Impact of Market Crises on Real Options

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Abstract: We study the impact of market crises on investment decisions through real option theory. The framework we consider involves a Brownian motion and a Poisson process, the jumps characterizing the crisis effects. We first analyze the consequences of different modeling choices. We then provide the real option characteristics and establish the existence of an optimal discount rate. We also characterize the optimal time to invest and derive some properties of its Laplace Transform (bounds, monotonicity, robustness). At last we specify the consequences of some wrong model specifications on the investment decision.

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1. Introduction

Investment has always been a crucial question for firms. Should a given project be undertaken? And, if so, when is it the best time to invest? In order to answer these questions, the neo-classical criterion of Net Present Value (N.P.V.) is still widely used. It consists in investing if and only if the sum of the project discounted benefits is higher than the sum of its discounted costs. Such a criterion does however have several weaknesses. Among many others, the following facts are often mentioned:

- The N.P.V. method does not take into account potential uncertainty of future cash flows;

- It uses an explicit calculation for the cost of the risk;

- It focuses on present time: the investment decision can only be taken now or never.

But, reality is often more complex and flexible including, for instance, optional components for the project: a firm may have the opportunity (but not the obligation) to undertake the project, not only at a precise and given time, but during a whole period of time (or even without any time limit). In that sense, these characteristics may be related to that of an American call option, the underlying asset being, for example, the ratio discounted benefits/discounted costs, and the strike level "1". Therefore, the N.P.V. criterion implies that the American option has to be exercised as soon as it is in the money, which is obviously a sub-optimal strategy.

The use of a method based on option theory, such as the real option theory would improve the optimality of the investment decision. Several articles appear as benchmark in this field. The seminal studies of Brennan et al. (1985), Mc Donald et al. (1986), Pindyck (1991) or Trigeorgis (1996) are often quoted as they present the fundamentals of this method, using particularly dynamic programming and arbitrage techniques. The literature on real options has been prolific from very technical papers to case studies and manuals for practitioners (see among many references, the book edited by Brennan and Trigeorgis (2000) or that of Schwartz and Trigeorgis (2001)). Such an approach better suits reality by taking into account project optional characteristics such as withdrawal, sequential investment, delocalization, crisis management... In that sense real option theory leads to a decision criterion that adapts to each particular project assessment.
But real options have also some specific characteristics compared to "classical" financial options. In particular, the "risk-neutral" logic widely used in option pricing cannot apply here: the real options' underlying asset corresponds to the investment project flows and is generally not quoted on financial markets. Any replicating strategy of the option payoff is then impossible. So the pricing is made under a prior probability measure (the historical probability measure or another measure chosen according to the investor's expectations and beliefs). Moreover, a specific feature of a real option framework is the key points of interest for the investor. More precisely, she is interested in:

- the cash flows generated by the project. They are represented by the "price" of the real option. Note that the notion of "price" is not so obvious in this framework. It corresponds rather to the value a particular investor gives to this project. However, for the sake of simplicity in the notations, we will use the terminology "price" in the rest of the paper.

- But also, the optimal time to invest. This optimal time corresponds to the exercising time of the real option.

Therefore, it is important noticing that real options are above all a management tool for decision taking. Once the investment project has been well-specified, the major concern for the investor is indeed summarized in the following question: "When is it optimal to invest in the project?" In that sense, knowing the value of the option is less important than knowing its optimal exercising time. For that reason, in this paper, we focus especially on the properties of this optimal time. Moreover, real options studies are usually written in a continuous framework for the underlying dynamics. But the existence of crises and shocks on investment markets generates discontinuities. The impact of these crises on the decision process is then an important feature to consider. This is especially relevant when some technical innovations may lead to instabilities in production fields.

For all these reasons, this paper is dedicated to the analysis of the exercising time properties in an unstable framework. The modeling of the underlying dynamics involves a mixed-diffusion (made up of Brownian motion and Poisson process). The jumps are negative as to represent troubles and difficulties occurring in the underlying market.
In the second section of this paper, we describe the framework of the study and analyze the consequences of different modeling choices. The crisis effect may be expressed via a Poisson process or the compensated martingale associated with it. Of course, there is an obvious relation between these models and they are equivalent from a static point of view. But when studying the real option characteristics and their sensitivity towards the jump size, these models lead to various outcomes.
After analyzing the real option characteristics in the third section, we focus on the discount rate. We prove the existence of an optimal discount rate, considering the maximization of the Laplace transform of the optimal time to invest as a choice criterion. We also characterize the average waiting time.
In the fifth section we study the robustness of the element decision characteristics. We first specify the robustness of the optimal time to enter the project with respect to the jump size. We establish in particular that its Laplace transform is a decreasing function.
Then, assuming that the investor only knows the expected value of the random jump size, we prove that this imperfect knowledge leads her to undertake the project too early.
In the last section, we focus on the impact of a wrong model specification, assuming that the investor believes in continuous underlying dynamics. In such a framework, we specify the error made in the optimal investment time.

All proofs are delayed in Appendix.

2. The model

2.1. Notation

In this paper, we consider a particular investor evolving in a universe, defined as a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. She has to decide whether she will undertake a given investment project and, if so, when it is optimal to invest. We assume that the investor has no time limit to take her decision. Consequently, the time horizon we consider is infinite. The investment opportunity value at time $t = 0$ is then on the form

$$C_0 = \sup_{\tau \in \Upsilon} \mathbb{E}\left[\exp(-\mu\tau)(S_\tau - 1)^+\right]$$

where $\mathbb{E}$ denotes the expectation with respect to the prior probability measure $\mathbb{P}$, Υ is the set of the $(\mathcal{F}_t)$- stopping times and $(S_t, t \geq 0)$ is the process of the profits/costs ratio.
It is worthwhile noticing that the discount rate μ is usually different from the instantaneous risk-free rate. We will come back later to the real meaning of discount rate in such a framework and to the problem related to its choice.
The profits/costs ratio related to the investment project is characterized by the following dynamics

$$\begin{cases} dS_t = S_{t^-}\left[\alpha dt + \sigma dW_t + \varphi dM_t\right] \\ S_0 = s_0 \end{cases} \tag{A}$$

where $(W_t, t \geq 0)$ is a standard $(\mathbb{P}, (\mathcal{F}_t))$-Brownian motion and $(M_t, t \geq 0)$ is the compensated martingale associated with a $(\mathbb{P}, (\mathcal{F}_t))$-Poisson process N. The Poisson process is assumed to have a constant intensity λ and the considered filtration is defined by $\mathcal{F}_t = \sigma\left(W_s,\ M_s, 0 \leq s \leq t\right)$. Equivalently, the process $(S_t,\ t \geq 0)$ may be written on the form:

$$S_t = s_0 \exp(X_t)$$

where $(X_t,\ t \geq 0)$ is a Lévy process with the Lévy exponent Ψ

$$\mathbb{E}\left(\exp\left(\xi X_t\right)\right) = \exp\left(t\Psi\left(\xi\right)\right)$$

with

$$\Psi\left(\xi\right) = \xi^2 \frac{\sigma^2}{2} + \xi\left(\alpha - \lambda\varphi - \frac{\sigma^2}{2}\right) - \lambda\left(1 - \left(1+\varphi\right)^{\xi}\right) \tag{2.1}$$

Hence, we have

$$\begin{aligned} \mathbb{E}\left(\exp\left(iX_1\right)\right) &= \exp\left(i\xi\left(\alpha - \lambda\varphi - \frac{\sigma^2}{2}\right) - \xi^2\frac{\sigma^2}{2} + \lambda\left(e^{i\xi\ln(1+\varphi)} - 1\right)\right) \\ &= \exp\left(-\Phi\left(\xi\right)\right) \end{aligned}$$

Therefore, the Lévy measure associated with the characteristic exponent Φ is expressed in terms of the Dirac measure δ as:

$$\nu\left(dx\right) = \lambda\delta_{\ln(1+\varphi)}\left(dx\right)$$

Assumptions
In the rest of the paper, the following hypothesis (H) holds.

$$\begin{cases} i) & 0 < s_0 < 1, \\ ii) & \sigma > 0 \\ iii) & 0 > \varphi > -1. \end{cases} \tag{H}$$

Assumption i) states that s_0 is (strictly) less than 1: this is not a restrictive hypothesis, since the problem we study is a "true" decision problem. In fact, delaying the project realization is only relevant in the case where the profits/costs ratio is less than one.
Assumption iii) states that the jump size is negative as we study a crisis situation. The jump process allows to take into account falls in the project business field. These negative jumps may be induced, for instance, by a brutal introduction of a direct substitute into the market, leading to a decrease in the potential sales. Moreover, we assume that the jump size is greater that -1. This hypothesis together with the identity

$$S_t = s_0\left(1+\varphi\right)^{N_t} \times e^{(\alpha-\lambda\varphi)t} \times e^{\sigma W_t - \frac{1}{2}\sigma^2 t}$$

ensure that the process S remains strictly positive.

We also impose the integrability condition

$$\mu > \sup\left(\alpha; 0\right) \tag{2.2}$$

There exists an optimal frontier L_φ^* such that

$$\sup_{\tau\in\Upsilon} \mathbb{E}\left(e^{-\mu\tau}\left(S_\tau - 1\right)^+\right) = \mathbb{E}\left(e^{-\mu\tau_{L_\varphi^*}}\left(S_{\tau_{L_\varphi^*}} - 1\right)^+\right)$$

where τ_L is the first hitting time of the boundary L by the process S, defined as

$$\tau_L = \inf\{t \geq 0; S_t \geq L\} \tag{2.3}$$

(For the proof, see for instance Darling et al. (1972) or Mordecki (1999)).
Before the profits/costs ratio S reaches the optimal boundary L_φ^*, it is optimal for the investor not to undertake the investment project and to wait. However, as soon as S goes beyond this threshold, it is optimal for her to invest.

2.2. Consequence of the modeling choice

In the framework previously described, we may work *a priori* with either of the two following models:

$$(A) \qquad \begin{cases} dS_t = S_{t^-}\left[\alpha dt + \sigma dW_t + \varphi dM_t\right] \\ S_0 = s_0 \end{cases}$$

$$(B) \qquad \begin{cases} dS_t = S_{t^-}\left[\overline{\alpha} dt + \sigma dW_t + \varphi dN_t\right] \\ S_0 = s_0 \end{cases}$$

In the case where all the parameters are constant, these models are obviously equivalent and writing

$$\alpha = \overline{\alpha} + \lambda\varphi \tag{2.4}$$

is sufficient to see why. Note that the integrability condition for model (B) is expressed as

$$\mu > \max\left(\overline{\alpha} + \lambda\varphi; 0\right)$$

However, when studying the sensitivity of the different option characteristics with respect to the jump size, choosing (A) or (B) really matters. Indeed, monotonicity properties are significantly different in both frameworks, as underlined below.

• Let us first focus on the *optimal time to enter the project*, characterized by its Laplace transform defined as $\mathbb{E}\left(\exp\left(-\mu\tau_{L_\varphi^*}\right)\right)$.

Considering model (A), if the initial value of the profits/costs ratio is not "too small", the Laplace transform of the optimal investment time is monotonic (this result is proved in Proposition 5.1). But this monotonicity property does not hold any more for model (B) as it is illustrated in Figure 1, which is done for the following set of parameters:

$$s_0 = 0.8 \qquad \lambda = 0.1 \qquad \overline{\alpha} = 0.05 \qquad \mu = 0.15 \qquad \sigma = 0.2$$

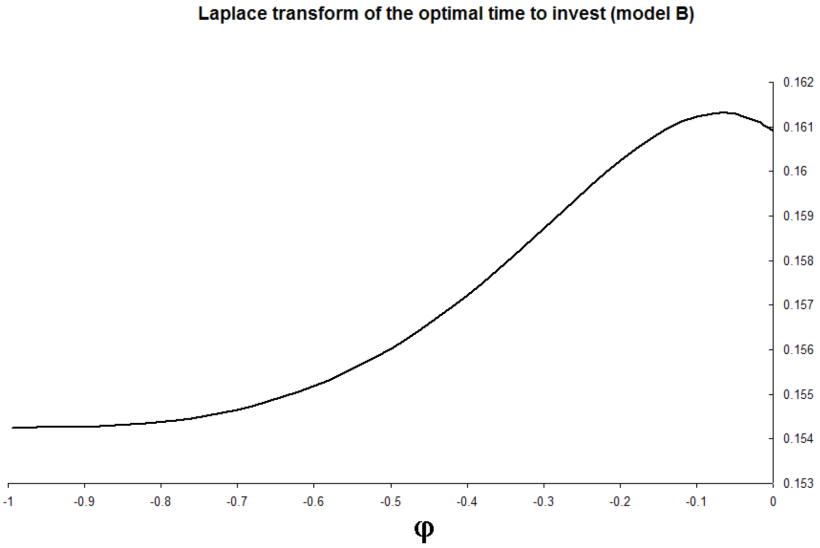

Figure 1

- We now focus on the *investment opportunity value* C_0.

Proposition 2.1. *Let us consider model* (B)*. Then the investment opportunity value is an increasing function of the jump size.*

Figure 2 illustrates Proposition 2.1. It represents the variations of the investment value with respect to the jump size for different values of the jump intensity and for the following set of parameters:

$$s_0 = 0.8 \qquad \overline{\alpha} = 0.05 \qquad \mu = 0.15 \qquad \sigma = 0.2$$

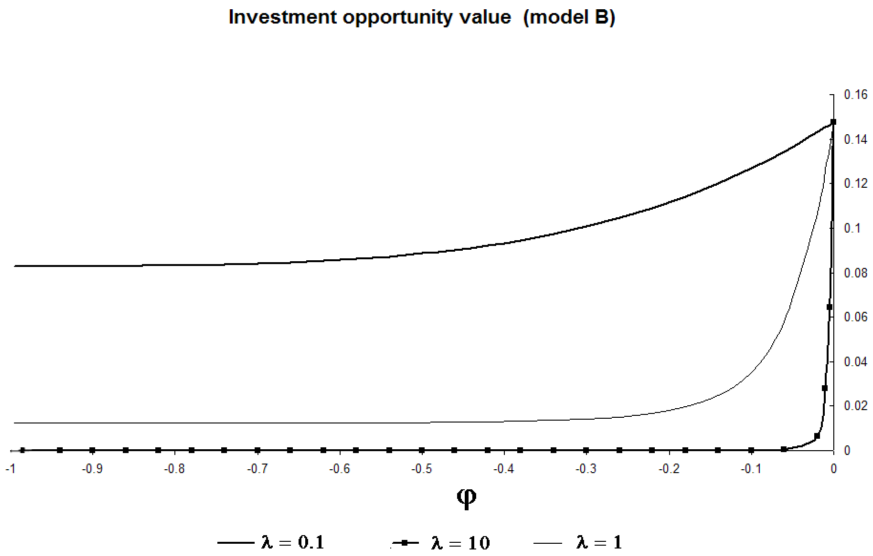

Figure 2

However, this property of the investment opportunity value does not hold any more when considering model (A). Intuitively, the studied model leads to a double effect of the jump size on the underlying level: φ has a positive effect on the underlying by increasing the drift but it also has a negative effect on the underlying by acting on the Poisson process level:

$$dS_t = S_{t^-}\left((\alpha - \lambda\varphi)\, dt + \sigma dW_t + \varphi dN_t\right)$$

This double effect explains the differences between models (A) and (B), and in particular accounts for the following result: in setting (A), the maximum value of C_0 is not necessarily obtained for $\varphi = 0$.
As a conclusion, it cannot be said that one of these models is better or more relevant than the other one. From a static point of view (with respect to the parameter φ), both are mathematically equivalent. In particular, given condition (2.4), they lead to the same first and second moments for S. But, from a dynamic point of view with respect to the jump size, they are different.
In the setting (B), crisis is only detected as the spread between the level of S before and after a shock while on the other hand, in the setting (A), there is an additional effect of the shocks on the drift term of S. Economically speaking, both have their own interests and motivations. However, once a model is chosen, the consequences of this choice must be kept in mind, especially the implications for the monotonicity properties of the real option characteristics.
In this study, since we are especially interested in the optimal time to invest, we choose a martingale representation for the stochastic part of $\dfrac{dS_t}{S_{t^-}}$, therefore the model defined by (A) prevails in the following.

3. The real option characteristics

In this section, we first recall the classical formulae for the optimal time to invest and for the investment opportunity.
We denote by k_φ the unique real number defined in terms of the Lévy exponent Ψ defined in euqation (2.1) since it satisfies:

$$k_\varphi > 1 \quad \text{and} \quad \Psi(k_\varphi) = \mu$$

Then the optimal profits/costs ratio L_φ^* satisfies:

$$L_\varphi^* = \frac{k_\varphi}{k_\varphi - 1}$$

The investment opportunity value at time 0 is given by:

$$C_0 = \left(\frac{s_0}{k_\varphi}\right)^{k_\varphi} \left(\frac{1}{k_\varphi - 1}\right)^{1-k_\varphi} \tag{3.1}$$

and the optimal investment time is characterized by its Laplace transform:

$$\mathbb{E}\left(\exp\left(-\mu\tau_{L_\varphi^*}\right)\right) = \left(\frac{s_0(k_\varphi - 1)}{k_\varphi}\right)^{k_\varphi} \tag{3.2}$$

(For detail proofs, see among others Gerber and Shiu (1994), Bellamy (1999), Mordecki (1999) and (2002)).
It can be noticed that k_φ, as well as the optimal profits/costs ratio L_φ^*, depend on φ, λ and μ.

Remark 1. *In the framework we deal with, the so-called principle of smooth pasting is satisfied. Such a principle is always satisfied in a continuous framework but if the model is driven by discontinuous Lévy processes, this property can fail. In the model we consider, however, the smooth pasting principle still holds (see for instance Chan (2003) and (2004), Boyarchenko and Levendorskii (2002), Alili and Kyprianou (2004) or Avram et al. (2004))*

It is also easy to check that the optimal profits/costs ratio satisfies $L_\varphi^* > 1$. This underlines the interest of waiting before undertaking the project, as well as the gain in optimality obtained from considering a real option approach rather than the standard N.P.V. method (see, for instance, Dixit et al. (1993)).
The value of the optimal ratio may be much greater than the limit value "1". This fact is at variance with the N.P.V. criterion and perfectly illustrates what Mc Donald et al. (1986) have called "The value of waiting to invest".

As an illustration, the optimal ratio L_φ^* is calculated in the table below for the following set of parameters:

$$\mu = 0.15 \qquad \lambda = 1 \qquad \alpha = 0.1 \qquad s_0 = 0.8$$

Values of the optimal benefits/costs ratio L_φ^* as a function of σ and φ

$\sigma \setminus \varphi$	-0.9	-0.5	-0.3	-0.1
0.1	16.25	6.59	4.39	3.29
0.2	17.13	6.93	4.74	3.69
0.4	18.24	8.19	6.08	5.12
0.6	20.23	10.26	8.19	7.29

Note that high values for the volatility coefficient σ are also considered in this study. This is relevant since the underlying market related to the investment project may be more highly volatile than traditional financial markets (for instance, markets related to new technology).

4. Optimal discount rate and average waiting time

4.1. Optimal discount rate

We now focus on the discount rate μ and present some general comments about its choice, which is indeed crucial in this study. The rate μ does not correspond to the instantaneous risk-free rate, traditionally used in the pricing of standard financial options. In fact, in this real option framework, the rate μ characterizes the preference of the investor for the present or her aversion for the future. Choosing the "right" μ is extremely difficult. Many different authors have been interested in this question (among many others, M. Weitzman (1998)). Some have also proved the existence of a specific relationship between discount rate and future growth rate (C. Gollier (1999), C. Gollier et al. (1998) and M.S. Kimball (1990)). The optimal choice criterion for the rate μ depends however on the considered framework. We present here a relevant criterion for this particular problem, corresponding to the maximization of the Laplace transform of the optimal investment time.

Proposition 4.1. *i) There exists a unique real number $\widehat{\mu}$ strictly positive such that*

$$\mathbb{E}\left(\exp\left(-\widehat{\mu}\tau_{L^*_{\widehat{\mu}}}\right)\right) = \max_{\mu} \mathbb{E}\left(\exp\left(-\mu\tau_{L^*_{\mu}}\right)\right)$$

The real number $\widehat{\mu}$ agrees with an optimal choice of the discount rate μ.
ii) The optimal discount rate $\widehat{\mu}$ increases with the jumps intensity and decreases with the jumps size.

This optimal discount rate is increasing with the absolute value of the jump size and with the intensity of the jumps. Such a behaviour seems rather logical as the occurrence and the frequency of negative jumps in the future make the value of the project decrease and represent an additional risk for the investor. The more important the jump intensity and size in absolute value are, the more the investor favors the present. Thus, she will choose a higher discount rate. Figure 3 shows the variations of the optimal rate $\widehat{\mu}$ with respect to φ for different values of λ and for the following set of parameters:

$$s_0 = 0.8 \qquad \alpha = 0.1 \qquad \sigma = 0.2$$

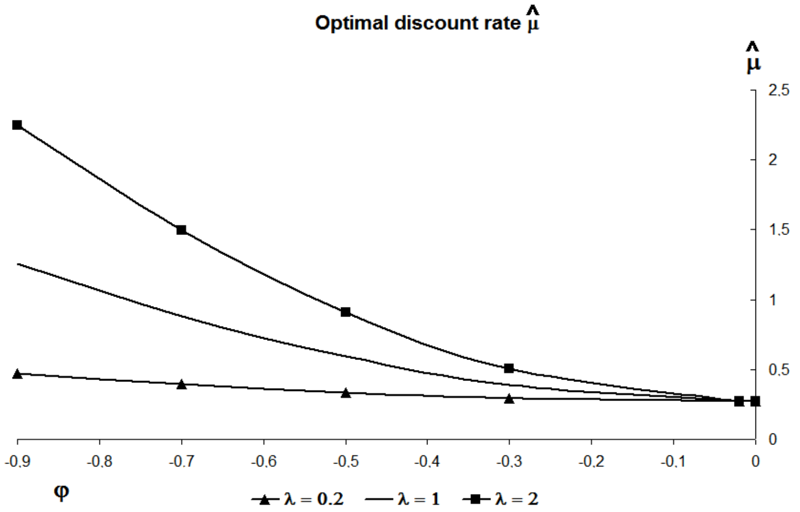

Figure 3

Remark 2. *Other criteria may have been considered in order to choose an optimal rate. For instance, the maximization of* C_0 *could appear as an alternative. But it is not a relevant criterion, since the function*

$$\mu \mapsto C_0 = \left(\frac{k_\varphi}{k_\varphi - 1} - 1\right)\left(\frac{s_0(k_\varphi - 1)}{k_\varphi}\right)^{k_\varphi}$$

is strictly decreasing.

4.2. Average waiting time

Another question relative to the best time to invest is of course that of the characterization of an average waiting time. If we denote it by T_c, it is defined as the unique element of $\mathbb{R}_+^*$ such that:

$$\mathbb{E}\left(\exp\left(-\widehat{\mu}\tau_{L^*_{\widehat{\mu}}}\right)\right) = \exp\left(-\widehat{\mu}T_c\right)$$

Hence T_c corresponds to the average waiting time. In fact, it is the certainty equivalent of τ_L when the utility criterion is exponential and the risk aversion coefficient is $\widehat{\mu}$. As previously seen, this rate $\widehat{\mu}$ can easily be interpreted as a future aversion coefficient (or a present preference coefficient) and T_c may be explicitly determined as:

$$T_c = -\frac{1}{\widehat{\mu}} \ln \mathbb{E}\left(\exp\left(-\widehat{\mu}\tau_{L^*_{\widehat{\mu}}}\right)\right)$$

From Proposition 4.1 we deduce that the average waiting time decreases with respect to the jump intensity as well as to the absolute value of the jump size. This mathematical property

can be economically understood as previously. In fact jumps induce additional risks, increasing with previous jump intensity and the jump size absolute value.
The average waiting time can be related to an exponential utility criterion. Therefore, the investor we consider appears to be risk averse, with an exponential utility function and a risk aversion coefficient of $\widehat{\mu}$. So, in her decision process, she will take into account the expected profit as well as the associated risk. She will tend to reduce the risk induced by the business field by entering earlier in the project. Obviously, the more she waits, the greater the probability of jumps and then the risk are.
Figure 4 highlights this fact. It represents the variations of the average waiting time with respect to the jump size. The graphs are done for different values of the jump intensity. All these curves converge to the same point as the jump size tends to zero: this point corresponds to the average waiting time in the model without jump, or, in other words, in an universe without crisis. The following set of parameters has been used:

$$s_0 = 0.8 \qquad \alpha = 0.1 \qquad \sigma = 0.2$$

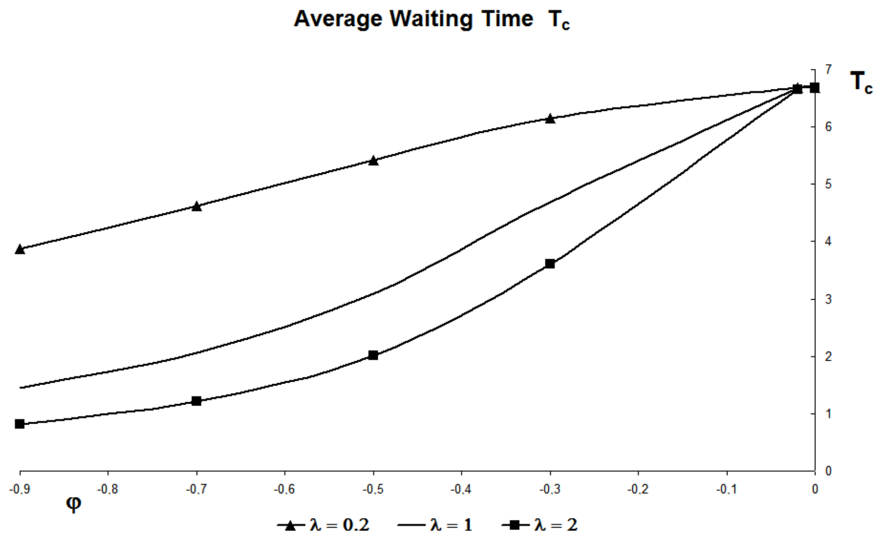

Figure 4

5. Robustness of the investment decision characteristics

All the different parameters of the model have to be estimated using historical data or strategic anticipations. Every estimation and calibration may lead to an error on the choice of the input parameters. Some stability (or robustness) of the results is an essential condition for a real practical use of a model.

5.1. Robustness of the optimal time to invest

As it has already been underlined, the optimal time to invest is the major concern of the investor. Hence, the robustness of its Laplace transform appears as a key point to be checked. We particularly focus on the study of the sensitivity of this quantity with respect to the jump size.
We study the behaviour of the Laplace transform of the optimal time to invest when the jump size is not perfectly known: the investor only knows that there exists $\underline{\varphi}$ and $\overline{\varphi}$ such that

$$-1 < \underline{\varphi} \leq \varphi \leq \overline{\varphi} < 0$$

We first provide a monotonicity result.

Proposition 5.1. *Let $\widehat{s_0}$ be the level defined as $\widehat{s_0} = \frac{k_0}{k_0-1}\exp\left(-\frac{1}{k_0-1}\right)$. We assume that s_0 satisfies*

$$\widehat{s_0} < s_0 < 1 \tag{5.1}$$

Then the Laplace transform of the optimal time to invest is an increasing function of the jump size.

Proposition 5.1 can be heuristically interpreted as follows: the more the jump size increases (hence decreases in absolute value), the more the investor delays entering the investment project. The maximum waiting time is attained in the lack of jump.

Remark 3. *The Assumption $\widehat{s_0} < s_0$ amounts to consider investment project only if the initial value is not "too small". From an economic point of view, such an assumption is not very restrictive. In fact, the investor will stop being interested in the project as soon as s_0 is below a given threshold. If, for example we consider the following standard set of parameters*

$$\alpha = 0.10 \quad ; \quad \sigma = 0.20 \quad ; \quad \mu = 0.15,$$

then we get

$$\widehat{s_0} = 0.276$$

Note that this level $\widehat{s_0}$ is far from the strike value 1.

Figure 5 shows the changes in the Laplace transform of the optimal time to invest time with respect to φ for different values of λ. The following set of parameters is used:

$$s_0 = 0.8 \quad ; \quad \alpha = 0.10 \quad ; \quad \sigma = 0.20 \quad ; \quad \mu = 0.15$$

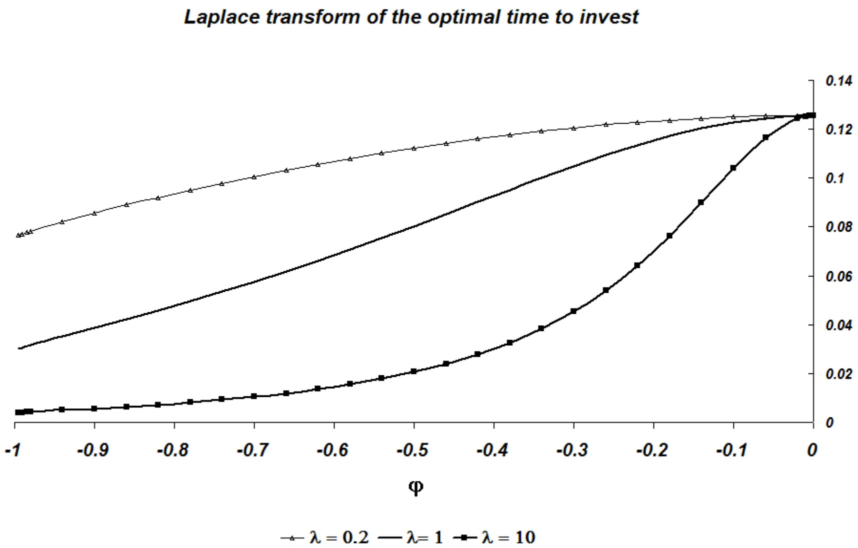

Figure 5

The robustness property of the Laplace transform is a straightforward consequence of Proposition 5.1.

Corollary 5.2. *We assume that condition (5.1) holds and*

$$-1 < \underline{\varphi} \leq \varphi \leq \overline{\varphi} < 0$$

Then we have

$$\mathbb{E}\left(\exp\left(-\mu\tau_{L^*_{\underline{\varphi}}}\right)\right) \leq \mathbb{E}\left(\exp\left(-\mu\tau_{L^*_{\varphi}}\right)\right) \leq \mathbb{E}\left(\exp\left(-\mu\tau_{L^*_{\overline{\varphi}}}\right)\right)$$

This result underlines the model robustness as far as the Laplace transform of the optimal time to invest is concerned. More precisely, if the investor does not know exactly the size of the jump, in other words the impact of the market crisis on the project, but knows however some boundaries for it, then she has an idea of the optimal time to enter the project. More precisely, the Laplace transform boundaries are expressed in terms of the boundaries for the market crisis impact. Equivalently, having some control or knowledge of the crisis impact enables the investor to have some control of her optimal time to invest.

5.2. Random jump size

We now consider the situation where the jump size is an unknown random variable Φ. We focus on the impact that this additional hazard may have on the investor decision.
Assuming that the investor estimates the jump size Φ by its expected value $\mathbb{E}(\Phi)$, we focus on the impact of such an error on her decision. Will she invest too early or too late? In order to

answer these questions, we compare the "true" Laplace transform of the optimal time to invest, with the Laplace transform estimated by means of $\mathbb{E}(\Phi)$.
The dynamics of the process of the project is now:

$$dS_t^{\Phi} = S_{t^-}^{\Phi}\left(\alpha dt + \sigma dW_t + \Phi dM_t\right) \quad ; \quad S_0^{\Phi} = s_0$$

and the investor builds her strategy from $S_t^{\mathbb{E}(\Phi)}$ where

$$dS_t^{\mathbb{E}(\Phi)} = S_{t-}^{\mathbb{E}(\Phi)}\left(\alpha dt + \sigma dW_t + \mathbb{E}(\Phi)\, dM_t\right) \quad ; \quad S_0^{\mathbb{E}(\Phi)} = s_0$$

We assume that the random variable Φ is independent of the filtration generated by the Brownian motion and the Poisson process.
Let L_{Φ}^* be the true optimal benefit-cost ratio. If the investor only knows $\mathbb{E}(\Phi)$, she estimates this ratio by $L_{\mathbb{E}(\Phi)}^*$. The next proposition provides a comparison between this two quantities.

Proposition 5.3. *We assume that condition (5.1) holds. Then the wrong specification in the model leads the investor to underestimate the optimal profits/costs ratio.*

Moreover we can precise the consequences of this error on the decision taking. We assume that the investor undertakes the project when the observed process of the benefits/costs ratio reaches what she supposes to be the optimal level. Therefore, her strategy is determined by the first hitting time of $L_{\mathbb{E}(\Phi)}^*$, instead of the first hitting time of L_{Φ}^* by process S. This proposition can be interpreted as follows: when the investor only knows $\mathbb{E}(\Phi)$, she tends to undertake the project too early.

6. Continuous model versus discontinuous model

In this section, we focus on the impact of a wrong model choice. This part extends the previous study of robustness. We suppose that the investor believes in a continuous underlying dynamics for S, while its true dynamics is given by (A). As a consequence, the investor governs her strategy according to the following process:

$$d\widetilde{S}_t = \widetilde{S}_t\left(\widetilde{\alpha} dt + \widetilde{\sigma} dW_t\right) \tag{$\widetilde{A}$}$$

where

$$\begin{cases} \widetilde{S}_0 & = s_0 \\ \widetilde{\alpha} & = \alpha \\ \widetilde{\sigma}^2 & = \sigma^2 + \lambda\varphi^2 \end{cases}$$

These equalities come directly from the calibration of both model (A) and $(\widetilde{A})$ on the same data set, leading to the same first and second moments for S and $\widetilde{S}$. The volatility parameter of the model without jump is different from that of the model with jumps: the absence of jump in the dynamics is indeed compensated by a higher volatility. In order to obtain the "equivalent" volatility, the right brackets of S and $\widetilde{S}$ have to be equal. The process $\widetilde{S}$ is called "equivalent process without jump".
We now focus on the impact of such a wrong specification on the investment time. To this end, we first consider the error in the optimal profits/costs ratio.

6.1. Error in the optimal profit-cost ratio

We denote by $\widetilde{L}_\varphi^*$ the optimal profits/costs ratio in the model defined by $(\widetilde{A})$. More precisely, using the same arguments as in Subsection 3, $\widetilde{L}_\varphi^*$ is given by the following ratio

$$\widetilde{L}_\varphi^* = \frac{\widetilde{k}_\varphi}{\widetilde{k}_\varphi - 1}$$

where $\widetilde{k}_\varphi$ is solution of

$$\widetilde{\psi}\,(k) = \frac{\sigma^2 + \lambda\varphi^2}{2} k^2 + \left(\alpha - \frac{\sigma^2 + \lambda\varphi^2}{2}\right) k = \mu \tag{6.1}$$

Note that this optimal ratio depends on the volatility parameter of the model, or equivalently on both jump parameters φ and λ. For the sake of simplicity, as we are especially interested in the sensitivity with respect to the jump size, we use the notation $\widetilde{L}_\varphi^*$.

Proposition 6.1. *The previous wrong specification of the model leads the investor to underestimate the optimal profits/costs ratio if and only if*

$$\sigma^2 + \lambda\varphi^2 + 2\alpha \geq \mu \tag{6.2}$$

Note that for usual values of the parameters, inequality (6.2) often holds. For instance, if we consider $\lambda = 1$, $\alpha = 0.1$, $\sigma = 0.2$, $\mu = 0.15$, then $\sigma^2 + \lambda\varphi^2 + 2\alpha \geq \mu$ is true for all φ in $]-1, 0[$.

As an illustration, the relative error (expressed in percentage) on the optimal profits/costs ratio

$$RE(L^*, \varphi) = 100 \times \left(\frac{L_\varphi^* - \widetilde{L}_\varphi^*}{L_\varphi^*}\right)$$

is calculated in the table below for different values of the jump size φ and for the standard set of parameters:

$$s_0 = 0.8 \quad ; \quad \alpha = 0.1 \quad ; \quad \sigma = 0.20 \quad ; \quad \mu = 0.15 \quad ; \quad \lambda = 1$$

Relative error on the optimal profits/costs ratio as a function of φ

φ	-0.995	-0.7	-0.5	-0.3	-0.1	-0.01
$RE(L^*, \varphi)$	38.30	15.81	7.11	1.87	0.08	0.01

Very naturally, the relative error becomes negligible as the jump size tends to zero. This error is still managable when the jump size is not too large (up to -0.5). For larger values however, the relative error becomes quite important to reach more than a third of the value of the ratio when the jump size is maximal.

Using the same argument as in the previous section, we can precise the consequences this wrong specification has on the investor's strategy. The investor's waiting time is determined by $\widetilde{L}_\varphi^*$ instead of L_φ^*. So, if condition (6.2), we can assert that the error in the model leads the investor to undertake the project too early.

This fact is brought to the fore by Figure 6. The optimal time to enter the project for a well-informed investor as well as that of the previous investor are respectively characterized by the Laplace transforms $\mathbb{E}\left(\exp\left(-\mu\tau_{L_\varphi^*}\right)\right)$ and $\mathbb{E}\left(\exp\left(-\mu\tau_{\widetilde{L}_\varphi^*}\right)\right)$.
Figure 6 represents the variations of these Laplace transforms with respect to the jump size φ. This is done for the following values:

$$s_0 = 0.8 \quad ; \quad \alpha = 0.1 \quad ; \quad \sigma = 0.20 \quad ; \quad \mu = 0.15 \quad ; \quad \lambda = 1$$

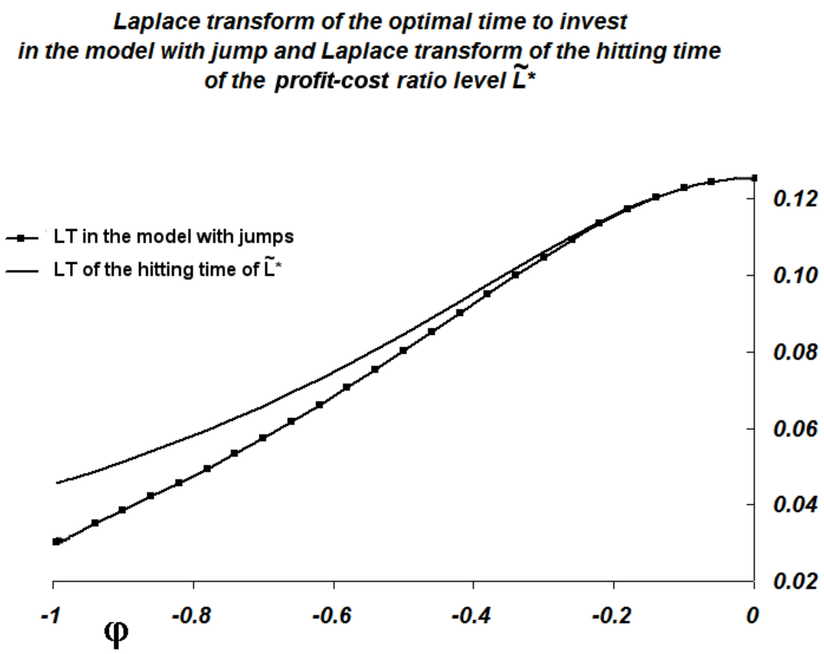

Figure 6

As another illustration, the relative error (expressed in percentage) on the Lapace transform of the optimal time to invest

$$RE(LT,\varphi) = 100 \times \left(\frac{LT - \widetilde{LT}}{LT}\right)$$

is calculated in the table below for different values of the jump size φ and for the previous set of parameters:

Relative error on the Laplace transform of the optimal time to invest as a function of φ

φ	-0.995	-0.7	-0.5	-0.3	-0.1	-0.02
$RE(LT,\varphi)$	-51.77	-14.68	-5.58	-1.26	-0.04	-0.01

The interpretation of these results is very similar to those associated with the relative error on the optimal profits/costs ratio. It can be noticed however that for large values of the jump size, the relative error becomes quite important to reach more than a half of the Laplace transform when the jump size is maximal. Hence, the impact of a wrong model specification could be important if the investor focuses on the optimal time to invest in the project.

6.2. Error in the investment opportunity value

In the "true" model with jumps, the investment opportunity value is C_0. If we assume that the investor becomes involved in the project when the "true" process S reaches the level $\widetilde{L}^*_\varphi$, then her investment opportunity value is

$$\widetilde{C}_0 = \left(\widetilde{L}^*_\varphi - 1\right) \mathbb{E}\left(\exp\left(-\mu\tau_{\widetilde{L}^*_\varphi}\right)\right)$$

where

$$\widetilde{C}_0 = \left(\widetilde{L}^*_\varphi - 1\right) \times \left(\frac{s_0}{\widetilde{L}^*_\varphi}\right)^{\widetilde{k}_\varphi}$$

where $\widetilde{k}_\varphi$ is solution of equation (6.1).
The following graph represents the variations of C_0 and $\widetilde{C}_0$ with respect to the jump size φ. Of course, since $\widetilde{L}^*_\varphi$ defers from the optimal frontier L^*, we have for any φ,

$$\widetilde{C}_0 \leq C_0$$

and the loss $C_0 - \widetilde{C}_0$ comes from a wrong investment time. This loss tends to zero when the jump size tends to zero and this fact was expected as $\widetilde{L}^*_\varphi$ tends to the optimal frontier L^* when φ tends to 0.
Figure 7 is done for the following values:

$$s_0 = 0.8 \quad ; \quad \alpha = 0.1 \quad ; \quad \sigma = 0.20 \quad ; \quad \mu = 0.15 \quad ; \quad \lambda = 1$$

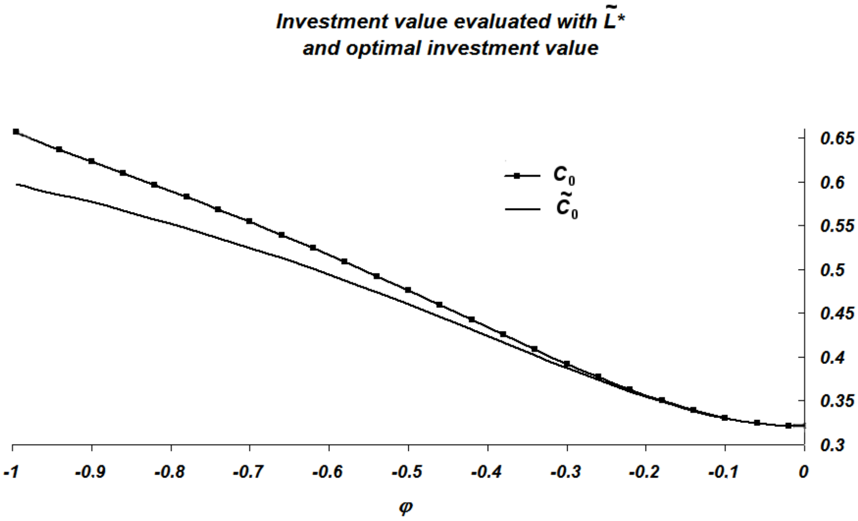

Figure 7

As another illustration, the relative error (expressed in percentage) on the investment opportunity value

$$RE(C,\varphi) = 100 \times \left(\frac{C_0 - \widetilde{C}_0}{C_0}\right)$$

is calculated in the table below for different values of the jump size φ and for the previous set of parameters:

Relative error on the investment opportunity value as a function of φ

φ	-0.995	-0.7	-0.5	-0.3	-0.1	-0.02
$RE(C,\varphi)$	9.02	5.33	3.19	1.14	0.06	0.01

The relative error remains managable even for large values of the jump size since it is always less than 10%. Therefore, the impact of a wrong model specification is relatively not so important if the investor focuses on the value of the investment opportunity.

7. Conclusion

In this paper, we study the impact of market crises on investment decision via real option theory. The investment project, modeled by its profits/costs ratio, is characterized by a mixed diffusion process, whose jumps represent the consequences of crises on the investment field.
After having analyzed the implications of different model choices, we study the real option associated with this investment project.
We establish the existence of an optimal discount rate, given a criterion based on this investment time and we characterize the average waiting time.

We study in details the properties of the optimal investment time, through its Laplace transform, and focus in particular on its robustness when the underlying dynamics of the project is not well-known or is wrongly specified. We interpret the results in terms of the investment decision. More precisely, when the investor bases her decision on the expected value of the random jump size, she tends to undertake the project too early. The same property holds if she believes in a continuous dynamics for the underlying project.
In this paper, we focus on a single investor. The complexity of reality suggests however that different other aspects, in particular strategic relationships between the economic agents, may play an important role. Investigating more general models involving strategic dimensions and game theory is a topic for future research.

8. Appendix

Proof of Proposition 2.1
Let S be defined by equation (B). We define $\mathcal{C}(\varphi, L)$ as
$\mathcal{C}(\varphi, L) = (L-1) \times \mathbb{E}\left(\exp\left(-\mu \tau_L^{\varphi}\right)\right)$ where $\tau_L^{\varphi} = \inf\{t \geq 0; S_t \geq L\}$. Hence

$$C_0(\varphi) = \mathcal{C}\left(\varphi, L_{\varphi}^*\right)$$

where L_{φ}^* is the optimal frontier, that is to say, the optimal benefit-cost ratio.
Let φ_2 and φ_1 be such that $-1 < \varphi_2 < \varphi_1 < 0$. We have

$$C_0(\varphi_1) = \mathcal{C}\left(\varphi_1, L_{\varphi_1}^*\right) \geq \mathcal{C}\left(\varphi_1, L_{\varphi_2}^*\right)$$

Then inequality $\varphi_1 > \varphi_2$ leads to

$$\forall t \geq 0, S_t(\varphi_1) \geq S_t(\varphi_2)$$

and consequently

$$\mathbb{E}\left(\exp\left(-\mu \tau_{L_{\varphi_2}^*}^{\varphi_1}\right)\right) \geq \mathbb{E}\left(\exp\left(-\mu \tau_{L_{\varphi_2}^*}^{\varphi_2}\right)\right)$$

Finally we get $C_0(\varphi_1) \geq \mathcal{C}(\varphi_1, L^*(\varphi_2)) \geq \mathcal{C}(\varphi_2, L^*(\varphi_2)) = C_0(\varphi_2)$ ■

Proof of Proposition 4.1
i) The function $k \in]1, +\infty[\rightarrow \left(\frac{s_0(k-1)}{k}\right)^k$ admits a maximum for $k = \widehat{k}$, defined by:

$$\ln s_0 + \ln \frac{\widehat{k}-1}{\widehat{k}} + \frac{1}{\widehat{k}-1} = 0 \tag{8.1}$$

The study of the Lévy exponent Ψ leads to the existence of a unique value of μ, denoted $\widehat{\mu}$, such that $\widehat{\mu} > \alpha$ and $k_{\varphi}^{(\widehat{\mu})} = \widehat{k}$. Moreover $\widehat{\mu}$ satisfies:

$$\mathbb{E}\left(\exp\left(-\widehat{\mu} \tau_{L_{\widehat{\mu}}^*}\right)\right) = \max_{\mu} \mathbb{E}\left(\exp\left(-\mu \tau_{L_{\mu}^*}\right)\right)$$

Assertion ii) comes from the definition of $\widehat{\mu}$ and the following properties of the Lévy exponent:

$$\forall k \in \left]1, \widehat{k}\right] \quad \forall \varphi \in \left]-1, 0\right], \qquad \lambda \to \Psi\left(k\right)$$

is increasing and

$$\forall k \in \left]1, \widehat{k}\right] \quad \forall \lambda > 0, \qquad \varphi \to \Psi\left(k\right)$$

is decreasing. ■

Proof of Proposition 5.1
Let $\widehat{k}$ be defined by equation (8.1). We have

$$k_0 \leq \widehat{k} \Longleftrightarrow s_0 \geq \widehat{s_0}$$

where

$$\widehat{s_0} = \frac{k_0}{k_0 - 1} \exp\left(-\frac{1}{k_0 - 1}\right)$$

and where k_0 is the limit: $k_0 = \lim_{\varphi \to 0} k_\varphi$.
In order to get the conclusion, it suffices to prove that k_φ is strictly increasing with respect to the jump size φ.
Let $F : \left]-1; 0\right[\times \left]1; +\infty\right[\to \mathbb{R}$ be the function defined as: $F\left(\varphi, k\right) = \Psi\left(k\right) - \mu$ where Ψ is given by euqtaion (2.1).
For any $\left(\varphi, k\right) \in \left]-1; 0\right[\times \left]1; +\infty\right[$ such that $F\left(\varphi, k\right) = 0$, we can easily check that $\Psi^{'}\left(k\right) > 0$.
Using the implicit function theorem, we get:

$$\frac{\partial k}{\partial \varphi} = -\frac{\frac{\partial F}{\partial \varphi}\left(\varphi, k\right)}{\frac{\partial F}{\partial k}\left(\varphi, k\right)}$$

and the inequality $\frac{\partial F}{\partial \varphi}\left(\varphi, k\right) < 0$ implies $\dfrac{\partial k}{\partial \varphi} > 0$. Hence the function $\varphi \mapsto k_\varphi$ is strictly increasing.
■

Proof of Proposition 5.3
We denote by Ψ_Φ and $\Psi_{\mathbb{E}(\Phi)}$ the Lévy exponents of the processes $\left(X_t^\Phi\right)_{t\geq 0}$ and $\left(X_t^{\mathbb{E}(\Phi)}\right)_{t\geq 0}$, where $X_t^\Phi = \ln(\dfrac{S_t^\Phi}{s_0})$ and $X_t^{\mathbb{E}(\Phi)} = \ln\left(\dfrac{S_t^{\mathbb{E}(\Phi)}}{s_0}\right)$.
Let k_Φ (resp. $k_{\mathbb{E}(\Phi)}$) be the unique real number strictly greater than 1 such that $\Psi_\Phi\left(k_\Phi\right) = \mu$ (resp. $\Psi_{\mathbb{E}(\Phi)}\left(k_{\mathbb{E}(\Phi)}\right) = \mu$).
We have

$$\begin{cases} \Psi_\Phi\left(k\right) = \lambda \mathbb{E}\left[\left(1 + \Phi\right)^k\right] - \lambda k \mathbb{E}\left(\Phi\right) + g\left(k\right) \\ \Psi_{\mathbb{E}(\Phi)}\left(k\right) = \lambda\left(1 + \mathbb{E}\left(\Phi\right)\right)^k - \lambda k \mathbb{E}\left(\Phi\right) + g\left(k\right) \end{cases}$$

where $g(k) = \frac{\sigma^2}{2}k^2 + \left(\alpha - \frac{\sigma^2}{2}\right)k - \lambda$.
Jensen inequality implies that

$$\forall k > 1,\ \Psi_{\mathbb{E}(\Phi)}(k) \leq \Psi_{\Phi}(k)$$

Hence

$$k_{\Phi} \leq k_{\mathbb{E}(\Phi)}$$

and from this last inequality, we conclude $L^*_{\Phi} \geq L^*_{\mathbb{E}(\Phi)}$. ■

Proof of Proposition 6.1
Let $\widetilde{\Psi}$ be the Lévy exponent of the process $\left(\widetilde{X}_t\right)_{t\geq 0}$ where $\widetilde{X}_t = \ln(\dfrac{\widetilde{S}_t}{s_0})$ and $\widetilde{k}_{\varphi}$ be the unique real number such that

$$\widetilde{k}_{\varphi} > 1 \qquad \widetilde{\Psi}\left(\widetilde{k}_{\varphi}\right) = \mu.$$

Then, from the equalities

$$\Psi(0) = \widetilde{\Psi}(0) = 0 \text{ and } \Psi(2) = \widetilde{\Psi}(2) = \sigma^2 + \lambda\varphi^2 + 2\alpha,$$

we get

$$\widetilde{k}_{\varphi} \geq k_{\varphi}$$

if and only if

$$\sigma^2 + \lambda\varphi^2 + 2\alpha \geq \mu.$$

and therefore we have

$$L^*_{\varphi} \geq \widetilde{L}^*_{\varphi}$$

■

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