

## Solutions to Extra Exercise 4

1.

$$\begin{aligned}
 E\left(\bar{a}_{\max\{T_x, 5\}}\right) &= E\left(\bar{a}_{\overline{T_x \vee 5}}\right) \\
 &= \int_0^5 e^{-rt} dt + \int_5^\infty e^{-rt} {}_t p_x dt \\
 &= \bar{a}_{\overline{5}} + e^{-5r} {}_5 p_x \int_5^\infty e^{-r(t-5)} {}_{t-5} p_{x+5} dt \\
 &= \bar{a}_{\overline{5}} + v^5 {}_5 p_x \int_0^\infty e^{-ru} {}_u p_{x+5} du \\
 &= \bar{a}_{\overline{5}} + v^5 {}_5 p_x \bar{a}_{x+5}
 \end{aligned}$$

2.

$$\begin{aligned}
 E\left(\bar{a}_{\overline{T_x+5}}\right) &= \int_0^5 e^{-rt} dt + \int_5^\infty e^{-rt} {}_{t-5} p_x dt \\
 &= \bar{a}_{\overline{5}} + v^5 \int_0^\infty e^{-r(t-5)} {}_{t-5} p_x dt \\
 &= \bar{a}_{\overline{5}} + v^5 \int_0^\infty e^{-ru} {}_u p_x du \\
 &= \bar{a}_{\overline{5}} + v^5 \bar{a}_x
 \end{aligned}$$

To calculate the variances:

1.

$$\bar{a}_{\overline{T_x \vee 5}} = \frac{1 - e^{-r(T_x \vee 5)}}{r}$$

$$\begin{aligned}
 Var\left(\bar{a}_{\overline{T_x \vee 5}}\right) &= Var\left(\frac{1 - e^{-r(T_x \vee 5)}}{r}\right) \\
 &= \frac{1}{r^2} Var\left(e^{-r(T_x \vee 5)}\right)
 \end{aligned}$$

$$\begin{aligned}
 E\left(e^{-2r(T_x \vee 5)}\right) &= e^{-2r \times 5} (1 - {}_5 p_x) + \int_5^\infty e^{-2rt} {}_t p_x \mu_{x+t} dt \\
 &= v^{10} (1 - {}_5 p_x) + \int_5^\infty e^{-2rt} {}_t p_x \mu_{x+t} dt
 \end{aligned}$$

and

$$E\left(e^{-r(T_x \vee 5)}\right) = e^{-5r} (1 - {}_5 p_x) + \int_5^\infty e^{-rt} {}_t p_x \mu_{x+t} dt$$

So

$$Var\left(\bar{a}_{T_x \vee 5}\right) = \frac{1}{r^2} \left[ E\left(e^{-2r(T_x \vee 5)}\right) - \left(E\left(e^{-r(T_x \vee 5)}\right)\right)^2 \right]$$

2.

$$\bar{a}_{T_x+5} = \frac{1 - e^{-r(T_x+5)}}{r}$$

$$\begin{aligned} Var\left(\bar{a}_{T_x+5}\right) &= Var\left(\frac{1 - e^{-r(T_x+5)}}{r}\right) \\ &= \frac{e^{-10r}}{r^2} Var\left(e^{-rT_x}\right) \\ &= \frac{e^{-10r}}{r^2} \left(\bar{A}_x^{\text{at } 2r} - \left(\bar{A}_x^{\text{at } r}\right)^2\right) \end{aligned}$$