Two-side Parisian Option with single barrier

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Abstract

In this paper, we study the excursion times of a Brownian motion with drift below and above a given level by using a simple two states semi-Markov model. In mathematical finance, these results have an important application in the valuation of path dependent options such as Parisian options. Based on our results, we introduce a new type of Parisian options, single barrier two-sided Parisian options, and give an explicit expression for the Laplace transform of its price formula.

Keywords: Excursion time, Two states Semi-Markov model, Path dependent options, Parisian options, Laplace transform.

1 Introduction

The concept of Parisian options was first introduced by Chesney, Jeanblanc-Picqué and Yor [7]. A Parisian option is a special case of path dependent options. Its payoff does not only depend on the final price of the underlying asset, but also its price trajectory during the whole life span of the option. More precisely, a Parisian option will be either initiated or terminated upon the price reaching a predetermined barrier level L and staying above or below the barrier for a predetermined time D before the maturity date T.

There are two different ways of measuring the time spent above or below the barrier, corresponding to the excursion time and the occupation time defined below. The excursion time below (above) the barrier starts counting from 0 each time the process crosses the barrier from above (below) and stops counting when the process crosses the barrier from below (above). The occupation time up to a specific time t adds up all the time the process spend below (above) the barrier; it is therefore the summation of all excursion time intervals before time t. In [7] the Parisian options related to the occupation time are called *cumulative Parisian options*. In this paper, we focus on the Parisian options based on excursion time.

The owner of a *Parisian down-and-out option* loses the option if the underlying asset price S reaches the level L and remains constantly below this level for a time interval longer than D. For a *Parisian down-and-in option* the same event gives the owner the right to exercise the option. The owner of a *cumulative Parisian down-and-out option* loses the option if the total time the underlying asset price S stays below L up to the end of the contract for longer than D. For details on the pricing of Parisian options see [7], [12], [15] and [11]. For cumulative Parisian options see [7] and since these are related to the occupation times and hence the quantiles of the process, also see [1], [9] and [13]. In this paper, we focus on the Parisian option defined upon the excursion time.

From the description above, we can see that the key for pricing a Parisian option is the derivation of the distribution of the excursion time. As in [7] we reduce the problem to finding the Laplace transform of the first time the length of the excursion reaches level D. In [7] this was obtained by using the Brownian meander and the Azéma martingale (see [3]). A restriction of this technique is that it relies heavily on the properties of standard Brownian motions; therefore the result cannot be extended to other processes easily. It is also hard to see how it can be used for the pricing of the more complicated options that we will introduce.

In this paper, we are going to study the excursion time in a more general framework using a simple semi-Markov model consisting of two states indicating whether the process is above or below a fixed level L . By applying the model to a Brownian motion, we can, for the first time, get the explicit form of the Laplace transforms for the prices of the Parisian options defined in [7]. One can then invert the Laplace transform using techniques as in [12].

Furthermore, we introduce a new type of Parisian options, named *single-barrier two-sided Parisian option.* In contrast to the Parisian options mentioned above, we consider the excursions both below and above the barrier. Let us look at two examples, depending on whether the condition is that the required excursions above and below the barrier have to both happen before the maturity date or that either one of them happens before the maturity. In one example, the owner of a *Parisian Max Out option* loses the option if the underlying asset price S has both an excursion above the barrier for longer than d_1 and below the barrier for longer than d_2 before the maturity of the option. In another example, the owner of a *Parisian Min Out option* loses the right to exercise the option if there is either an excursion above the barrier for longer than d_1 or below the barrier for longer than d_2 before the maturity. Later on, we will give the explicit forms of the Laplace transforms for the prices of this type of options.

In Section 2 we give the mathematical definitions and set out the model. We also introduce a pair of new processes, perturbed Brownian motions, which have the same behavior as a Brownian motion except that each time when they hit 0, they jump towards the other side of 0 by size ϵ. In Section 3 we present an important lemma for the perturbed Brownian motions together with its proof, which will be used in the following sections. We give our main results applied to Brownian motions in Section 4, including the Laplace transforms for the stopping times we define for both Brownian motions with drift and standard Brownian motions, which are vital for the pricing. In Section 5 we focus on pricing our newly defined Parisian options by using the results in Section 4. As a special case, we also give the explicit form of the Laplace transform for the price of the Parisian options studied in [7] for the first time. In [7] these were given in the form of double integrals. Using a different approach yields explicit results in our paper (see remark after corollary 4.3 later).

2 Definitions

We are going to use the same definition for the excursion as in [7], [8] and [14]. Let L be the level of the barrier and assume S is the price of the underlying asset following a geometric Brownian motion:

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t \mathrm{d}W_t, \quad S_0 = x, \quad x > 0, \tag{1}$$

where W_t with $W_0 = 0$ is a standard Brownian motion under a risk neutral measure Q. As in [7], we define

$$g^S_{L,t} = \sup\{s \le t \mid S_s = L\}, \qquad d^S_{L,t} = \inf\{s \ge t \mid S_s = L\} \tag{2}$$

with the usual convention, $\sup\{\emptyset\} = 0$ and $\inf\{\emptyset\} = \infty$. The trajectory between $g^S_{L,t}$ and $d^S_{L,t}$ is the excursion of process S, which straddles time t. Assuming $d_1 > 0$, $d_2 > 0$, we now define

$$\tau^S_{1,L} = \inf\left\{t > 0 \mid \mathbf{1}_{\{S_t > L\}}\left(t - g^S_{L,t}\right) \ge d_1\right\}, \tag{3}$$

$$\tau^S_{2,L} = \inf\left\{t > 0 \mid \mathbf{1}_{\{S_t < L\}}\left(t - g^S_{L,t}\right) \ge d_2\right\}, \tag{4}$$

$$\tau^S_L = \tau^S_{1,L} \wedge \tau^S_{2,L}. \tag{5}$$

$\tau^S_{1,L}$ is therefore the first time that the length of the excursion of the process S above the barrier L reaches given level d_1; $\tau^S_{2,L}$ corresponds to the one below L; and τ^S_L is the smaller of $\tau^S_{1,L}$ and $\tau^S_{2,L}$.

Assume r is the risk-free rate, T is the term of the option, K is the strike price, S is the underlying asset price defined as above. If we have an up-out Parisian call option with the barrier L, its price can be expressed as:

$$P_{up-out-call} = e^{-rT} E_Q\left(\mathbf{1}_{\left\{\tau^S_{1,L} > T\right\}}\left(S_T - K\right)^+\right);$$

and the price of a down-in Parisian put option with the barrier L is:

$$P_{down-in-put} = e^{-rT} E_Q\left(\mathbf{1}_{\left\{\tau^S_{2,L} < T\right\}}\left(K - S_T\right)^+\right).$$

Without loss of generality, from now on, we assume $L = 0$. We simplify the expressions of τ^S_0, $\tau^S_{1,0}$ and $\tau^S_{2,0}$ by τ^S, τ^S_1 and τ^S_2.

From (1) we can see that in order to study the excursion of the asset price S we just need to study the excursion of the Brownian motion W. However, the peculiar properties of the sample path of the Brownian motion result in many difficulties. A major problem is the occurrence of an infinite number of very small excursions. In order to solve these problems we introduce a pair of new processes, *perturbed Brownian motions*, $X^{\pm}$ as follow. Assume W^{μ} is a Brownian motion with non-negative drift and it starts from 0 and set $\epsilon = \pm\eta$, where $\eta > 0$. Define a sequence of stopping times

$$\begin{array}{lcl} \delta_0 & = & 0, \\ \sigma_n & = & \inf\left\{t > \delta_n \,|\, W_t^{\mu} = -\epsilon\right\}, \\ \delta_{n+1} & = & \inf\left\{t > \sigma_n \,|\, W_t^{\mu} = 0\right\}, \end{array}$$

where $n = 0, 1, \cdots$. Now define

$$X_t^{\pm} = \begin{cases} W_t^{\mu} + \epsilon, & \text{if} \quad \delta_n \le t < \sigma_n \\ W_t^{\mu}, & \text{if} \quad \sigma_n \le t < \delta_{n+1} \end{cases}.$$

When $\epsilon = \eta$, we denote the process by X^+ and in the case when $\epsilon = -\eta$, we have process X^- (see Figure 1 and Figure 2). By introducing the jumps to the original Brownian motion, we get this pair of processes $X^{\pm}$ which have a very clear structure of excursions above and below 0, i.e. the excursions above and below 0 alternate with the length of each excursion greater than 0. In the later section we prove that the Laplace transforms of the variables defined based on $X^{\pm}$ converge to those based on W^{μ} as η goes to 0. As a result, we can obtain the results for the Brownian Motion by carrying out the calculations for $X^{\pm}$ and taking the limit $\eta \to 0$; for more details see Theorem 4.1. Hence we will focus on studying the excursions of $X^{\pm}$ in the rest of this section and next section.

From the description of the excursion above, it is clear that we are actually considering two states, the state when the process is above the barrier and the state when it is below. For each state, we are interested in the time the process spends in it. We introduce a pair of new processes based on $X^{\pm}$.

$$Z_t^{\pm} = \begin{cases} 1, & \text{if } X_t^{\pm} > L \\ 2, & \text{if } X_t^{\pm} < L \end{cases}.$$

In this definition, we deliberately ignore the situation when $Z_t^{\pm} = L$. It is because the processes $Z^{\pm}$ satisfy

$$\int_0^t \mathbf{1}_{\{Z_u^{\pm} = L\}} \mathrm{d}u = 0.$$

We can now express the variables defined above in terms of $Z^{\pm}$:

$$g_{L,t}^{\pm} = \sup\left\{s \le t \mid Z_s^{\pm} \ne Z_t^{\pm}\right\}, \tag{6}$$

$$d_{L,t}^{\pm} = \inf\left\{s \ge t \mid Z_s^{\pm} \ne Z_t^{\pm}\right\}, \tag{7}$$

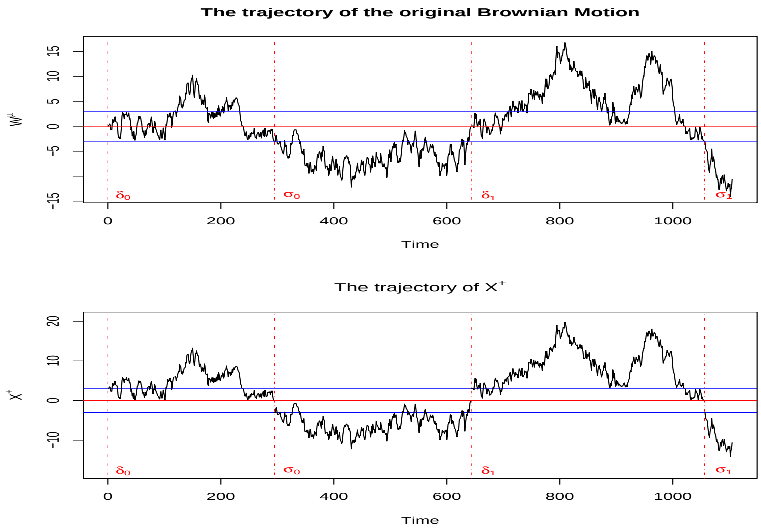

Figure 1: The Sample Path of X^+

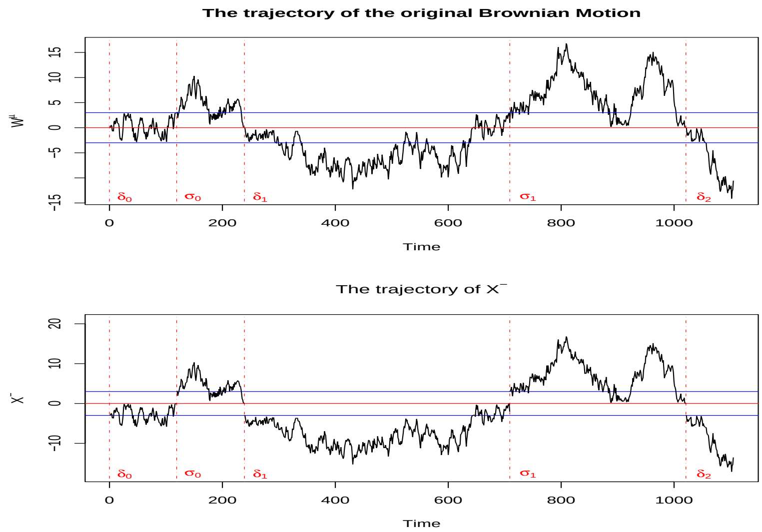

Figure 2: The Sample Path of X^-

$$\tau_1^{\pm} = \inf\left\{t > 0 \mid \mathbf{1}_{\left\{Z_t^{\pm}=1\right\}}\left(t - g_{L,t}^{\pm}\right) \geq d_1\right\}, \tag{8}$$

$$\tau_2^{\pm} = \inf\left\{t > 0 \mid \mathbf{1}_{\left\{Z_t^{\pm}=2\right\}}\left(t - g_{L,t}^{\pm}\right) \geq d_2\right\}, \tag{9}$$

$$\tau^{\pm} = \tau_1^{\pm} \wedge \tau_2^{\pm}. \tag{10}$$

We then define

$$V_t^{\pm} = t - g_{L,t}^{\pm},$$

the time $Z^{\pm}$ have spent in their current states. It is easy to see that both (Z_t^+, V_t^+) and (Z_t^-, V_t^-) are Markov processes. $Z^{\pm}$ are therefore semi-Markov processes with the state space $\{1,2\}$, where 1 stands for the state when $Z^{\pm}$ are above the barrier and 2 corresponds to the state below the barrier.

Furthermore, we set $U_{i,k}^{\pm}$, $i = 1, 2$ and $k = 1, 2, \cdots$ to be the time $Z^{\pm}$ spend in state i when they visit i for the kth time. And we have, for each given i and k,

$$U_{i,k}^{\pm} = V_{d_{L,t}^{\pm}}^{\pm} = d_{L,t}^{\pm} - g_{L,t}^{\pm}, \quad \text{for some } t.$$

Notice that given i, $U_{i,k}^{\pm}$, $k = 1, 2, \cdots$, are i.i.d. We therefore define the transition densities for $Z^{\pm}$:

$$p_{ij}^{\pm}(t) = \lim_{\Delta t \to 0} \frac{P(t < U_{i,k}^{\pm} < t + \Delta t)}{\Delta t},$$

$$P_{ij}^{\pm}(t) = P(U_{i,k}^{\pm} < t), \quad \bar{P}_{ij}^{\pm}(t) = P(U_{i,k}^{\pm} > t).$$

We have

$$P_{ij}^{\pm}(t) = \int_0^t p_{ij}^{\pm}(s)\mathrm{d}s = 1 - \bar{P}_{ij}^{\pm}(t),$$

which is actually the probability that the process will stay in state i for no more than time t. More precisely, according to the definition of $Z^{\pm}$, we actually have the transition densities for $Z^{\pm}$ as follows:

$$p_{12}^{+}(s) = p_{12}^{-}(s) = \frac{\eta}{\sqrt{2\pi s^3}} \exp\left\{-\frac{(\eta + \mu s)^2}{2s}\right\}, \tag{11}$$

$$p_{21}^{+}(s) = p_{21}^{-}(s) = \frac{\eta}{\sqrt{2\pi s^3}} \exp\left\{-\frac{(\eta - \mu s)^2}{2s}\right\}. \tag{12}$$

For simplicity, we set

$$p_{12}(s) = p_{12}^{+}(s) = p_{12}^{-}(s), \quad p_{21}(s) = p_{21}^{+}(s) = p_{21}^{-}(s).$$

Similarly, we have

$$P_{ij}(t) = P_{ij}^{+}(t) = P_{ij}^{-}(t), \quad \bar{P}_{ij}(t) = \bar{P}_{ij}^{+}(t) = \bar{P}_{ij}^{-}(t).$$

3 An Important Lemma

In this section, we will present an important lemma for $X^{\pm}$ together with its proof.

Lemma 1 *For the perturbed Brownian motion X^{+}, we have the following results:*

$$\begin{aligned}
&E\left(\exp\left\{-\alpha_1\tau_1^{+}-\alpha_2\tau_2^{+}\right\}\boldsymbol{1}_{\left\{\tau_1^{+}<\tau_2^{+}\right\}}\right) \\
&= \frac{e^{-\alpha_1 d_1-\alpha_2 d_2}\bar{P}_{21}(d_2)\int_{d_1}^{\infty}e^{-\alpha_2 s}p_{12}(s)\mathrm{d}s}{G(d_1,d_2)},
\end{aligned} \tag{13}$$

$$\begin{aligned}
&E\left(\exp\left\{-\alpha_1\tau_1^{+}-\alpha_2\tau_2^{+}\right\}\boldsymbol{1}_{\left\{\tau_1^{+}>\tau_2^{+}\right\}}\right) \\
&= \frac{e^{-\alpha_1 d_1-\alpha_2 d_2}\bar{P}_{12}(d_1)\int_{d_2}^{\infty}e^{-\alpha_1 s}p_{21}(s)\mathrm{d}s\int_0^{d_1}e^{-(\alpha_1+\alpha_2)s}p_{12}(s)\mathrm{d}s}{G(d_1,d_2)};
\end{aligned} \tag{14}$$

and for X^{-} we have

$$\begin{aligned}
&E\left(\exp\left\{-\alpha_1\tau_1^{-}-\alpha_2\tau_2^{-}\right\}\boldsymbol{1}_{\left\{\tau_1^{-}<\tau_2^{-}\right\}}\right) \\
&= \frac{e^{-\alpha_1 d_1-\alpha_2 d_2}\bar{P}_{21}(d_2)\int_{d_1}^{\infty}e^{-\alpha_2 s}p_{12}(s)\mathrm{d}s\int_0^{d_2}e^{-(\alpha_1+\alpha_2)s}p_{21}(s)\mathrm{d}s}{G(d_1,d_2)},
\end{aligned} \tag{15}$$

$$\begin{aligned}
&E\left(\exp\left\{-\alpha_1\tau_1^{-}-\alpha_2\tau_2^{-}\right\}\boldsymbol{1}_{\left\{\tau_1^{-}>\tau_2^{-}\right\}}\right) \\
&= \frac{e^{-\alpha_1 d_1-\alpha_2 d_2}\bar{P}_{12}(d_1)\int_{d_2}^{\infty}e^{-\alpha_1 s}p_{21}(s)\mathrm{d}u}{G(d_1,d_2)},
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
G(d_1,d_2) \quad = \quad & \left\{1-\int_0^{d_1}e^{-(\alpha_1+\alpha_2)s}p_{12}(s)\mathrm{d}s\int_0^{d_2}e^{-(\alpha_1+\alpha_2)s}p_{21}(s)\mathrm{d}s\right\} \\
& \left\{1-\int_0^{\infty}e^{-\alpha_2 s}p_{12}(s)\mathrm{d}s\int_0^{d_2}e^{-\alpha_2 s}p_{21}(s)\mathrm{d}s\right\}.
\end{aligned}$$

Proof: Let A_j^i denotes the event that the first time the length of the excursion above L reaches d_1 happens during the ith excursion above L, and the first time the length of the excursion below L reaches d_2 happens during the jth excursion below L. So we have,

$$\begin{aligned}
&E\left(\exp\left\{-\alpha_1\tau_1^{+}-\alpha_2\tau_2^{+}\right\}\mathbf{1}_{\left\{\tau_1^{+}<\tau_2^{+}\right\}}\right) \\
&=\sum_{j=1}^{\infty}\sum_{i=1}^{j}E\left(\exp\left\{-\alpha_1\tau_1^{+}-\alpha_2\tau_2^{+}\right\}\Big|A_j^i\right)P\left(A_j^i\right),
\end{aligned}$$

and

$$
\begin{aligned}
&E\left(\exp\left\{-\alpha_1\tau_1^+ - \alpha_2\tau_2^+\right\}\mathbf{1}_{\{\tau_1^+>\tau_2^+\}}\right)\\
&=\sum_{j=1}^{\infty}\sum_{i=j+1}^{\infty} E\left(\exp\left\{-\alpha_1\tau_1^+ - \alpha_2\tau_2^+\right\}\Big|A_j^i\right)P\left(A_j^i\right).
\end{aligned}
$$

Since excursions above and below alternate, given event A_j^i, τ_1^+ is comprised of $i-1$ full excursions below L with the length less than d_2, $i-1$ full excursions above barrier L with the length less than d_1 and the last one with the length d_1. We have

$$
\tau_1^+\Big|A_j^i = U_{1,1}^+ + U_{1,2}^+ + \cdots + U_{1,i-1}^+ + U_{2,1}^+ + U_{2,2}^+ + \cdots + U_{2,i-1}^+ + d_1,
$$

where $U_{1,k}^+ < d_1$ for $k=1,\cdots,i-1$, $U_{2,k}^+ < d_2$ for $k=1,\cdots,j-1$, $U_{1,i}^+ \geq d_1$ and $U_{2,j}^+ \geq d_2$. For simplicity, we denote the above condition of $U_{n,k}^+$'s by C. Similarly, for τ_2^+, we have

$$
\tau_2^+\Big|A_j^i = U_{1,1}^+ + U_{1,2}^+ + \cdots + U_{1,j}^+ + U_{2,1}^+ + U_{2,2}^+X + \cdots + U_{2,j-1}^+ + d_2,
$$

where $U_{n,k}^+$'s satisfy the condition C.

More importantly, due to the Markov property of X^+, these excursions are independent of each other. $U_{1,n}^+$'s have distribution P_{12}; $U_{2,n}^+$'s have distribution P_{21}. As a result, when $i \leq j$,

$$
\begin{aligned}
& E\left(\exp\left\{-\alpha_1\tau_1^+ - \alpha_2\tau_2^+\right\}\Big|A_j^i\right)\\
= \;& E\left(\exp\left\{-\alpha_1\left\{\sum_{k=1}^{i-1}\left(U_{1,k}^+ + U_{2,k}^+\right) + d_1\right\}\right.\right.\\
& \left.\left.-\alpha_2\left\{\sum_{k=1}^{j-1}\left(U_{1,k}^+ + U_{2,k}^+\right) + U_{1,j}^+ + d_2\right\}\right\}\Big|C\right)\\
= \;& e^{-\alpha_1 d_1 - \alpha_2 d_2}\left\{\int_0^{d_1} e^{-(\alpha_1+\alpha_2)s}\frac{p_{12}(s)}{P_{12}(d_1)}\mathrm{d}s\right\}^{i-1}\left\{\int_{d_1}^{\infty} e^{-\alpha_2 s}\frac{p_{12}(s)}{\bar{P}_{12}(d_1)}\mathrm{d}s\right\}\\
& \left\{\int_0^{\infty} e^{-\alpha_2 s}p_{12}(s)\mathrm{d}s\right\}^{j-i}\left\{\int_0^{d_2} e^{-\alpha_2 s}\frac{p_{21}(s)}{P_{21}(d_2)}\mathrm{d}s\right\}^{j-i}\\
& \left\{\int_0^{d_2} e^{-(\alpha_1+\alpha_2)s}\frac{p_{21}(s)}{P_{21}(d_2)}\mathrm{d}s\right\}^{i-1},
\end{aligned}
$$

and

$$
P(A_j^i) = P_{12}(d_1)^{i-1}P_{21}(d_2)^{j-1}\bar{P}_{12}(d_1)\bar{P}_{21}(d_2).
$$

We have therefore

$$
\begin{aligned}
& E\left(\exp\left\{-\alpha_1\tau_1^+ - \alpha_2\tau_2^+\right\}\mathbf{1}_{\{\tau_1^+<\tau_2^+\}}\right) \\
= & \sum_{j=1}^{\infty}\sum_{i=1}^{j} E\left(\exp\left\{-\alpha_1\tau_1^+ - \alpha_2\tau_2^+\right\}\Big| A_j^i\right) P\left(A_j^i\right) \\
= & \frac{e^{-\alpha_1 d_1-\alpha_2 d_2}\bar{P}_{21}(d_2)\int_{d_1}^{\infty} e^{-\alpha_2 u} p_{12}(s)\mathrm{d}s}{G(d_1,d_2)}.
\end{aligned}
$$

The proof of the case when $\tau_1^+ > \tau_2^+$ and the proof of (15) and (16) follow the same steps.

□

Remark: We can get $E\left(\exp\left\{-\alpha_1\tau_1^+ - \alpha_2\tau_2^+\right\}\right)$ by adding up (13) and (14) and $E\left(\exp\left\{-\alpha_1\tau_1^- - \alpha_2\tau_2^-\right\}\right)$ by adding up (15) and (16).

4 Main Results

In this section we show how to obtain results for standard Brownian motions through $X^{\pm}$.

In order to simplify the expressions, we define

$$\Psi(x) = 2\sqrt{\pi}x\mathscr{N}\left(\sqrt{2}x\right) - \sqrt{\pi}x + e^{-x^2},$$

where $\mathscr{N}(.)$ is the cumulative distribution function for the standard Normal distribution.

Theorem 1 *For a Brownian motion W^μ with $W_0^\mu = 0$, $\mu \geq 0$, $\tau_1^{W^\mu}$, $\tau_2^{W^\mu}$ and τ^{W^μ} defined as in (3), (4) and (5) with $S_t = W_t^\mu$, we have following Laplace transforms:*

$$E\left(e^{-\beta\tau^{W^\mu}}\mathbf{1}_{\{\tau_1^{W^\mu}<\tau_2^{W^\mu}\}}\right) = \frac{e^{-\beta d_1}\left\{\sqrt{d_2}\Psi\left(\mu\sqrt{\frac{d_1}{2}}\right) + \mu\sqrt{\frac{d_1 d_2\pi}{2}}\right\}}{\sqrt{d_2}\Psi\left(\sqrt{\frac{(2\beta+\mu^2)d_1}{2}}\right) + \sqrt{d_1}\Psi\left(\sqrt{\frac{(2\beta+\mu^2)d_2}{2}}\right)}, \tag{17}$$

$$E\left(e^{-\beta\tau^{W^\mu}}\mathbf{1}_{\{\tau_1^{W^\mu}>\tau_2^{W^\mu}\}}\right) = \frac{e^{-\beta d_2}\left\{\sqrt{d_1}\Psi\left(\mu\sqrt{\frac{d_2}{2}}\right) - \mu\sqrt{\frac{d_1 d_2\pi}{2}}\right\}}{\sqrt{d_2}\Psi\left(\sqrt{\frac{(2\beta+\mu^2)d_1}{2}}\right) + \sqrt{d_1}\Psi\left(\sqrt{\frac{(2\beta+\mu^2)d_2}{2}}\right)}, \tag{18}$$

$$
\begin{aligned}
& E\left(e^{-\beta\tau^{W^\mu}}\right) \qquad (19) \\
= & \frac{e^{-\beta d_1}\left\{\sqrt{d_2}\Psi\left(\mu\sqrt{\frac{d_1}{2}}\right) + \mu\sqrt{\frac{d_1 d_2\pi}{2}}\right\} + e^{-\beta d_2}\left\{\sqrt{d_1}\Psi\left(\mu\sqrt{\frac{d_2}{2}}\right) - \mu\sqrt{\frac{d_1 d_2\pi}{2}}\right\}}{\sqrt{d_2}\Psi\left(\sqrt{\frac{(2\beta+\mu^2)d_1}{2}}\right) + \sqrt{d_1}\Psi\left(\sqrt{\frac{(2\beta+\mu^2)d_2}{2}}\right)}.
\end{aligned}
$$

For a standard Brownian motion, the special case when $\mu = 0$, we have

$$E\left(e^{-\beta\tau^W}\mathbf{1}_{\left\{\tau_1^W<\tau_2^W\right\}}\right) = \frac{\sqrt{d_2}e^{-\beta d_1}}{\sqrt{d_2}\Psi\left(\sqrt{\beta d_1}\right)+\sqrt{d_1}\Psi\left(\sqrt{\beta d_2}\right)}, \tag{20}$$

$$E\left(e^{-\beta\tau^W}\mathbf{1}_{\left\{\tau_1^W>\tau_2^W\right\}}\right) = \frac{\sqrt{d_1}e^{-\beta d_2}}{\sqrt{d_2}\Psi\left(\sqrt{\beta d_1}\right)+\sqrt{d_1}\Psi\left(\sqrt{\beta d_2}\right)}, \tag{21}$$

$$E\left(e^{-\beta\tau^W}\right) = \frac{\sqrt{d_2}e^{-\beta d_1}+\sqrt{d_1}e^{-\beta d_2}}{\sqrt{d_2}\Psi\left(\sqrt{\beta d_1}\right)+\sqrt{d_1}\Psi\left(\sqrt{\beta d_2}\right)}. \tag{22}$$

Proof: According to the definitions of $X^{\pm}$ we have

$$X_t^- \leq W_t^\mu \leq X_t^+.$$

Furthermore, for any two processes satisfying $Y^{(1)} \leq Y^{(2)}$, the longest excursion of $Y^{(1)}$ above a barrier before any fixed time is not longer than the one of $Y^{(2)}$; and the longest excursion of $Y^{(1)}$ below a barrier before any fixed time is not shorter than the one of $Y^{(2)}$. Together with the definition of $\tau_1^{\pm}$ and $\tau_2^{\pm}$ we have therefore

$$\tau_1^+ \leq \tau_1^{W^\mu} \leq \tau_1^-, \quad \tau_2^+ \geq \tau_2^{W^\mu} \geq \tau_2^-.$$

Notice that $E\left(e^{-\beta\tau_1^S}\mathbf{1}_{\left\{\tau_1^S<\tau_2^S\right\}}\right)$ is a decreasing function of τ_1^S and an increasing function of τ_2^S; and $E\left(e^{-\beta\tau_2^S}\mathbf{1}_{\left\{\tau_1^S>\tau_2^S\right\}}\right)$ is a decreasing function of τ_2^S and an increasing function of τ_1^S, we have therefore

$$E\left(e^{-\beta\tau_1^+}\mathbf{1}_{\left\{\tau_1^+<\tau_2^+\right\}}\right) \geq E\left(e^{-\beta\tau_1^{W^\mu}}\mathbf{1}_{\left\{\tau_1^{W^\mu}<\tau_2^{W^\mu}\right\}}\right) \geq E\left(e^{-\beta\tau_1^-}\mathbf{1}_{\left\{\tau_1^-<\tau_2^-\right\}}\right) \tag{23}$$

and

$$E\left(e^{-\beta\tau_2^+}\mathbf{1}_{\left\{\tau_1^+>\tau_2^+\right\}}\right) \leq E\left(e^{-\beta\tau_2^{W^\mu}}\mathbf{1}_{\left\{\tau_1^{W^\mu}>\tau_2^{W^\mu}\right\}}\right) \leq E\left(e^{-\beta\tau_2^-}\mathbf{1}_{\left\{\tau_1^->\tau_2^-\right\}}\right). \tag{24}$$

According to (11), (12) and Lemma 3.1, we can actually calculate that

$$E\left(e^{-\beta\tau_1^+}\mathbf{1}_{\left\{\tau_1^+<\tau_2^+\right\}}\right) = \frac{e^{-\beta d_1}\bar{P}_{12}(d_1)}{1-\int_0^{d_1}e^{-\beta s}p_{12}(s)\mathrm{d}s\int_0^{d_2}e^{-\beta s}p_{21}(s)\mathrm{d}s},$$

$$E\left(e^{-\beta\tau_1^-}\mathbf{1}_{\left\{\tau_1^-<\tau_2^-\right\}}\right) = \frac{e^{-\beta d_1}\bar{P}_{12}(d_1)\int_0^{d_2}e^{-\beta s}p_{21}(s)\mathrm{d}s}{1-\int_0^{d_1}e^{-\beta s}p_{12}(s)\mathrm{d}s\int_0^{d_2}e^{-\beta s}p_{21}(s)\mathrm{d}s},$$

where

$$\bar{P}_{12}(d_1) = 1 - e^{-2\eta\mu}\mathscr{N}\left(\mu\sqrt{d_1}-\frac{\eta}{\sqrt{d_1}}\right) - \mathscr{N}\left(-\mu\sqrt{d_1}-\frac{\eta}{\sqrt{d_1}}\right),$$

$$\int_0^{d_1} e^{-\beta u} p_{12}(u)\mathrm{d}u = e^{-\left(\mu+\sqrt{2\beta+\mu^2}\right)\eta}\mathcal{N}\left(\sqrt{(2\beta+\mu^2)\,d_1}-\frac{\eta}{\sqrt{d_1}}\right) + e^{\left(\sqrt{2\beta+\mu^2}-\mu\right)\eta}\mathcal{N}\left(-\sqrt{(2\beta+\mu^2)\,d_1}-\frac{\eta}{\sqrt{d_1}}\right),$$

$$\int_0^{d_2} e^{-\beta u} p_{21}(u)\mathrm{d}u = e^{\left(\mu-\sqrt{2\beta+\mu^2}\right)\eta}\mathcal{N}\left(\sqrt{(2\beta+\mu^2)\,d_2}-\frac{\eta}{\sqrt{d_2}}\right) + e^{\left(\mu+\sqrt{2\beta+\mu^2}\right)\eta}\mathcal{N}\left(-\sqrt{(2\beta+\mu^2)\,d_2}-\frac{\eta}{\sqrt{d_2}}\right).$$

By taking the limit as $\eta \to 0$ we have

$$\lim_{\eta\to 0} E\left(e^{-\beta\tau_1^+}\mathbf{1}_{\{\tau_1^+<\tau_2^+\}}\right) = \lim_{\eta\to 0} E\left(e^{-\beta\tau_1^-}\mathbf{1}_{\{\tau_1^-<\tau_2^-\}}\right) = \frac{e^{-\beta d_1}\left\{\sqrt{d_2}\Psi\left(\mu\sqrt{\frac{d_1}{2}}\right)+\mu\sqrt{\frac{d_1 d_2\pi}{2}}\right\}}{\sqrt{d_2}\Psi\left(\sqrt{\frac{(2\beta+\mu^2)d_1}{2}}\right)+\sqrt{d_1}\Psi\left(\sqrt{\frac{(2\beta+\mu^2)d_2}{2}}\right)}.$$

Both bounds suggested by (23) have the same limit, so

$$\begin{aligned} E\left(e^{-\beta\tau^{W^\mu}}\mathbf{1}_{\{\tau_1^{W^\mu}<\tau_2^{W^\mu}\}}\right) &= E\left(e^{-\beta\tau_1^{W^\mu}}\mathbf{1}_{\{\tau_1^{W^\mu}<\tau_2^{W^\mu}\}}\right) \\ &= \lim_{\eta\to 0} E\left(e^{-\beta\tau_1^+}\mathbf{1}_{\{\tau_1^+<\tau_2^+\}}\right) = \lim_{\eta\to 0} E\left(e^{-\beta\tau_1^-}\mathbf{1}_{\{\tau_1^-<\tau_2^-\}}\right) \\ &= \frac{e^{-\beta d_1}\left\{\sqrt{d_2}\Psi\left(\mu\sqrt{\frac{d_1}{2}}\right)+\mu\sqrt{\frac{d_1 d_2\pi}{2}}\right\}}{\sqrt{d_2}\Psi\left(\sqrt{\frac{(2\beta+\mu^2)d_1}{2}}\right)+\sqrt{d_1}\Psi\left(\sqrt{\frac{(2\beta+\mu^2)d_2}{2}}\right)}. \end{aligned}$$

The equation (18) can be proved using the same arguments. Adding up (17) and (18) gives (19).
□

Remark: A similar result for a standard Brownian motion, i.e. $\mu = 0$ in the case when double barriers are considered can be found in [2].

If we let $\beta \to 0$, we get the following remarkable results.

Corollary 1.1 *The probability that W^μ achieves an excursion above 0 with length as least d_1 before it achieves an excursion below 0 with length at least d_2 is*

$$P\left(\tau_1^{W^\mu} < \tau_2^{W^\mu}\right) = \frac{\sqrt{d_2}\Psi\left(\mu\sqrt{\frac{d_1}{2}}\right)+\mu\sqrt{\frac{d_1 d_2\pi}{2}}}{\sqrt{d_2}\Psi\left(\mu\sqrt{\frac{d_1}{2}}\right)+\sqrt{d_1}\Psi\left(\mu\sqrt{\frac{d_2}{2}}\right)}. \tag{25}$$

Similarly, for a standard Brownian motion we have

$$P\left(\tau_1^W < \tau_2^W\right) = \frac{\sqrt{d_2}}{\sqrt{d_1}+\sqrt{d_2}}, \tag{26}$$

$$P\left(\tau_1^W > \tau_2^W\right) = \frac{\sqrt{d_1}}{\sqrt{d_1}+\sqrt{d_2}}. \tag{27}$$

Remark 1: The result stated by (26) has also been obtained in [2]. However, the result for Brownian motions with drift, (25) is presented here for the first time.

Remark 2: If we set $d_1 = d_2 = d$ in (25), we have for a standard Brownian motion

$$P\left(\tau_1^W < \tau_2^W\right) = P\left(\tau_1^W > \tau_2^W\right) = \frac{1}{2},$$

which can be explained by the symmetry of standard Brownian motions;

Remark 3: For a Brownian motion with positive drift, by setting $d_1 = d_2 = d$ in (26) and (27), we have

$$P\left(\tau_1^{W^\mu} < \tau_2^{W^\mu}\right) = \frac{1}{2} + \frac{\mu\sqrt{\frac{d\pi}{2}}}{\Psi\left(\frac{\mu^2 d}{2}\right)} > \frac{1}{2}, \quad P\left(\tau_1^{W^\mu} > \tau_2^{W^\mu}\right) = \frac{1}{2} - \frac{\mu\sqrt{\frac{d\pi}{2}}}{\Psi\left(\frac{\mu^2 d}{2}\right)} < \frac{1}{2},$$

because it has a tendency to move upwards.

If we only consider the excursion below 0, we have the following results.

Corollary 1.2 *For a Brownian motion W^μ with $W_0^\mu = 0$ and $\tau_2^{W^\mu}$ defined as in (4) with $S_t = W_t^\mu$, we the have the Laplace transform for $\tau_2^{W^\mu}$:*

$$E\left(e^{-\beta\tau_2^{W^\mu}}\right) = \frac{e^{-\beta d_2}\left\{\Psi\left(\mu\sqrt{\frac{d_2}{2}}\right) - \mu\sqrt{\frac{d_2\pi}{2}}\right\}}{\Psi\left(\sqrt{\frac{(2\beta+\mu^2)d_2}{2}}\right) + \sqrt{\frac{(2\beta+\mu^2)d_2}{2}}}. \tag{28}$$

When $\mu = 0$, we have the result for a standard Brownian motion:

$$E\left(e^{-\beta\tau_2^W}\right) = \frac{e^{-\beta d_2}}{\Psi\left(\sqrt{\beta d_2}\right) + \sqrt{\pi\beta d_2}}. \tag{29}$$

Proof: When $d_1 \to \infty$, we have $\tau_1 \to \infty$, therefore $\tau^S \to \tau_2^S$.

As a result, we have

$$E\left(e^{-\beta\tau_2^S}\right) = \lim_{d_1\to\infty} E\left(e^{-\beta\tau^S}\right).$$

□

Remark: As one of the most important results, (29) has been obtained in [7]. But the result for Brownian motions with drift, (28) is presented here for the first time.

So far we have been considering the case when the process starts from 0 and the barrier level is set to be 0. In practice, however, the barrier is different from the starting point of the underlying asset price in most cases. Therefore, in order to price the options, we introduce the follower theorems and corollaries.

Theorem 2 *For a Brownian motion W^μ with $W_0^\mu = 0$ and barrier $L = l$, the Laplace transform of $\tau_l^{W^\mu}$ is given by*
when $l < 0$,

$$
\begin{aligned}
& E\left(e^{-\beta\tau_l^{W^\mu}}\right) \qquad (30)\\
= \; & e^{-\beta d_1}\left\{1 - e^{2\mu l}\mathcal{N}\left(\mu\sqrt{d_1} + \frac{l}{\sqrt{d_1}}\right) - \mathcal{N}\left(-\mu\sqrt{d_1} + \frac{l}{\sqrt{d_1}}\right)\right\}\\
& + \left\{e^{\left(\mu+\sqrt{2\beta+\mu^2}\right)l}\mathcal{N}\left(\sqrt{\left(2\beta+\mu^2\right)d_1} + \frac{l}{\sqrt{d_1}}\right)\right.\\
& \left. + e^{\left(\mu-\sqrt{2\beta+\mu^2}\right)l}\mathcal{N}\left(-\sqrt{\left(2\beta+\mu^2\right)d_1} + \frac{l}{\sqrt{d_1}}\right)\right\}\\
& \frac{e^{-\beta d_1}\sqrt{d_2}\left\{\Psi\left(\mu\sqrt{\frac{d_1}{2}}\right) + \mu\sqrt{\frac{d_1 d_2 \pi}{2}}\right\} + e^{-\beta d_2}\sqrt{d_1}\left\{\Psi\left(\mu\sqrt{\frac{d_2}{2}}\right) - \mu\sqrt{\frac{d_1 d_2 \pi}{2}}\right\}}{\sqrt{d_2}\Psi\left(\sqrt{\frac{(2\beta+\mu^2)d_1}{2}}\right) + \sqrt{d_1}\Psi\left(\sqrt{\frac{(2\beta+\mu^2)d_2}{2}}\right)}.
\end{aligned}
$$

when $l > 0$,

$$
\begin{aligned}
& E\left(e^{-\beta\tau^{W^\mu}}\right) \qquad (31)\\
= \; & e^{-\beta d_2}\left\{1 - \mathcal{N}\left(\mu\sqrt{d_2} - \frac{l}{\sqrt{d_2}}\right) - e^{2\mu l}\mathcal{N}\left(-\mu\sqrt{d_2} - \frac{l}{\sqrt{d_2}}\right)\right\}\\
& + \left\{e^{\left(\mu-\sqrt{2\beta+\mu^2}\right)l}\mathcal{N}\left(\sqrt{\left(2\beta+\mu^2\right)d_2} - \frac{l}{\sqrt{d_2}}\right)\right.\\
& \left. + e^{\left(\mu+\sqrt{2\beta+\mu^2}\right)l}\mathcal{N}\left(-\sqrt{\left(2\beta+\mu^2\right)d_2} - \frac{l}{\sqrt{d_2}}\right)\right\}\\
& \frac{e^{-\beta d_1}\sqrt{d_2}\left\{\Psi\left(\mu\sqrt{\frac{d_1}{2}}\right) + \mu\sqrt{\frac{d_1 d_2 \pi}{2}}\right\} + e^{-\beta d_2}\sqrt{d_1}\left\{\Psi\left(\mu\sqrt{\frac{d_2}{2}}\right) - \mu\sqrt{\frac{d_1 d_2 \pi}{2}}\right\}}{\sqrt{d_2}\Psi\left(\sqrt{\frac{(2\beta+\mu^2)d_1}{2}}\right) + \sqrt{d_1}\Psi\left(\sqrt{\frac{(2\beta+\mu^2)d_2}{2}}\right)}.
\end{aligned}
$$

Proof: We only prove the case when $l < 0$. The same arguments apply to the case when $l > 0$. Define

$$T_l = \inf\left\{t \geq 0 \mid W_t^\mu = l\right\}.$$

The left hand side of (30) can be expressed as follow

$$E\left(e^{-\beta\tau_l^{W^\mu}}\right) = E\left(e^{-\beta\tau_l^{W^\mu}}\mathbf{1}_{\{T_l \geq d_1\}}\right) + E\left(e^{-\beta\tau_l^{W^\mu}}\mathbf{1}_{\{T_l < d_1\}}\right).$$

Moreover, we have

$$\begin{aligned} E\left(e^{-\beta\tau_l^{W^\mu}}\mathbf{1}_{\{T_l\geq d_1\}}\right) &= e^{-\beta d_1}P\left(T_l\geq d_1\right) \\ &= e^{-\beta d_1}\left\{1-e^{2\mu l}\mathscr{N}\left(\mu\sqrt{d_1}+\frac{l}{\sqrt{d_1}}\right)-\mathscr{N}\left(-\mu\sqrt{d_1}+\frac{l}{\sqrt{d_1}}\right)\right\}. \end{aligned}$$

$$\begin{aligned} E\left(e^{-\beta\tau_l^{W^\mu}}\mathbf{1}_{\{T_l<d_1\}}\right) &= E\left(e^{-\beta\left(T_l+\tau_l^{\widetilde{W}^\mu}\right)}\mathbf{1}_{\{T_l<d_1\}}\right) \\ &= E\left(e^{-\beta T_l}\mathbf{1}_{\{T_l<d_1\}}\right)E\left(e^{-\beta\tau_l^{\widetilde{W}^\mu}}\right)=E\left(e^{-\beta T_l}\mathbf{1}_{\{T_l<d_1\}}\right)E\left(e^{-\beta\tau^{W^\mu}}\right), \end{aligned}$$

where $\widetilde{W}^\mu$ stands for the Brownian motion starting from l. We have obtained $E\left(e^{-\beta\tau^{W^\mu}}\right)$ in Theorem 4.1. We also have that

$$\begin{aligned} & E\left(e^{-\beta T_l}\mathbf{1}_{\{T_l<d_1\}}\right) \\ = & \int_0^{d_1} e^{-\beta s}\frac{-l}{\sqrt{2\pi s^3}}\exp\left\{-\frac{(l-\mu s)^2}{2s}\right\}\mathrm{d}s \\ = & e^{\left(\mu+\sqrt{2\beta+\mu^2}\right)l}\mathscr{N}\left(\sqrt{\left(2\beta+\mu^2\right)d_1}+\frac{l}{\sqrt{d_1}}\right)+e^{\left(\mu-\sqrt{2\beta+\mu^2}\right)l}\mathscr{N}\left(-\sqrt{\left(2\beta+\mu^2\right)d_1}+\frac{l}{\sqrt{d_1}}\right). \end{aligned}$$

We have therefore proved (30).

□

We will now extend Theorem 4.4 to obtain the distribution of W at an exponential time. This will be an application of (30), (31) and Girsanov's theorem.

Theorem 3 *For a standard Brownian motion W with $W_0=0$, and τ_l^W defined as in (5) with $S_t=W_t$, we have the following result:*

For the case $l\geq 0$, when $x\geq l$,

$$P\left(W_{\widetilde{T}}\in\mathrm{d}x,\tau^W<\widetilde{T}\right)=\left\{a(d_2)e^{-\sqrt{2\gamma}(x-l)}+b_1p(x-l,d_1,d_2)\right\}\mathrm{d}x; \tag{32}$$

when $x<l$

$$P\left(W_{\widetilde{T}}\in\mathrm{d}x,\tau^W<\widetilde{T}\right)=\left\{a(d_2)e^{\sqrt{2\gamma}(x-l)}+b_1p(l-x,d_2,d_1)\right\}\mathrm{d}x; \tag{33}$$

For the case $l<0$, when $x\geq l$,

$$P\left(W_{\widetilde{T}}\in\mathrm{d}x,\tau^W<\widetilde{T}\right)=\left\{a(d_1)e^{-\sqrt{2\gamma}(x-l)}+b_2p(x-l,d_1,d_2)\right\}\mathrm{d}x; \tag{34}$$

when $x<l$

$$P\left(W_{\widetilde{T}}\in\mathrm{d}x,\tau^W<\widetilde{T}\right)=\left\{a(d_1)e^{\sqrt{2\gamma}(x-l)}+b_2p(l-x,d_2,d_1)\right\}\mathrm{d}x; \tag{35}$$

where $\widetilde{T}$ is a random variable independent of W, with an exponential distribution of parameter γ and

$$a(x) = \sqrt{\frac{2}{\gamma}} \left\{ e^{-\sqrt{2\gamma} l} \mathcal{N}\left(-\frac{l}{\sqrt{x}} + \sqrt{2\gamma x} \right) - e^{\sqrt{2\gamma} l} \mathcal{N}\left(\frac{l}{\sqrt{x}} + \sqrt{2\gamma x} \right) \right\},$$

$$b_1 = e^{-\sqrt{2\gamma} l} \mathcal{N}\left(-\frac{l}{\sqrt{d_2}} + \sqrt{2\gamma d_2} \right) + e^{\sqrt{2\gamma} l} \mathcal{N}\left(-\frac{l}{\sqrt{d_2}} - \sqrt{2\gamma d_2} \right),$$

$$b_2 = e^{\sqrt{2\gamma} l} \mathcal{N}\left(\frac{l}{\sqrt{d_1}} + \sqrt{2\gamma d_1} \right) + e^{-\sqrt{2\gamma} l} \mathcal{N}\left(\frac{l}{\sqrt{d_2}} - \sqrt{2\gamma d_1} \right),$$

$$\begin{aligned} p(x,y,z) \quad = \quad & \frac{\gamma \sqrt{2\pi y z} e^{-\sqrt{2\gamma}(x-l)}}{\sqrt{z}\Psi\left(\sqrt{\gamma y}\right) + \sqrt{y}\Psi\left(\sqrt{\gamma z}\right)} \left\{ \frac{e^{-\gamma y}}{2\sqrt{\pi \gamma y}} + \frac{e^{-\gamma z}}{2\sqrt{\pi \gamma z}} + \mathcal{N}\left(\frac{x-l}{\sqrt{y}} - \sqrt{2\gamma y} \right) \right. \\ & \left. - \mathcal{N}\left(-\sqrt{2\gamma y} \right) - \mathcal{N}\left(-\sqrt{2\gamma z} \right) - e^{2\sqrt{2\gamma}(x-l)} \mathcal{N}\left(-\frac{x-l}{\sqrt{y}} - \sqrt{2\gamma y} \right) \right\}. \end{aligned}$$

Proof: see appendix.

□

Similarly, we can obtain the result when we only consider the excursion below the barrier by taking the limit $d_1 \to \infty$.

Corollary 3.1 *For a standard Brownian motion W with $W_0 = 0$ and τ_2^W defined as in (4) with $S_t = W_t$, we have the following results:*

For the case $l \geq 0$, when $x \geq l$,

$$P\left(W_{\widetilde{T}} \in \mathrm{d}x, \tau^W < \widetilde{T} \right) = \left\{ a_2' e^{-\sqrt{2\gamma}(x-l)} + b_1' q_1(x-l) \right\} \mathrm{d}x; \tag{36}$$

when $x < l$

$$P\left(W_{\widetilde{T}} \in \mathrm{d}x, \tau^W < \widetilde{T} \right) = \left\{ a_2' e^{\sqrt{2\gamma}(x-l)} + b_1' q_2(x-l) \right\} \mathrm{d}x; \tag{37}$$

For the case $l < 0$, when $x \geq l$,

$$P\left(W_{\tilde{T}} \in \mathrm{d}x, \tau^W < \tilde{T} \right) = \left\{ a_1' e^{-\sqrt{2\gamma}(x-l)} + b_2' q_1(x-l) \right\} \mathrm{d}x; \tag{38}$$

when $x < l$

$$P\left(W_{\widetilde{T}} \in \mathrm{d}x, \tau^W < \widetilde{T} \right) = \left\{ a_1' e^{\sqrt{2\gamma}(x-l)} + b_2' q_2(x-l) \right\} \mathrm{d}x; \tag{39}$$

where

$$a_1' = \frac{2}{\gamma} \left\{ e^{-\sqrt{2\gamma} l} - e^{\sqrt{2\gamma} l} \right\}, \quad a_2' = a\left(d_2\right),$$

$$b_1' = b_1, \quad b_2' = e^{\sqrt{2\gamma} l},$$

$$q_1(x) = \sqrt{\frac{\gamma}{2}} e^{-\sqrt{2\gamma}x} \left(1 - \frac{2\sqrt{\pi\gamma d_2}}{2\sqrt{\pi\gamma d_2}\mathcal{N}\left(\sqrt{2\gamma d_2}\right) + e^{-\gamma d_2}}\right),$$

$$\begin{aligned} q_2(x) &= \frac{\gamma e^{\sqrt{2\gamma}x}\sqrt{2\pi d_2}}{2\sqrt{\pi\gamma d_2}\mathcal{N}\left(\sqrt{2\gamma d_2}\right) + e^{-\gamma d_2}} \left\{ \frac{e^{-\gamma d_2}}{2\sqrt{\pi\gamma d_2}} + \mathcal{N}\left(-\frac{x}{\sqrt{d_2}} - \sqrt{2\gamma d_2}\right) \right. \\ & \left. -\mathcal{N}\left(-\sqrt{2\gamma d_2}\right) - e^{-2\sqrt{2\gamma}x}\mathcal{N}\left(\frac{x}{\sqrt{d_2}} - \sqrt{2\gamma d_2}\right) \right\}; \end{aligned}$$

and where $\widetilde{T}$ is a random variable , independent of W, with an exponential distribution of parameter γ.

Remark: By using this result, we can calculate the explicit form of the Laplace transform of the price of the Parisian option defined in [7]. This approach is different from [7], where they try to find the Laplace transform of τ_2^W and the density of $W_{\tau_2^W}$, and the Laplace transform is given in form of double integral. Our approach produces explicit expressions without integrals.

5 Pricing Parisian Options

The result presented by (29) has been obtained in [7] and used to price Parisian options which consider the excursions at only one side of the barrier. Here we want to introduce the new Parisian options, considering the excursions at both sides of the barrier.

For example, we want to price a Parisian call option, the owner of which will obtain the right to exercise it when either the length of the excursion above the barrier reaches d_1, or the length of the excursion below the barrier reaches d_2 before T. Its price formula is given by

$$P_{min-call-in} = e^{-rT} E_Q \left((S_T - K)^+ \mathbf{1}_{\{\tau_L^S < T\}} \right),$$

where S is the underlying stock price, L is the barrier level, Q denotes the risk neutral measure. The subscript *min-call-in* means it is a Call option which will be triggered when the minimum of two stopping times, $\tau_{1,L}^S$ and $\tau_{2,L}^S$, is less than T, i.e. $\tau_L^S < T$. We assume S is a geometric Brownian motion defined as in (1). Set

$$m = \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right), \quad b = \frac{1}{\sigma}\ln\left(\frac{K}{x}\right), \quad l = \frac{1}{\sigma}\ln\left(\frac{L}{x}\right), \quad Y_t = mt + W_t.$$

We have

$$S_t = x\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\} = x\exp\left\{\sigma(mt + W_t)\right\} = xe^{\sigma Y_t}.$$

By applying Girsanov's Theorem, we have

$$P_{min-call-in} = e^{-(r+\frac{1}{2}m^2)T} E_P\left[\left(xe^{\sigma Y_T} - K\right)^+ e^{mY_T}\mathbf{1}_{\{\tau_l^Y < T\}}\right],$$

where P is a new measure, under which Y_t is a standard Brownian motion with $Y_0 = 0$. And we define

$$P^*_{min-call-in} = e^{\left(r+\frac{1}{2}m^2\right)T} P_{min-call-in}.$$

We are going to show that we can obtain the Laplace transform of $P^*_{min-call-in}$ w.r.t T, denoted by $\mathscr{L}_T$.

First of all, we have

$$\begin{aligned}
& E_P\left[\left(xe^{\sigma Y_{\widetilde{T}}} - K\right)^+ e^{mY_{\widetilde{T}}}\mathbf{1}_{\{\tau_l^Y < \widetilde{T}\}}\right] \\
= & \int_b^\infty (xe^{\sigma y} - K)\, e^{my} P\left(Y_{\widetilde{T}} \in \mathrm{d}y, \tau_l^Y < \widetilde{T}\right) \\
= & \int_0^\infty \gamma e^{-\gamma T} \int_b^\infty (xe^{\sigma y} - K)\, e^{my} P\left(Y_T \in \mathrm{d}y, \tau_l^Y < T\right) \mathrm{d}T \\
= & \gamma \int_0^\infty e^{-\gamma T} E_P\left[\left(xe^{\sigma Y_T} - K\right)^+ e^{mY_T}\mathbf{1}_{\{\tau_l^Y < T\}}\right] \mathrm{d}T \\
= & \gamma \mathscr{L}_T
\end{aligned}$$

Hence we have

$$\mathscr{L}_T = \frac{1}{\gamma}\int_b^\infty (xe^{\sigma y} - K)\, e^{my} P\left(Y_{\widetilde{T}} \in \mathrm{d}y, \tau_l^Y < \widetilde{T}\right).$$

By using the results in Theorem 4.5, this Laplace transform can be calculated explicitly.

When $b \geq 0$, i.e. $L \geq x$, we have

$$\mathscr{L}_T = \frac{xf(\sigma+m) - Kf(m)}{\sqrt{d_2}\Psi\left(\sqrt{\gamma d_1}\right) + \sqrt{d_1}\Psi\left(\sqrt{\gamma d_2}\right)},$$

where

$$\begin{aligned}
f(x) = & \frac{\sqrt{2\pi d_1 d_2}e^{b\left(x-\sqrt{2\gamma}\right)}}{\sqrt{2\gamma}-x}\left\{\frac{e^{-\gamma d_1}}{2\sqrt{\pi\gamma d_1}} + \frac{e^{-\gamma d_2}}{2\sqrt{\pi\gamma d_2}}\right. \\
& \left. + \mathscr{N}\left(\frac{b}{\sqrt{d_1}} - \sqrt{2\gamma d_1}\right) - \mathscr{N}\left(-\sqrt{2\gamma d_1}\right) - \mathscr{N}\left(-\sqrt{2\gamma d_2}\right)\right\} \\
& + \sqrt{2\pi d_1 d_2}\left\{\frac{e^{\left(x+\sqrt{2\gamma}\right)b}}{\sqrt{2\gamma}+x}\mathscr{N}\left(-\frac{b}{\sqrt{d_1}} - \sqrt{2\gamma d_1}\right)\right. \\
& \left. + \frac{2xe^{\frac{\left(x^2-2\gamma\right)d_1}{2}}}{2\gamma - x^2}\mathscr{N}\left(x\sqrt{d_1} - \frac{b}{\sqrt{d_1}}\right)\right\};
\end{aligned}$$

when $b < 0$, i.e. $L < x$, we have

$$\mathscr{L}_T = \frac{xg(\sigma+m) - Kg(m)}{\sqrt{d_2}\Psi\left(\sqrt{\gamma d_1}\right) + \sqrt{d_1}\Psi\left(\sqrt{\gamma d_2}\right)},$$

where

$$
\begin{aligned}
g(x) \;=\; & \sqrt{2\pi d_1 d_2}\left\{\frac{e^{b\left(x+\sqrt{2\gamma}\right)}}{\sqrt{2\gamma}+x}\left[\mathscr{N}\left(-\sqrt{2\gamma d_1}\right)+\mathscr{N}\left(-\sqrt{2\gamma d_2}\right)\right.\right.\\
& -\mathscr{N}\left(-\frac{b}{\sqrt{d_2}}-\sqrt{2\gamma d_2}\right)-\frac{e^{-\gamma d_1}}{2\sqrt{\pi\gamma d_1}}-\left.\frac{e^{-\gamma d_2}}{2\sqrt{\pi\gamma d_2}}\right]\\
& -\frac{e^{\left(x-\sqrt{2\gamma}\right)b}}{\sqrt{2\gamma}-x}\mathscr{N}\left(\frac{b}{\sqrt{d_2}}-\sqrt{2\gamma d_2}\right)\\
& +\frac{2x}{2\gamma-x^2}\left[e^{\frac{\left(x^2-2\gamma\right)d_2}{2}}\left(\mathscr{N}\left(x\sqrt{d_2}-\frac{b}{\sqrt{d_2}}\right)-\mathscr{N}\left(x\sqrt{d_2}\right)\right)\right.\\
& \left.\left.-e^{\frac{\left(x^2-2\gamma\right)d_1}{2}}\mathscr{N}\left(x\sqrt{d_1}\right)\right]+\frac{2\sqrt{2\gamma}}{2\gamma-x^2}\left[\frac{e^{-\gamma d_1}}{2\sqrt{\pi\gamma d_1}}+\frac{e^{-\gamma d_2}}{2\sqrt{\pi\gamma d_2}}\right]\right\}.
\end{aligned}
$$

A special case is when we only consider the excursions below the barrier. The results can be calculated based on corollary 4.6.

When $L \geq x$, we have

$$
\mathscr{L}_T=\left(\frac{1}{\sqrt{2\gamma}}-\frac{\sqrt{2\pi d_2}}{2\sqrt{\pi\gamma d_2}\mathscr{N}\left(\sqrt{2\gamma d_2}\right)+e^{-\gamma d_2}}\right)\left(\frac{xe^{\left(\sigma+m-\sqrt{2\gamma}\right)b}}{\sqrt{2\gamma}-\sigma-m}-\frac{Ke^{\left(m-\sqrt{2\gamma}\right)b}}{\sqrt{2\gamma}-m}\right);
$$

when $L < x$, we have

$$
\mathscr{L}_T=\frac{xh(\sigma+m)-Kh(m)}{2\sqrt{\pi\gamma d_2}\mathscr{N}\left(\sqrt{2\gamma d_2}\right)+e^{-\gamma d_2}},
$$

where

$$
\begin{aligned}
h(x) \;=\; & \frac{e^{b\left(x+\sqrt{2\gamma}\right)}}{\sqrt{2\gamma}+x}\left\{\sqrt{2\pi d_2}\left[\mathscr{N}\left(-\sqrt{2\gamma d_2}\right)-\mathscr{N}\left(-\frac{b}{\sqrt{d_2}}-\sqrt{2\gamma d_2}\right)\right]-\frac{e^{-\gamma d_2}}{\sqrt{2\gamma}}\right\}\\
& +\frac{2e^{-\gamma d_2}}{2\gamma-x^2}-\sqrt{2\pi d_2}\left\{\frac{e^{\left(x-\sqrt{2\gamma}\right)b}}{\sqrt{2\gamma}-x}\mathscr{N}\left(\frac{b}{\sqrt{d_2}}-\sqrt{2\gamma d_2}\right)\right.\\
& \left.+\frac{2xe^{\frac{\left(x^2-2\gamma\right)d_2}{2}}}{2\gamma-x^2}\left[\mathscr{N}\left(x\sqrt{d_2}-\frac{b}{\sqrt{d_2}}\right)-\mathscr{N}\left(x\sqrt{d_2}\right)\right]\right\}.
\end{aligned}
$$

Remark 1: It is the first time we manage to get the explicit expressions for the Laplace transforms of the option prices even for the one-sided excursion case. In [7] an expression involving double integrals is provided.

Remark 2: The prices can be calculated by numerical inversion of the Laplace transforms.

So far, we have shown how to obtain the Laplace transform of

$$
P^*_{min-call-in}=e^{\left(r+\frac{1}{2}m^2\right)T}P_{min-call-in}.
$$

For

$$P_{min-call-out} = e^{-rT} E_Q \left((S_T - K)^+ \mathbf{1}_{\{\tau_L^S > T\}} \right),$$

we can get the result from the relationship that

$$P_{min-call-out} = e^{-rT} E_Q \left\{ (S_T - K)^+ \right\} - P_{min-call-in}.$$

Furthermore, if we set

$$\tilde{\tau}_L^Y = \tau_{1,L}^Y \vee \tau_{2,L}^Y,$$

we can define another type of Parisian options by $\tilde{\tau}_L^Y$:

$$P_{max-call-in} = e^{-rT} E_Q \left((S_T - K)^+ \mathbf{1}_{\left\{\tilde{\tau}_L^S < T\right\}} \right).$$

In order to get its pricing formula, we should use the following relationship:

$$\mathbf{1}_{\left\{\tilde{\tau}_L^S < T\right\}} = \mathbf{1}_{\left\{\tau_{1,L}^S < T\right\}} + \mathbf{1}_{\left\{\tau_{2,L}^S < T\right\}} - \mathbf{1}_{\left\{\tau_L^S < T\right\}}.$$

We have therefore

$$P_{max-call-in} = P_{up-in-call} + P_{down-in-call} - P_{min-call-in}.$$

Similarly, from

$$P_{max-call-out} = e^{-rT} E_Q \left\{ (S_T - K)^+ \right\} - P_{max-call-in},$$

we can work out $P_{max-call-out}$.

6 Appendix

We prove Theorem 4.5 in this section. Let T be the final time. According to the definition of $\Psi(x)$, we have

$$\Psi(x) = 2\sqrt{\pi} x \mathcal{N}\left(\sqrt{2}x\right) - \sqrt{\pi}x + e^{-x^2} = \sqrt{\pi}x - \sqrt{\pi}x \mathrm{Erfc}\,(x) + e^{-x^2}.$$

It is not difficult to show that

$$E\left(e^{-\beta \tau^{W^\mu}}\right) = E\left(\int_0^\infty \beta e^{-\beta T} \mathbf{1}_{\left\{\tau^{W^\mu} < T\right\}} \mathrm{d}T\right).$$

By Girsanov's theorem, this is equal to

$$\int_0^\infty \beta e^{-(\beta + \frac{1}{2}\mu^2)T} E\left(e^{\mu W_T} \mathbf{1}_{\{\tau^W < T\}}\right) \mathrm{d}T.$$

Setting $\gamma = \beta + \frac{1}{2}\mu^2$ gives

$$\begin{aligned} E\left(e^{-\beta \tau^{W^\mu}}\right) &= \int_0^\infty (\gamma - \frac{1}{2}\mu^2) e^{-\gamma T} E\left(e^{\mu W_T} \mathbf{1}_{\{\tau^W < T\}}\right) \mathrm{d}T \\ &= \frac{\gamma - \frac{1}{2}\mu^2}{\gamma} E\left(e^{\mu W_{\tilde{T}}} \mathbf{1}_{\left\{\tau^W < \tilde{T}\right\}}\right), \end{aligned}$$

where $\widetilde{T}$ is a random variable, independent of W, with an exponential distribution of parameter γ. Assume $\mu > 0$. We have therefore when $l \geq 0$

$$
\begin{aligned}
& E\left(e^{\mu W_{\tilde{T}}}\mathbf{1}_{\{\tau^W<\widetilde{T}\}}\right) \\
= \quad & \frac{\gamma}{\gamma-\frac{1}{2}\mu^2}E\left(e^{-\beta\tau^{W^\mu}}\right) \\
= \quad & \frac{\gamma e^{-\gamma d_2}}{\gamma-\frac{1}{2}\mu^2}e^{\frac{d_2}{2}\mu^2}\left\{\mathcal{N}\left(-\mu\sqrt{d_2}+\frac{l}{\sqrt{d_2}}\right)-e^{2l\mu}\mathcal{N}\left(-\mu\sqrt{d_2}-\frac{l}{\sqrt{d_2}}\right)\right\} \\
& +\frac{\gamma\left\{e^{-\sqrt{2\gamma}l}\mathcal{N}\left(\sqrt{2\gamma d_2}-\frac{l}{\sqrt{d_2}}\right)+e^{\sqrt{2\gamma}l}\mathcal{N}\left(-\sqrt{2\gamma d_2}-\frac{l}{\sqrt{d_2}}\right)\right\}e^{\mu l}}{\left(\gamma-\frac{1}{2}\mu^2\right)\left\{\sqrt{d_2}\Psi\left(\sqrt{\gamma d_1}\right)+\sqrt{d_1}\Psi\left(\sqrt{\gamma d_2}\right)\right\}} \\
& \left[e^{-(\gamma-\frac{\mu^2}{2})d_1}\left\{\sqrt{d_2}\Psi\left(\mu\sqrt{\frac{d_1}{2}}\right)+\mu\sqrt{\frac{d_1d_2\pi}{2}}\right\}\right. \\
& \left.+e^{-(\gamma-\frac{\mu^2}{2})d_2}\left\{\sqrt{d_1}\Psi\left(\mu\sqrt{\frac{d_2}{2}}\right)-\mu\sqrt{\frac{d_1d_2\pi}{2}}\right\}\right] \\
= \quad & \frac{\gamma e^{-\gamma d_2}}{\gamma-\frac{1}{2}\mu^2}e^{\frac{d_2}{2}\mu^2}\left\{\mathcal{N}\left(-\mu\sqrt{d_2}+\frac{l}{\sqrt{d_2}}\right)-e^{2l\mu}\mathcal{N}\left(-\mu\sqrt{d_2}-\frac{l}{\sqrt{d_2}}\right)\right\} \\
& +\frac{\gamma\left\{e^{-\sqrt{2\gamma}l}\mathcal{N}\left(\sqrt{2\gamma d_2}-\frac{l}{\sqrt{d_2}}\right)+e^{\sqrt{2\gamma}l}\mathcal{N}\left(-\sqrt{2\gamma d_2}-\frac{l}{\sqrt{d_2}}\right)\right\}e^{\mu l}}{\left(\gamma-\frac{1}{2}\mu^2\right)\left\{\sqrt{d_2}\Psi\left(\sqrt{\gamma d_1}\right)+\sqrt{d_1}\Psi\left(\sqrt{\gamma d_2}\right)\right\}} \\
& \left[e^{-\gamma d_1}\left\{\sqrt{2\pi d_1d_2}\mu e^{\frac{d_1}{2}\mu^2}+\sqrt{d_2}\left\{1-\sqrt{\frac{d_1}{2}}\pi\mu e^{\frac{d_1}{2}\mu^2}\mathrm{Erfc}\left(\sqrt{\frac{d_1}{2}}\mu\right)\right\}\right\}\right. \\
& \left.+e^{-\gamma d_2}\sqrt{d_1}\left\{1-\sqrt{\frac{d_2}{2}}\pi\mu e^{\frac{d_2}{2}\mu^2}\mathrm{Erfc}\left(\sqrt{\frac{d_2}{2}}\mu\right)\right\}\right].
\end{aligned}
$$

We will now invert the moment generating function above. We have that

$$
\begin{aligned}
e^{\frac{d_2}{2}\mu^2}\mathcal{N}\left(-\mu\sqrt{d_2}+\frac{l}{\sqrt{d_2}}\right) &= \int_l^\infty e^{\mu x}\frac{1}{\sqrt{2\pi d_2}}e^{-\frac{x^2}{2d_2}}\mathrm{d}x, \\
e^{\frac{d_2}{2}\mu^2}e^{2l\mu}\mathcal{N}\left(-\mu\sqrt{d_2}-\frac{l}{\sqrt{d_2}}\right) &= \int_l^\infty e^{\mu x}\frac{1}{\sqrt{2\pi d_2}}e^{-\frac{(x-2l)^2}{2d_2}}\mathrm{d}x, \\
\frac{\mu}{\gamma-\frac{\mu^2}{2}} &= \int_0^\infty e^{\mu x}e^{-\sqrt{2\gamma}x}\mathrm{d}x-\int_{-\infty}^0 e^{\mu x}e^{\sqrt{2\gamma}x}\mathrm{d}x, \\
\frac{1}{\gamma-\frac{\mu^2}{2}} &= \int_0^\infty e^{\mu x}\frac{1}{\sqrt{2\gamma}}e^{-\sqrt{2\gamma}x}\mathrm{d}x+\int_{-\infty}^0 e^{\mu x}\frac{1}{\sqrt{2\gamma}}e^{\sqrt{2\gamma}x}\mathrm{d}x, \\
e^{\frac{d_1}{2}\mu^2} &= \int_{-\infty}^\infty e^{\mu x}\frac{1}{\sqrt{2\pi d_1}}\exp\left\{-\frac{x^2}{2d_1}\right\}\mathrm{d}x,
\end{aligned}
$$

$$1-\sqrt{\frac{d_i}{2}}\pi\mu e^{\frac{d_i}{2}\mu^2}\mathrm{Erfc}\left(\sqrt{\frac{d_i}{2}}\mu\right)=\int_{-\infty}^{0}e^{\mu x}\frac{-x}{d_i}e^{-\frac{x^2}{2d_i}}\mathrm{d}x.$$

The inversion of $\frac{e^{\frac{d_2}{2}\mu^2}}{\gamma-\frac{\mu^2}{2}}\mathscr{N}\left(-\mu\sqrt{d_2}+\frac{l}{\sqrt{d_2}}\right)$ is given below.

For $x\geq l$,

$$\int_l^\infty \frac{1}{\sqrt{2\pi d_2}}e^{-\frac{y^2}{2d_2}}\frac{1}{\sqrt{2\gamma}}e^{-\sqrt{2\gamma}(x-y)}\mathrm{d}y=\frac{e^{\gamma d_2}e^{-\sqrt{2\gamma}x}}{\sqrt{2\gamma}}\mathscr{N}\left(-\frac{l}{\sqrt{d_2}}+\sqrt{2\gamma d_2}\right);$$

For $x<l$,

$$\int_l^\infty \frac{1}{\sqrt{2\pi d_2}}e^{-\frac{y^2}{2d_2}}\frac{1}{\sqrt{2\gamma}}e^{\sqrt{2\gamma}(x-y)}\mathrm{d}y=\frac{e^{\gamma d_2}e^{\sqrt{2\gamma}x}}{\sqrt{2\gamma}}\mathscr{N}\left(-\frac{l}{\sqrt{d_2}}-\sqrt{2\gamma d_2}\right).$$

The inversion of $\frac{e^{\frac{d_2}{2}\mu^2}e^{2l\mu}}{\gamma-\frac{\mu^2}{2}}\mathscr{N}\left(-\mu\sqrt{d_2}-\frac{l}{\sqrt{d_2}}\right)$ is given below.

For $x\geq l$,

$$\int_l^\infty \frac{1}{\sqrt{2\pi d_2}}e^{-\frac{(y-2l)^2}{2d_2}}\frac{1}{\sqrt{2\gamma}}e^{-\sqrt{2\gamma}(x-y)}\mathrm{d}y=\frac{e^{\gamma d_2}e^{2l\sqrt{2\gamma}}e^{-\sqrt{2\gamma}x}}{\sqrt{2\gamma}}\mathscr{N}\left(\frac{l}{\sqrt{d_2}}+\sqrt{2\gamma d_2}\right);$$

For $x<l$,

$$\int_l^\infty \frac{1}{\sqrt{2\pi d_2}}e^{-\frac{(y-2l)^2}{2d_2}}\frac{1}{\sqrt{2\gamma}}e^{\sqrt{2\gamma}(x-y)}\mathrm{d}y=\frac{e^{\gamma d_2}e^{-2l\sqrt{2\gamma}}e^{\sqrt{2\gamma}x}}{\sqrt{2\gamma}}\mathscr{N}\left(\frac{l}{\sqrt{d_2}}-\sqrt{2\gamma d_2}\right).$$

The inversion of $\frac{\mu e^{\frac{d_1}{2}\mu^2}}{\gamma-\frac{\mu^2}{2}}$ is

$$\begin{aligned}&\int_0^\infty e^{-\sqrt{2\gamma}y}\frac{1}{\sqrt{2\pi d_1}}e^{-\frac{(x-y)^2}{2d_1}}\mathrm{d}y-\int_{-\infty}^0 e^{\sqrt{2\gamma}y}\frac{1}{\sqrt{2\pi d_1}}e^{-\frac{(x-y)^2}{2d_1}}\mathrm{d}y\\ =\;& e^{\gamma d_1}\left\{e^{-\sqrt{2\gamma}x}\mathscr{N}\left(\frac{x}{\sqrt{d_1}}-\sqrt{2\gamma d_1}\right)-e^{\sqrt{2\gamma}x}\mathscr{N}\left(-\frac{x}{\sqrt{d_1}}-\sqrt{2\gamma d_1}\right)\right\}.\end{aligned}$$

The inversion of $\frac{1-\sqrt{\frac{d_i}{2}}\pi\mu e^{\frac{d_i}{2}\mu^2}\mathrm{Erfc}\left(\sqrt{\frac{d_i}{2}}\mu\right)}{\gamma-\frac{\mu^2}{2}}$ is given below.

For $x\geqslant 0$,

$$\int_{-\infty}^0 \frac{-y}{d_i}e^{-\frac{y^2}{2d_i}}\frac{1}{\sqrt{2\gamma}}e^{-\sqrt{2\gamma}(x-y)}\mathrm{d}y=\frac{e^{-\sqrt{2\gamma}x}}{\sqrt{2\gamma}}-e^{\gamma d_i-\sqrt{2\gamma}x}\sqrt{2\pi d_i}\mathscr{N}\left(-\sqrt{2\gamma d_i}\right);$$

For $x<0$,

$$\begin{aligned}&\int_{-\infty}^x \frac{-y}{d_i}e^{-\frac{y^2}{2d_i}}\frac{1}{\sqrt{2\gamma}}e^{-\sqrt{2\gamma}(x-y)}\mathrm{d}y+\int_x^0 \frac{-y}{d_i}e^{-\frac{y^2}{2d_i}}\frac{1}{\sqrt{2\gamma}}e^{\sqrt{2\gamma}(x-y)}\mathrm{d}y\\ =\;&\frac{e^{\sqrt{2\gamma}x}}{\sqrt{2\gamma}}-e^{\gamma d_i-\sqrt{2\gamma}x}\sqrt{2\pi d_i}\mathscr{N}\left(\frac{x}{\sqrt{d_i}}-\sqrt{2\gamma d_i}\right)\\ &+e^{\gamma d_i+\sqrt{2\gamma}x}\sqrt{2\pi d_i}\left\{\mathscr{N}\left(\sqrt{2\gamma d_i}\right)-\mathscr{N}\left(\frac{x}{\sqrt{d_i}}+\sqrt{2\gamma d_i}\right)\right\}.\end{aligned}$$

Consequently, we can get Theorem 4.5.

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