(t)|X(t) = i)
\end{aligned}
$$

Reserves at time u given we are in state i (for simplicity, $r(t) = r$):

$$
\begin{aligned}
V_i(u) &= \sum_{j=1}^{K} \int_u^n b_j(t) e^{-r(t-u)} P(X(t) = j|X(u) = i) dt \\
&+ \sum_{j=1}^{K} \sum_{k=1,k\neq j}^{K} \int_u^n B_{jk}(t) P(X(t) = j|X(u) = i) \mu_{jk}(u) e^{-r(t-u)} dt \\
&+ \sum_{j=1}^{K} A_j P(X(n) = j|X(u) = i) e^{-r(n-u)} \\
&- \sum_{j=1}^{K} \int_u^n \pi_j(t) e^{-r(t-u)} P(X(t) = j|X(u) = i) dt
\end{aligned}
$$

6.6 Thiele's equations for the multi-state model

$$
\begin{aligned}
V_i(t+dt) &= V_i(t)(1+rdt) - (b_i(t) - \pi_i(t))dt - \sum_{k=1,k\neq i}^{K} \mu_{ik} dt B_{ik} \\
&- \sum_{k=1,k\neq i}^{K} \mu_{ik} dt V_k(t) + \sum_{k=1,k\neq i}^{K} \mu_{ik} dt V_i(t) + o(dt) \\
\frac{V_i(t+dt) - V_i(t)}{dt} &= (r + \mu_{i.}) V_i(t) - \sum_{k=1}^{K} \mu_{ik} (B_{ik}(t) + V_k(t)) - b_i(t) + \pi_i(t) \\
\frac{dV_i(t)}{dt} &= (r + \mu_{i.}) V_i(t) - \sum_{k=1,k\neq i}^{K} \mu_{ik} (B_{ik}(t) + V_k(t)) - b_i(t) + \pi_i(t) \\
\frac{dV_i(t)}{dt} &= rV_i(t) - \sum_{k=1,k\neq i}^{K} \mu_{ik} B_{ik}(t) - b_i(t) + \pi_i(t) - \sum_{k=1,k\neq i}^{K} \mu_{ik}(t)(V_k(t) - V_i(t))
\end{aligned}
$$

for $i = 1, ..., K$ such that $V_i(n-) = A_i$. If there is no premium upfront, $V_i(0) = 0$, $\forall i$.

Example: 2 lives aged x, y. Premium payable in advance. Benefit of 1 payable on second death.

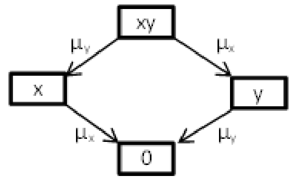

Figure 8: Two lives x, y

Thiele's equations for the reserves:

$$\begin{aligned}
\frac{dV_{xy}(t)}{dt} &= (r+\mu_{x+t}+\mu_{y+t})V_{xy}(t)-\mu_{x+t}V_y(t)-\mu_{y+t}V_x(t)\\
\frac{dV_x(t)}{dt} &= (r+\mu_{x+t})V_x(t)-\mu_{x+t}\\
\frac{dV_y(t)}{dt} &= (r+\mu_{y+t})V_y(t)-\mu_{y+t}
\end{aligned}$$

and the conditions

$$V_{xy}(n)=V_x(n)=V_y(n)=0, \qquad V_{xy}(0)=\pi$$

Suppose there is no premium upfront, but premium is payable continuously while both lives are alive. Then

$$\begin{aligned}
\frac{dV_{xy}(t)}{dt} &= (r+\mu_{x+t}+\mu_{y+t})V_{xy}(t)-\mu_{x+t}V_y(t)-\mu_{y+t}V_x(t)+\pi\\
\pi &= \frac{\bar{A}_{\overline{xy}}}{\bar{a}_{xy}}
\end{aligned}$$

Example: Health-Sickness model. A continuous benefit b is payable as long as the life is sick. Single premium payable upfront. The contract lasts for n years.

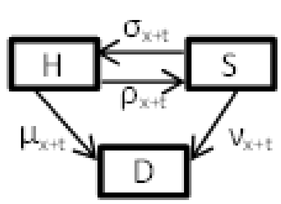

Figure 9: Health-Sickness Model

$$\begin{aligned}
V_H'(t) &= rV_H(t)+\mu_{x+t}V_H(t)-\sigma_{x+t}(V_S(t)-V_H(t))\\
V_S'(t) &= rV_S(t)+\nu_{x+t}V_s(t)-\rho_{x+t}(V_H(t)-V_S(t))-b
\end{aligned}$$

subject to the conditions

$$V_H(n)=0, \qquad V_S(n)=0$$

Solve and we have $V_H(0)=\pi$.

Example: Health-Sickness model. Same as above, but instead of a single premium we have a continuous premium π payable while healthy, then

$$\begin{aligned} V_H'(t) &= rV_H(t) + \mu_{x+t}V_H(t) - \sigma_{x+t}(V_S(t) - V_H(t)) + \pi \\ V_S'(t) &= rV_S(t) + \nu_{x+t}V_S(t) - \rho_{x+t}(V_H(t) - V_S(t)) - b \end{aligned}$$

and the conditions

$$V_H(n) = 0, \qquad V_S(n) = 0$$

π should be such that $V_H(0) = 0$. But how do we find π?

First, we want to find the value of a continuous annuity payable as long as a life is healthy and up to time n (this can be viewed as a continuous benefit of 1 payable as long as healthy). In order to find that, we solve:

$$\begin{aligned} W_H'(t) &= rW_H(t) + \mu_{x+t}W_H(t) - \sigma_{x+t}(W_S(t) - W_H(t)) - 1 \\ W_S'(t) &= rW_S(t) + \nu_{x+t}W_S(t) - \rho_{x+t}(W_H(t) - W_H(t)) \end{aligned}$$

subject to the conditions

$$W_H(n) = 0, \qquad W_S(n) = 0$$

$W_H(0)$ will be the value of this annuity (by the Principle of equivalence).

Now, consider the single premium model as in the previous example:

$$\begin{aligned} U_H'(t) &= rU_H(t) + \mu_{x+t}U_H(t) - \sigma_{x+t}(U_S(t) - U_H(t)) \\ U_S'(t) &= rU_S(t) + \nu_{x+t}U_S(t) - \rho_{x+t}(U_H(t) - U_S(t)) - b \end{aligned}$$

subject to the conditions

$$U_H(n) = 0, \qquad U_S(n) = 0$$

By the Principle of equivalence,

$$\begin{aligned} \pi W_H(0) &= U_H(0) \\ \pi &= \frac{U_H(0)}{W_H(0)} \end{aligned}$$

Then we substitute π into the differential equations for $V_H(t)$, $V_S(t)$ and solve to find the reserves. Check to see if $V_H(0) = 0$.

7 Higher Moments

We have

- $W(t)$ is a stochastic process.
- $V_i(t) = E(W(t)|\mathcal{F}_t) = E(W(t)|X(t) = i)$.
- $V_i^{(2)}(t) = E(W^2(t)|X(t) = i)$, which will lead to $Var(W(t)|X(t) = i)$.

Example: Life-Death Model - Continuous benefit b payable as long as alive until time n, one off payment B at time of death, and C at time n if alive.

$$W(t) = \int_t^n e^{-r(s-t)} bI_s ds + \int_t^n e^{-r(s-t)} B dN_s + Ce^{-r(n-t)} I_n$$

Let T_{x+t} be the remaining lifetime. So

$$\begin{aligned}
W(t) &= b\bar{a}_{\overline{T_{x+t}\wedge n}|} + Be^{-rT_{x+t}}\mathbf{1}_{\{T_{x+t}<n\}} + Ce^{-r(n-t)}\mathbf{1}_{\{T_{x+t}>n\}} \\
\text{Second moment} &= \int_0^\infty \left(b\frac{1-e^{-r(s\wedge n)}}{r} + e^{-rs}B\mathbf{1}_{\{s<n\}} + Ce^{-r(n-t)}\mathbf{1}_{\{s>n\}}\right)^2 {}_sp_{x+t}\mu_{x+t+s}ds
\end{aligned}$$

Alternatively, we can use differential equations.
The first moment:

$$\begin{aligned}
W(t) &= e^{-rdt}W(t+dt) + b(t)dt + (B(t) - W(t+dt))dN_t \\
V(t) &= E(W(t)|\mathcal{F}_t) \\
&= (1-rdt)V(t+dt) + b(t)dt + (B(t) - V(t+dt))\mu_{x+t}dt \\
\frac{dV^{(1)}(t)}{dt} &= rV^{(1)}(t) - b(t) - \left(B(t) - V^{(1)}(t)\right)\mu_{x+t}
\end{aligned}$$

such that $V^{(1)}(n) = C$.
The second moment:

$$\begin{aligned}
W^2(t) &= [W(t+dt) - rdtW(t+dt) + b(t)dt + (B(t) - W(t+dt))dN_t]^2 \\
&= W^2(t+dt) - 2rdtW^2(t+dt) + 2b(t)dtW(t+dt) + 2W(t+dt)(B(t) - W(t+dt))dN_t \\
&\quad +(B(t) - W(t+dt))^2dN_t + o(dt) \\
V^{(2)}(t) &= E(W^2(t)|\mathcal{F}_t) \\
&= V^{(2)}(t+dt) - 2rV^{(2)}dt + 2b(t)V^{(1)}(t+dt) + \left(2B(t)V^{(1)}(t+dt) - 2V^{(2)}(t+dt)\right)\mu_{x+t}dt \\
&\quad + \left(B(t)^2 - 2B(t)V^{(1)}(t+dt) + V^{(2)}(t+dt)\right)\mu_{x+t}dt + o(dt) \\
&= V^{(2)}(t+dt) - 2rdtV^{(2)}(t+dt)dt + 2b(t)V^{(1)}(t+dt) + \left(B(t)^2 - V^{(2)}(t+dt)\right)\mu_{x+t}dt \\
\frac{dV^{(2)}(t)}{dt} &= 2rV^{(2)}(t) - 2b(t)V^{(1)}(t) - \left(B^2(t) - V^{(2)}(t)\right)\mu_{x+t}
\end{aligned}$$

such that $V^{(2)}(n) = C^2$. Note that we have to solve $V^{(1)}(t)$ first. In MAPLE, we solve simultaneously (1) and (2). $V^{(2)}(0) - \left(V^{(1)}(0)\right)^2$ is the variance of the contract.

Example: A more general case

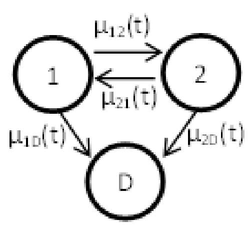

Figure 10: 3 state model

$$\begin{aligned}
W(t) &= \int_t^n \left(e^{-r(s-t)}b_1I_1(s)ds + e^{-r(s-t)}b_2I_2(s)ds + B_{12}e^{-r(s-t)}dN_{12}(s) + B_{21}e^{-r(s-t)}dN_{21}(s)\right. \\
&\quad \left.+B_{1D}e^{-r(s-t)}dN_{1D}(s) + B_{2D}e^{-r(s-t)}dN_{2D}(s)\right) + C_1e^{-r(n-s)}I_1(n) + C_2e^{-r(n-t)}I_2(n)
\end{aligned}$$

We denote $W_1(t) = W(t)|$ state 1 at t, $W_2(t) = W(t)|$ state 2 at t.
Thiele's equations for the first moments:

$$\begin{aligned}
W_1(t) &= (1 - r dt) W_1(t+dt) + b_1 dt + (B_{12} + W_2(t+dt) - W_1(t+dt))\, dN_{12}(t) \\
&\quad + (B_{1D} - W_1(t+dt))\, dN_{1D}(t) \\
V_1^{(1)}(t) &= (1 - r dt) V_1^{(1)}(t+dt) + b_1 dt + \left(B_{12} + V_2^{(1)}(t+dt) - V_1^{(1)}(t+dt)\right) \mu_{12}(t) dt \\
&\quad + \left(B_{1D} - V_1^{(1)}(t+dt)\right) \mu_{1D}(t) dt \\
\frac{dV_1^{(1)}(t)}{dt} &= r V_1^{(1)}(t) - b_1 - \mu_{12}(t) \left(B_{12} + V_2^{(1)}(t) - V_1^{(1)}(t)\right) - \mu_{1D}(t) \left(B_{1D} - V_1^{(1)}(t)\right)
\end{aligned}$$

Similarly,

$$\frac{dV_2^{(1)}(t)}{dt} = r V_2^{(1)}(t) - b_2 - \mu_{21}(t) \left(B_{21} + V_1^{(1)}(t) - V_2^{(1)}(t)\right) - \mu_{2D}(t) \left(B_{2D} - V_2^{(1)}(t)\right)$$

Solve subject to $V_1(n) = C_1$, $V_2(n) = C_2$.

To get the second moments,

$$\begin{aligned}
W_1^2(t) &= (1 - r dt)^2 W_1^2(t+dt) + (B_{12} + W_2(t+dt) - W_1(t+dt))^2\, dN_{12}(t) + 2 b_1 dt W_1(t+dt) \\
&\quad + (B_{1D} - W_1(t+dt) dt)^2\, dN_{1D}(t) + 2\,(B_{12} + W_2(t+dt) - W_1(t+dt))\, W_1(t+dt) dN_{12}(t) \\
&\quad + 2\,(B_{1D} - W_1(t+dt))\, W_1(t+dt) dN_{1D} + o(dt) \\
V_1^{(2)}(t) &= V_1^{(2)}(t+dt) - 2r V_1^{(2)}(t+dt) dt + \left(B_{1D}^2 - V_1^{(2)}(t+dt)\right) \mu_{1D}(t) dt \\
&\quad + \left(B_{12}^2 + 2 B_{12} V_2^{(1)}(t+dt) + V_2^{(2)}(t+dt) - V_1^{(2)}(t+dt)\right) \mu_{12}(t) dt \\
&\quad + 2 b_1 V_1^{(1)}(t+dt) dt \\
\frac{dV_1^{(2)}}{dt} &= 2r V_1^{(2)}(t) - \left(B_{1D}^2 - V_1^{(2)}(t)\right) \mu_{1D}(t) \\
&\quad - \left(B_{12}^2 + 2 B_{12} V_2^{(1)}(t) + V_2^{(2)}(t) - V_1^{(2)}(t)\right) \mu_{12}(t) - 2 b_1 V_1^{(1)}(t)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{dV_2^{(2)}}{dt} &= 2r V_2^{(2)}(t) - \left(B_{2D}^2 - V_2^{(2)}(t)\right) \mu_{2D}(t) \\
&\quad - \left(B_{21}^2 + 2 B_{21} V_1^{(1)}(t) + V_1^{(2)}(t) - V_2^{(2)}(t)\right) \mu_{21}(t) - 2 b_2 V_2^{(1)}(t)
\end{aligned}$$

Solve subject to $V_1^{(1)}(n) = C_1$, $V_1^{(2)}(n) = C_1^2$, $V_2^{(1)}(n) = C_2$, $V_2^{(2)} = C_2^2$.

8 Safety Margins

Principle of Equivalence: EPV(benefits) = EPV(premiums).

However, to make a profit, a safety margin is included so that premiums charged are higher than that implied by the principle of equivalence.

Example: Consider a one year temporary assurance, where benefit b is payable at the end of 1 year if life is dead.

$$EPV(benefit) = e^{-r} b q_x$$

and the single premium is $e^{-r} b q_x (1+\theta)$ where $\theta > 0$ is the loading factor. Suppose we have 10000 such policies, independent of each other.

$$\begin{aligned}
E(profits) &= 10000\left(e^{-r} b q_x (1+\theta) - e^{-r} b q_x\right) \\
& 10000\theta e^{-r} b q_x \\
E(profits^2) &= 10000 b^2 \theta^2 e^{-2r} q_x \\
Var(profits) &= 10000^2 b^2 \theta^2 e^{-2r} q_x p_x
\end{aligned}$$

By the Central limit theorem, the profits

$$Z \sim \mathcal{N}\left(10000\theta e^{-r} b q_x, 10000 b^2 \theta^2 e^{-2r} q_x p_x\right)$$

So we have

$$P(loss) = \Psi\left(\frac{-10000\theta e^{-r} b q_x}{100 b \theta e^{-r}\sqrt{q_x p_x}}\right) \approx 0$$

This is especially so if there is a large number of policies.