

This worksheet is to demonstrate the solution of Thiele's equations. We will use the standard health-sickness model and investigate a 30 year policy. We are going to assume that a continuous benefit of 20000 per annum is payable for as long as the life is sick. The continuous premium is payable while he is healthy. We start by defining the usual mortality force (this is now the force for healthy lives)

```
> m := t -> 0.0005+0.00007585775*10^(0.038*t);
```

$$m := t \rightarrow 0.0005 + 0.00007585775 10^{(0.038 t)}$$

We now define the mortality force for sick lives. There are three ways to adjust the mortality upwards and we apply all three of them; you could of course choose to use a totally different function.

```
> ms := t -> 0.0002+1.5*m(t+5);
```

$$ms := t \rightarrow 0.0002 + 1.5 m(t + 5)$$

```
> evalf(ms(45));
```

$$0.009988392898$$

We now define the force of transition from the healthy state to the sick state; It is defined as $si(x+t)$ where x is the age of the life. In general it does not have to be of this form, but you have by now been trained to think this way. It is appropriate that it is increasing.

```
> si := t -> 0.1*(1+t/50);
```

$$si := t \rightarrow 0.1 \left(1 + \frac{1}{50} t \right)$$

And finally the recovery force from sick to healthy. It is appropriate that it is decreasing. Again we use $re(x+t)$ where x is the age of the life and of course it does not have to be of this form.

```
> re := t -> 2*(1+t/50)^(-1);
```

$$re := t \rightarrow \frac{2}{1 + \frac{1}{50}t}$$

The force of interest is 0.05. In order to write down the equations we first need to calculate the premium. We will proceed as follows. Let us first pretend the policy is funded by a single premium payable upfront. That would be $wh(0)$ in the solution of

```
> dsys1 :=
{diff(wh(t), t) = (m(35+t) + si(35+t) + 0.05) * wh(t)
- si(35+t) * ws(t),
diff(ws(t), t) = (ms(35+t) + re(35+t) + 0.05) * ws(t)
- 20000 - re(35+t) * wh(t), wh(30) = 0,
ws(30) = 0};
```

$$dsys1 := \left\{ \begin{aligned} \frac{d}{dt} wh(t) &= (0.2205000000 \\ &+ 0.00007585775 \cdot 10^{(1.330 + 0.038 t)} + 0.002000000000 t) wh(t) \\ &- (0.1700000000 + 0.002000000000 t) ws(t), \frac{d}{dt} ws(t) = \\ &\left(0.05095 + 0.000113786625 \cdot 10^{(1.520 + 0.038 t)} + \frac{2}{\frac{17}{10} + \frac{t}{50}} \right) ws(t) \end{aligned} \right.$$

$$\left. -20000 - \frac{2 \text{ wh}(t)}{\frac{17}{10} + \frac{t}{50}}, \text{ wh}(30) = 0, \text{ ws}(30) = 0 \right\}$$

```
> dsol1 := dsolve(dsyst1, numeric, range = 0..30);
```

```
dsol1 := proc(x_rkf45) ... end proc
```

```
> dsol1(30);
```

```
[t = 30., wh(t) = 0., ws(t) = 0.]
```

```
> dsol1(0);
```

```
[t = 0., wh(t) = 42794.8333026865148,
ws(t) = 57067.8796379095293]
```

Let us now calculate in a similar way the value of an annuity of 1 payable continuously while the life is healthy. This would be wh(0) in the solution of

```
> dsyst2 :=
{diff(wh(t), t) = (m(35+t) + si(35+t) + 0.05) * wh(t)
- si(35+t) * ws(t) - 1,
diff(ws(t), t) = (ms(35+t) + re(35+t) + 0.05) * ws(t)
- re(35+t) * wh(t), wh(30) = 0, ws(30) = 0};
```

```
dsyst2 := { wh(30) = 0, ws(30) = 0,  $\frac{d}{dt} \text{wh}(t) = (0.2205000000$ 
```

```
 $+ 0.00007585775 \cdot 10^{(1.330 + 0.038 t)} + 0.002000000000 t) \text{wh}(t)$ 
```

$$- (0.1700000000 + 0.002000000000 t) \text{ws}(t) - 1, \frac{d}{dt} \text{ws}(t) =$$

$$\left(0.05095 + 0.000113786625 10^{(1.520 + 0.038 t)} + \frac{2}{\frac{17}{10} + \frac{t}{50}} \right) \text{ws}(t)$$

$$- \frac{2 \text{wh}(t)}{\frac{17}{10} + \frac{t}{50}} \Bigg\}$$

```
> dsol2 := dsolve(dsyst, numeric, range =
0..30);
```

```
dsol2 := proc(x_rkf45) ... end proc
```

```
> dsol2(0);
```

```
[t = 0., wh(t) = 12.4724100560195588,
ws(t) = 11.7303723907940558]
```

And so the continuous premium is

```
> evalf(42794.8333026865148/12.47241005601955
88);
```

3431.159904

Let us now solve Thiele's equations.

```
> dsyst3 :=
{diff(vh(t), t) = (m(35+t) + si(35+t) + 0.05) * vh(t)
- si(35+t) * vs(t) + 3431.159904,
diff(vs(t), t) = (ms(35+t) + re(35+t) + 0.05) * vs(t)
- re(35+t) * vh(t) - 20000, vh(30) = 0,
vs(30) = 0};
```

$$\begin{aligned}
 \text{dsys3} := & \left\{ \begin{aligned} & \text{vh}(30) = 0, \text{vs}(30) = 0, \frac{d}{dt} \text{vh}(t) = (0.2205000000 \\ & + 0.00007585775 \cdot 10^{(1.330 + 0.038 t)} + 0.002000000000 t) \text{vh}(t) \\ & - (0.1700000000 + 0.002000000000 t) \text{vs}(t) + 3431.159904, \\ & \frac{d}{dt} \text{vs}(t) = \\ & \left(0.05095 + 0.000113786625 \cdot 10^{(1.520 + 0.038 t)} + \frac{2}{\frac{17}{10} + \frac{t}{50}} \right) \text{vs}(t) \\ & - \frac{2 \text{vh}(t)}{\frac{17}{10} + \frac{t}{50}} - 20000 \end{aligned} \right\}
 \end{aligned}$$

```
> dsol3 := dsolve(dsys3, numeric, range = 0..30);
```

```
      dsol3 := proc(x_rkf45) ... end proc
```

```
> dsol3(0);
```

```
[t = 0., vh(t) = -0.000248198026326917900,
  vs(t) = 16819.0967395445478]
```

```
>
```

Note that $\text{vh}(30)$ is almost 0 as it should be. What does $\text{vs}(0)$ represent? Think.

We will also plot the two curves. We will discuss them in class. Can you spot a little problem and explain it?

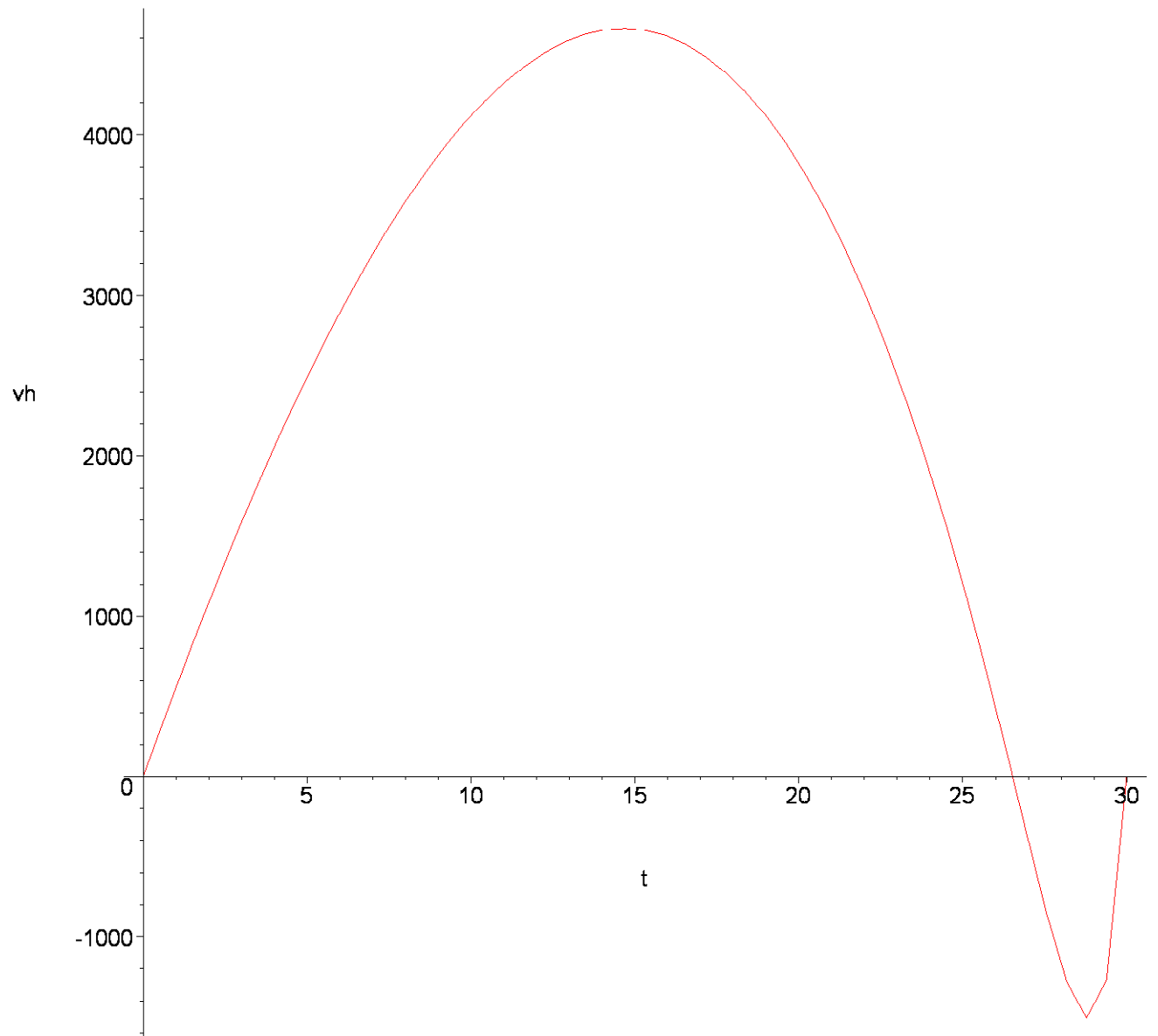
```
> with(plots);
```

```
>
```

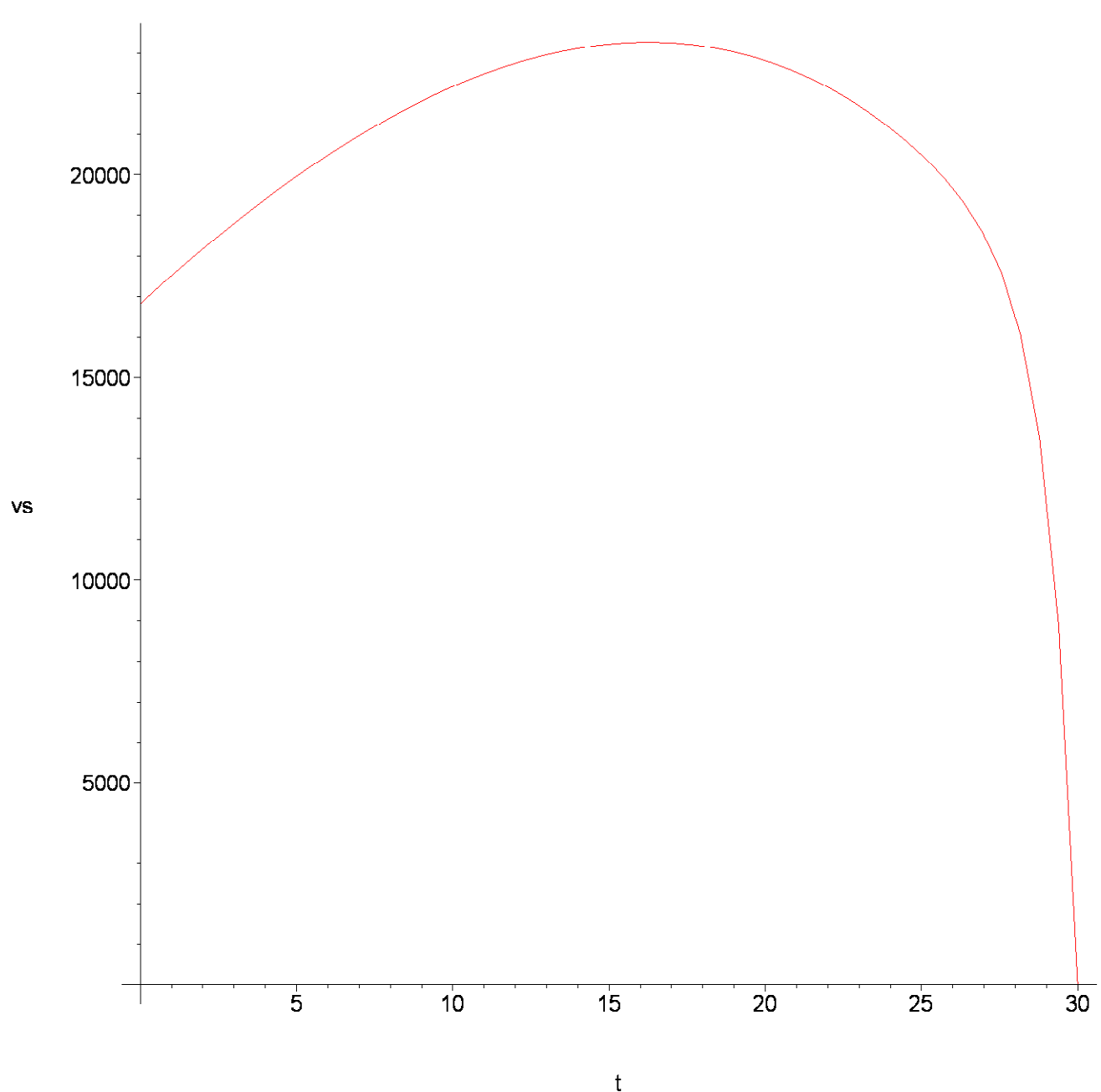
Warning, the name `changecoords` has been redefined

```
[animate, animate3d, animatecurve, arrow, changecoords,  
  complexplot, complexplot3d, conformal, conformal3d,  
  contourplot, contourplot3d, coordplot, coordplot3d, cylinderplot,  
  densityplot, display, display3d, fieldplot, fieldplot3d, gradplot,  
  gradplot3d, graphplot3d, implicitplot, implicitplot3d, inequal,  
  interactive, listcontplot, listcontplot3d, listdensityplot, listplot,  
  listplot3d, loglogplot, logplot, matrixplot, odeplot, pareto,  
  plotcompare, pointplot, pointplot3d, polarplot, polygonplot,  
  polygonplot3d, polyhedra_supported, polyhedraplot, replot,  
  rootlocus, semilogplot, setoptions, setoptions3d, spacecurve,  
  sparsematrixplot, sphereplot, surfdata, textplot, textplot3d,  
  tubeplot]
```

```
> odeplot(dsol3);
```



```
> odeplot(dsol3, [t,vs(t)]);
```



Let us fix the problem with the negative reserve. One way is to restrict premiums to the first 20 years with no premium payable thereafter. We need to calculate the value of the annuity payable for as long as the life is healthy and for no more than 20 years

```
> dsys4 :=  

  {diff(wh(t), t) = (m(35+t) + si(35+t) + 0.05) * wh(t)  

    - si(35+t) * ws(t) - 1,
```



```
diff (ws (t) , t) = (ms (35+t) + re (35+t) + 0.05) * ws (t)
- re (35+t) * wh (t) , wh (20) = 0, ws (20) = 0};
```

$$dsys4 := \left\{ \begin{aligned} \frac{d}{dt} wh(t) &= (0.2205000000 \\ &+ 0.00007585775 \cdot 10^{(1.330 + 0.038 t)} + 0.002000000000 t) wh(t) \\ &- (0.1700000000 + 0.002000000000 t) ws(t) - 1, \frac{d}{dt} ws(t) = \\ &\left(0.05095 + 0.000113786625 \cdot 10^{(1.520 + 0.038 t)} + \frac{2}{\frac{17}{10} + \frac{t}{50}} \right) ws(t) \\ &- \frac{2 wh(t)}{\frac{17}{10} + \frac{t}{50}}, wh(20) = 0, ws(20) = 0 \end{aligned} \right\}$$

```
> dsol4 := dsolve(dsys4, numeric, range =
0..30);
```

```
dsol4 := proc(x_rkf45) ... end proc
```

```
> dsol4(0);
```

```
[t = 0., wh(t) = 10.5309679479451362,
ws(t) = 9.79290824618128752]
```

The premium is now

```
> evalf(42794.8333026865148/10.53096794794513
62);
```

4063.713184

Note the premium is now payable for 20 years only. Thiele's

equations now are: (look at how we deal with the discontinuity of the premium)

```
> dsys5 :=
  {diff(vh(t), t) = (m(35+t) + si(35+t) + 0.05) * vh(t)
   - si(35+t) * vs(t) + 4063.713184 * 0.5 * (signum(20
   - t) + 1),
   diff(vs(t), t) = (ms(35+t) + re(35+t) + 0.05) * vs(t)
   - re(35+t) * vh(t) - 20000, vh(30) = 0,
   vs(30) = 0};
```

$$dsys5 := \left\{ \begin{array}{l} vh(30) = 0, vs(30) = 0, \frac{d}{dt} vs(t) = \\ \left(0.05095 + 0.000113786625 \cdot 10^{(1.520 + 0.038 t)} + \frac{2}{\frac{17}{10} + \frac{t}{50}} \right) vs(t) \\ - \frac{2 \cdot vh(t)}{\frac{17}{10} + \frac{t}{50}} - 20000, \frac{d}{dt} vh(t) = (0.2205000000 \\ + 0.00007585775 \cdot 10^{(1.330 + 0.038 t)} + 0.002000000000 t) \cdot vh(t) \\ - (0.1700000000 + 0.002000000000 t) \cdot vs(t) \\ - 2031.856592 \cdot \text{signum}(t - 20) + 2031.856592 \end{array} \right\}$$

```
> dsol5 := dsolve(dsys5, numeric, range =  
0..30);
```

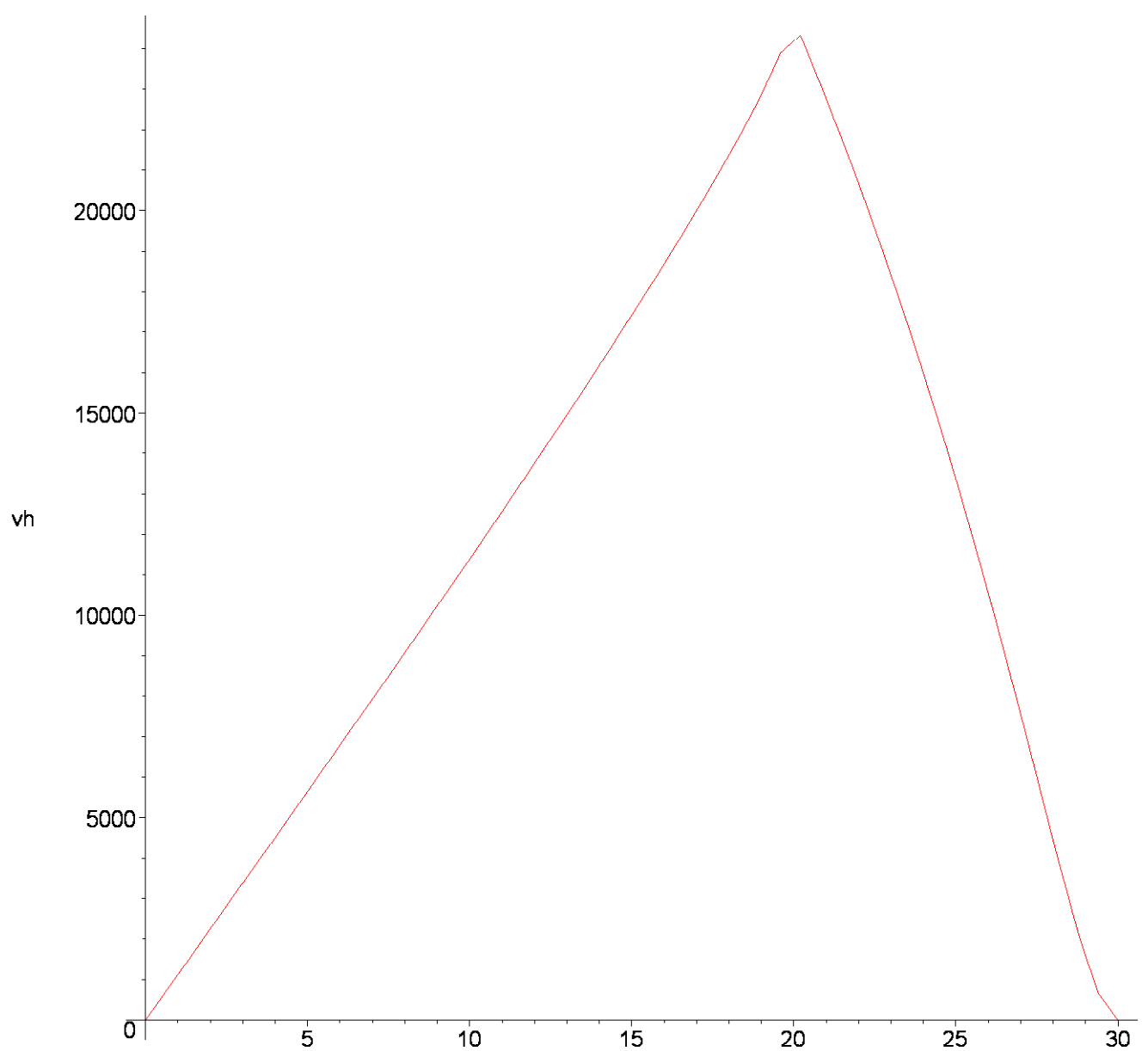
```
dsol5 := proc(x_rkf45) ... end proc
```

```
> dsol5(0);
```

```
[t = 0., vh(t) = -0.0455854216957050086,  
vs(t) = 17272.2593205819467]
```

The reserve at 0 is again almost 0. Here are the plots, now without any problems. Why are the peaks so sharp?

```
> odeplot(dsol5);
```



```
> odeplot(dsol15, [t, vs(t)]);
```

