# Optimal Risk Transfer

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February, 1st 2004
(Preliminary version)

**Abstract**

We develop a methodology to optimally design a financial issue to hedge non-tradable risk on financial markets, in the general framework of convex risk measures. The modelling involves a minimization of the risk borne by issuer given the constraint imposed by a buyer who enters the transaction if and only if her risk level remains below a given threshold. Both agents have also the opportunity to invest all their residual wealth on financial markets but may not have the same access to financial investments.

## 1 Introduction

In recent years, a new type of financial instruments (among them, the so-called "insurance derivatives") has appeared on financial markets. Even though they have all the features of financial contracts, they are very different from the classical structures. Their underlying risk is indeed related to a non-financial risk (natural catastrophe, weather event...), which may somehow be connected to more traditional financial risks. Their high level of illiquidity, deriving partly from the fact that the underlying asset is not traded on financial markets, makes them difficult to evaluate and to use.
This securitization phenomenon (i.e. the call on financial markets to manage non-financial risks) is part of a more general phenomenon of convergence and interplay between finance and insurance. The capacity of financial markets to absorb large losses is one of the key arguments for this process. However this global phenomenon of convergence and interplay between insurance and finance raises several questions about the classification of these new products but also about their pricing and management. The characterization of their price is very interesting as it questions the logic of these contracts itself. Indeed, standard techniques for derivatives pricing, using, for instance, replication, are not valid any more because of the specific nature of the underlying risk. Moreover, the determination of the contract structure is a problem in itself: on the one hand, the underlying market related to these risks is extremely illiquid, but on the other hand, the logic of these products itself is closer to that of an insurance policy. Consequently the question of the product design, unusual in finance, is raised.
Finally, this accrued complexity of financial products has lead to an increasing interest in quantitative methods of assessing the risk related to a given financial position. In particular, with

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the development of new diversification instruments for any investor, optimal wealth allocation becomes a more complex question and this question of an efficient quantitative risk assessment becomes crucial. Different authors have recently been interested in defining and constructing a coherent, in some sense, risk measure (see, for instance, Artzner *et al.* [1] or Föllmer and Schied [21]), using a systematic axiomatic approach. The framework developed by these authors will be that of this study.

More precisely, this paper focuses on these problems in a framework where economic agents may take positions on two types of risk: a purely financial risk (or market risk) and a (non-financial) non-tradable risk. The optimal structure of a contract depending on the non-tradable risk and its price are determined. Several authors (see, for instance, D. Becherer [8], M. Davis [15] or M. Musiela and T. Zariphopoulou [30]) have been interested in these new products. However, neither their impact on "classical" investments nor their optimal design are mentioned in the literature. As it is usually the case in finance, these papers focus on the pricing rule of these contracts. In that sense, this work presents a very different approach.

This paper is structured as follows: after having presented some results on convex risk measures and in particular on the inf-convolution of different convex risk measures, we focus on the impact of both the financial market and the non-tradable risk on risk measures and give an explicit characterization of the optimal structure in a particular case and a necessary and sufficient condition for its existence in a general framework. In the last section, we present some concluding remarks.

# 2 Risk measures: basic properties and new developments

When assessing the risk related to a given position, a first natural approach is based on the distribution of the risky position itself. In this framework, the most classical measure of risk is simply the *variance* (or the mean-variance analysis). However, it does not take into account the whole distribution's features (as asymmetry or skewness) and especially it does not focus on the "real" financial risk which is the downside risk. Therefore different methods have been developed to focus on the risk of losses: the most widely used (as it is recommended to bankers by many financial institutions) is the so-called *Value at Risk* (denoted by $V@R$), based on quantiles for the lower tail of the distribution. More precisely, the VaR associated with the position $X$ at a level $\varepsilon$ is defined as

$$V@R_{\varepsilon}(X) = \inf\{k : \mathbb{P}(X + k < 0) \leq \varepsilon\}$$

The $V@R$ corresponds to the minimal amount to be added to a given position to make it acceptable. Such a criterion has several key properties:
$a$) It is decreasing;
$b$) It satisfies the monetary property in the sense that it is translation invariant: $\forall m \in \mathbb{R}$, $V@R_{\varepsilon}(X + m) = V@R_{\varepsilon}(X) - m$;
$c$) It is positive homogeneous as $\forall \lambda \geq 0$, $V@R_{\varepsilon}(\lambda X) = \lambda V@R_{\varepsilon}(X)$.
This last property reflects the linear impact of the size of the position on the risk measure. However, as noticed by Artzner *et al.* [1], this criterion fails to meet a natural consistency requirement: it is not a convex risk measure while the convexity property translates the natural fact that diversification should not increase risk. In particular, any convex combination of "admissible" risks should be "admissible". The absence of convexity of the $V@R$ may lead to arbitrage opportunities inside the financial institution using such criterion as risk measure.

Based on this logic, Artzner *et al.* [1] have adopted a more general approach to risk measurement based. Their paper is essential as it has initiated a systematic axiomatic approach to risk measurement. A *coherent measure of risk* should be convex and satisfy the three key properties of the $V@R$.

## 2.1 Definition and properties

### 2.1.1 Definition

More recently, the axiom of positive homogeneity has been discussed. Indeed, such a condition does not seem to be compatible with the notion of liquidity risk existing on the market as it implies that the size of the risky position has simply a linear impact on the risk measure. To tackle this shortcoming, Föllmer and Schied ([21] and [22]) consider instead *convex risk measures* defined as follows:

**Definition 1** *The functional $\rho$, defined on the set of bounded positions $\mathcal{X}$, is a* convex risk measure in the sense of Föllmer and Schied *if, for any $X$ and $Y$ in $\mathcal{X}$, it satisfies the following properties:*
*a) Convexity:* $\forall \lambda \in [0,1] \quad \rho(\lambda X + (1-\lambda) Y) \leq \lambda \rho(X) + (1-\lambda) \rho(Y)$;
*b) Decreasing monotonicity;*
*c) Translation invariance;*
*d) Continuity from below:* $X_n \nearrow X \Rightarrow \rho(X_n) \searrow \rho(X)$.

Just as for the $V@R$ criterion, the property of translation invariance underlines the translation effect of adding a non-risky position to an existing portfolio (note that, in practice, it may be difficult to define what is a non-risky position).

**Example 2** *i) A natural extension of the $V@R$ criterion is the coherent risk measure, $AV@R$, defined by* $AV@R_\lambda(X) = \frac{1}{\lambda}\int_0^\lambda V@R_\varepsilon(X)\, d\varepsilon$. *It corresponds to the* Average Value at Risk *at level $\lambda \in (0,1)$.*
*ii) Another extension is the Excess $V@R$, or $EV@R$, which is also a coherent risk measure, being defined as* $EV@R_\varepsilon(X) = \mathbb{E}_\mathbb{P}(-X \mid X < -V@R_\varepsilon(X))$.

It is also possible to define $\rho$ in terms of its related acceptance set $\mathcal{A}_\rho$, i.e. the set of all positions which do not require additional capital:

$$\rho(X) = \inf\{m \in \mathbb{R}, m + X \in \mathcal{A}_\rho\} \tag{1}$$

Such a definition makes obvious the interpretation of the risk measure as the minimal amount of capital which, if added to the position, makes the position acceptable.

### 2.1.2 Dual representation of convex risk measures

Both $V@R$ and its extension $AV@R$ are closely related to a probability measure. This enables to quantify, or at least estimate, the risk associated with a particular exposure. A natural question is then: do we have such a representation in terms of probability measures for a general convex risk measure? The answer is positive due to the convexity of the considered framework. Indeed, as shown by Föllmer and Schied [22], any convex risk measure $\rho$ admits the following dual representation:

**Theorem 3** *A dual representation of the convex risk measure $\rho$ is given in terms of a penalty function, $\alpha(\mathbb{Q})$ taking values in $\mathbb{R} \cup \{+\infty\}$:*

$$\forall \Psi \in \mathcal{X} \quad \rho(\Psi) = \sup_{\mathbb{Q} \in \mathcal{M}_1} \{\mathbb{E}_{\mathbb{Q}}(-\Psi) - \alpha(\mathbb{Q})\} \tag{2}$$

*By duality,*

$$\forall \mathbb{Q} \in \mathcal{M}_1 \quad \alpha(\mathbb{Q}) = \sup_{\Psi \in \mathcal{X}} \{\mathbb{E}_{\mathbb{Q}}(-\Psi) - \rho(\Psi)\} \qquad (\geq \rho(0)) \tag{3}$$

*where $\mathcal{M}_1$ is the set of all probability measures on the considered space $(\Omega, \Im)$.*

**Example 4** *i) A classical example of convex risk measure is the* entropic risk measure *defined as:*

$$\forall \Psi \in \mathcal{X} \qquad e_\gamma(\Psi) = \sup_{\mathbb{Q} \in \mathcal{M}_1} (\mathbb{E}_{\mathbb{Q}}(-\Psi) - \gamma h(\mathbb{Q}/\mathbb{P})) = \gamma \ln \mathbb{E}_{\mathbb{P}} \left(\exp\left(-\frac{1}{\gamma}\Psi\right)\right)$$

*where $h(\mathbb{Q}/\mathbb{P})$ is the relative entropy of $\mathbb{Q}$ with respect to the prior probability $\mathbb{P}$, defined by*

$$h(\mathbb{Q}/\mathbb{P}) = \begin{cases} \mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}} \ln \frac{d\mathbb{Q}}{d\mathbb{P}}\right) & \text{if } \mathbb{Q} \ll \mathbb{P} \\ +\infty & \text{otherwise} \end{cases}$$

*Note that this risk measure is closely related to the exponential utility function. More precisely:*

$$e_\gamma(\Psi) = \gamma \ln \mathbb{E}_{\mathbb{P}}(-U(\Psi))$$

*is the opposite of the certainty equivalent of $\Psi$ considering the utility function $U(x) = -\exp\left(-\frac{1}{\gamma}x\right)$ and $\gamma$ the risk tolerance coefficient of the considered agent.*
*ii) As shown by Föllmer and Schied [22], the AV@R measure may also be written as*

$$AV@R_\lambda(\Psi) = \max_{\mathbb{Q} \in \mathcal{Q}_\lambda} \mathbb{E}_{\mathbb{Q}}(-\Psi)$$

*where $\mathcal{Q}_\lambda$ is the set of all probability measures $\mathbb{Q} \ll \mathbb{P}$ whose density $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is $\mathbb{P}$ almost surely bounded by $\frac{1}{\lambda}$.*

### 2.1.3 Super-hedging as risk measures

The definition of the convex risk measure $\rho$ in terms of its acceptance set (Equation (1)) puts a specific accent on the hedging. $\rho(X)$ appears indeed as the smallest amount to add to the position $X$ to obtain the perfect hedge. In particular, $\rho(X + \rho(X)) = \rho(X) - \rho(X) = 0$.
In this sense, In particular, $\rho(X)$ may be reinterpreted as the opposite of the price of the position $X$ for an agent having such a risk measure.

Considering more precisely the hedging problem, we may focus on a particular convex subset $\mathcal{H}$ of $\mathcal{X}$, consisting of potential hedges. The idea is then to dominate an element of $\mathcal{H}$, in other words to find the super-hedge. Hence $\mathcal{A}_{\mathcal{H}} = \{\Psi \in \mathcal{X}, \exists \xi \in \mathcal{H}, \quad \Psi \geq \xi\}$ is the set of all super-hedged positions.
Under some topological assumptions on $\mathcal{H}$, it is also possible to interpret the convex set $\mathcal{A}_{\mathcal{H}}$ as an acceptance set related to a convex risk measure $\nu^{\mathcal{H}}$ defined by:

$$\nu^{\mathcal{H}}(\Psi) = \inf\{m \in \mathbb{R}; \text{such that } \exists \xi \in \mathcal{H},\ m + \Psi \geq \xi\}$$

This measure has been widely studied when $\mathcal{H}$ is associated with a family of hedging strategies in incomplete markets. This corresponds to the buyer price (to within the sign) and the duality relationship was the crucial argument of the studies (see for instance, El Karoui-Quenez [19] or Kramkov [28]).

Note also that when $\mathcal{H}$ is a cone, $\nu^{\mathcal{H}}$ is a coherent risk measure and is simply defined as $\nu^{\mathcal{H}}(\Psi) = \sup_{\mathbb{Q}\in\mathcal{M}_{\mathcal{H}}} \mathbb{E}_{\mathbb{Q}}(-\Psi)$ where $\mathcal{M}_{\mathcal{H}}$ is the set of all probability measures such that $\forall\xi \in \mathcal{H}$, $\mathbb{E}_{\mathbb{Q}}(\xi) \geq 0$.
Note that this set $\mathcal{M}_{\mathcal{H}}$ is close to the familiar notion of "equivalent martingale measures", traditionally used in arbitrage pricing theory.

## 2.2 Inf-convolution

In this section, we focus on convex risk measures in a general framework and underline some key properties for our optimal design problem which will be studied in the next section. In particular, we construct new risk measures as solution of an inf-convolution program involving different convex risk measures. We then study some stability properties and present a particular family of convex risk measures which satisfies a semi-group property.

### 2.2.1 Inf-convolution of risk measures

The problems of the type $\inf\{\rho(\Psi - H), H \in \mathcal{H}\}$ appear very naturally when considering hedging. We will study them under a broader perspective.

**Theorem 5** *Let $\rho_1$ and $\rho_2$ be two convex risk measures with respective penalty functions $\alpha_1$ and $\alpha_2$. Let $\rho_{1,2}$ be the inf-convolution of $\rho_1$ and $\rho_2$ defined as*

$$\rho_{1,2}(\Psi) \triangleq \rho_1 \Box \rho_2(\Psi) = \inf_{H\in\mathcal{X}} \{\rho_1(\Psi - H) + \rho_2(H)\} \qquad \text{with } \rho_{1,2}(0) > -\infty$$

*i) Then $\rho_{1,2}$ is a convex risk measure, which is finite for all $\Psi \in \mathcal{X}$, with penalty function*

$$\forall\mathbb{Q}\in\mathcal{M}_1 \quad \alpha_{1,2}(\mathbb{Q}) = \alpha_1(\mathbb{Q}) + \alpha_2(\mathbb{Q})$$

*ii) If $\rho_2$ is the risk measure generated by a convex subset $\mathcal{H}$ of $\mathcal{X}$, $\nu^{\mathcal{H}}$, then*

$$\rho^{\mathcal{H}}(\Psi) \triangleq \rho\Box\nu^{\mathcal{H}}(\Psi) = \inf\{\rho(\Psi - H), H \in \mathcal{H}\}$$

*is a convex risk measure with penalty function*

$$\forall\mathbb{Q}\in\mathcal{M}_1 \quad \alpha^{\mathcal{H}}(\mathbb{Q}) = \alpha(\mathbb{Q}) + l^{\mathcal{H}}(\mathbb{Q})$$

*If $\mathcal{H}$ is a cone, $\rho^{\mathcal{H}}$ has the penalty function*

$$\alpha^{\mathcal{H}}(\mathbb{Q}) = \begin{cases} \alpha(\mathbb{Q}) & \text{if } \mathbb{Q} \in \mathcal{M}^{\mathcal{H}} \\ +\infty & \text{otherwise} \end{cases}$$

**Proof:**
Please refer to Barrieu-El Karoui [7]. ■

A typical application of Theorem 5*ii*) is the problem of optimal hedging. Let $\mathcal{V}_T$ be a convex subset of $\mathcal{X}$. It represents the set of the gain processes associated with financial investments. Hence, $\rho^m(\Psi) = \inf\{\rho(\Psi - \xi), \xi \in \mathcal{V}_T\}$ is a convex risk measure. This new risk measure corresponds to the risk measure the agent has after having optimally chosen her financial investment/hedge on the market. Hence, it is simply called *market modified risk measure.*
In other words, the introduction of a financial market leads to a modification of the risk measure of the considered agent. This seems very intuitive as financial markets offer the agent an opportunity to diversify their risks and, as a consequence, a new assessment of her initial risks.

### 2.2.2 Risk tolerance coefficients

We now focus on a particular situation where the different agents of the economy assess their respective risks using the same family of risk measures but with different coefficients, translating their respective risk tolerance. More precisely,

**Definition 6** *Let $\rho$ be a convex risk measure with penalty function $\alpha$ and $\gamma > 0$ a real parameter.*
*$\gamma$ is the* risk tolerance coefficient *of any agent having a risk measure $\rho_\gamma$ associated with the penalty function $\alpha_{\rho_\gamma} = \gamma\alpha$.*

As a consequence, the risk measure $\rho_\gamma$ may be expressed as

$$\rho_\gamma (\Psi) = \gamma\rho\left(\frac{1}{\gamma}\Psi\right) \tag{4}$$

and satisfies a dilatation property with respect to the size of the position. Hence, $\rho_\gamma$ is called the *dilated risk measure* associated with $\rho$.

A typical example of dilated risk measure is then the entropic risk measure $e_\gamma (\Psi) = \gamma \ln \mathbb{E}_\mathbb{P}\left(\exp\left(-\frac{1}{\gamma}\Psi\right)\right)$, which may be associated with $e_1$.

This class of risk measures satisfies some nice properties:

**Proposition 7** *Let $\left(\rho_\gamma, \gamma > 0\right)$ be the family of $\rho$-dilated risk measures. Then, the following properties hold:*
*$i)$ For any $\gamma, \gamma' > 0$, $\rho_\gamma \square \rho_{\gamma'} = \rho_{\gamma+\gamma'}$*
*$ii)$ Moreover, $\rho_\gamma \square \rho_{\gamma'} (X) = \inf_F \left\{\rho_\gamma (F) + \rho_{\gamma'} (X - F)\right\} = \rho_\gamma (F^*) + \rho_{\gamma'} (X - F^*)$*
*for $F^* = \frac{\gamma}{\gamma+\gamma'}X$.*
*$iii)$ $\rho$ is a coherent risk measure if and only if $\rho_\gamma \equiv \rho$.*

**Proof:**
$i)$ is an immediate consequence of the definition and characterization of dilated risk measures.
$ii)$ The problem to be solved is simply to find $F^*$ such that the inf-convolution is realized:

$$\inf_F \left\{\rho_\gamma (F) + \rho_{\gamma'} (X + F)\right\} = \rho_{\gamma+\gamma'}(X) = (\gamma + \gamma') .\rho\left(\frac{1}{\gamma + \gamma'}X\right)$$

This is obtained for $F = \frac{\gamma}{\gamma+\gamma'}X$ and

$$\rho_\gamma (F) + \rho_{\gamma'} (X - F) = \gamma\rho\left(\frac{F}{\gamma}\right) + \gamma'\rho\left(\frac{X - F}{\gamma'}\right) = (\gamma + \gamma') .\rho\left(\frac{1}{\gamma + \gamma'}X\right)$$

$iii)$ is directly obtained from both the definition of dilated risk measures and the characterization of coherent risk measures. ■

# 3 Optimal risk transfer in insurance

In this section, we focus on the question of an optimal risk transfer between two economic agents, one of them being exposed towards a non-tradable risk. There is no other investment available for them. A transaction between them is the only way they have to improve their situation.

## 3.1 Framework

More precisely, these agents, respectively denoted $A$ and $B$, are evolving in an uncertain universe modelled by a probability space $(\Omega, \Im, \mathbb{P})$. At a fixed future date $T$, agent $B$ is exposed towards a non-tradable risk $\Theta$ for an amount $X \triangleq X(\Theta, \omega)$ in the scenario $\omega$. She wants to buy a protection. In other words, she calls on agent $A$, who will issue a financial product $F \triangleq F(\Theta, \omega)$. Agent $B$ will buy this structure for a forward price at time $T$ denoted by $\pi$ as to reduce her exposure. We assume that $X$ and $F$ are bounded (i.e. they belong to $\mathcal{X}$).
For the sake of a better understanding, we consider that the terminal wealth of agent $B$ if she does the $F$-transaction is $X - F - \pi$, with a minus sign before $F$ to insist on the fact that $F$ reduces her exposure. Hence, the terminal wealth of agent $A$ will be $F + \pi$.

Both agents assess the risk associated with their respective positions by a convex risk measure, denoted respectively $\rho_A$ and $\rho_B$ (with associated penalty functions $\alpha_A$ and $\alpha_B$).
The issuer, agent $A$, wants to determine the optimal structure $(F, \pi)$ she will sell as to minimize her global risk measure

$$\inf_{F \in \mathcal{X}, \pi} \rho_A(\pi + F)$$

while her constraint related to the buyer's interest in doing the transaction may be written as

$$\rho_B(X - F - \pi) \leq \rho_B(X)$$

This constraint simply imposes a threshold to the risk the buyer accepts to bear. The level $\rho_B(X)$ corresponds indeed to her risk measure when she does not buy any protection.

## 3.2 Optimal pricing rule

Given the convexity of the program, the constraint is bounded for the optimal structure. Using the translation invariance property of the risk measure $\rho_B$, the optimal pricing rule $\pi^*(F)$ of the financial product $F$ is entirely determined by the buyer as

$$\pi^*(F) = \rho_B(X) - \rho_B(X - F)$$

Agent $B$ determines the minimal pricing rule, ensuring the existence of the transaction. $\pi^*(F)$ corresponds to the maximal amount agent $B$ is ready to pay to enter the $F$-transaction and bear the associated risk. In other words, $\pi^*(F)$ corresponds to the *indifference pricing rule* of $F$ for the risk measure $\rho_B$. Agent $B$ is then indifferent, from her risk measure, between doing the $F$-transaction and not doing it.

**Remark 1** *The notion of indifference price has been widely studied in the literature, especially when replicating a terminal cash flow using a utility criterion (cf., for instance, the articles of S.D. Hodges and A. Neuberger [25] or of N. El Karoui and R. Rouge [20]).*

## 3.3 Optimal structure

Using the optimal pricing rule and the translation invariance property of $\rho_A$, the optimization program

$$\inf_{F \in \mathcal{X}, \pi} \rho_A(\pi + F) \qquad \text{subject to} \qquad \rho_B(X - F - \pi) \leq \rho_B(X)$$

may be rewritten as

$$\inf_{F \in \mathcal{X}} \{\rho_A(F) + \rho_B(X - F)\} = \rho_A \square \rho_B(X) \triangleq R_{AB}(X)$$

Solving the optimal risk transfer problem is reduced to solving an inf-convolution problem of two convex risk measure. As a direct consequence of Theorem 5, the value function of the program is a convex risk measure $R_{AB}(X)$ which corresponds to the residual risk measure after the transaction.

### 3.3.1 Dilated risk measures

When both agents simply differ in their risk tolerance but assess their respective exposure using the same type of risk measures, in other words, when they have dilated risk measures associated with the same initial risk measure, the optimal risk transfer is explicitly given as:

**Proposition 8** *If both agents have dilated risk measures associated with the respective risk tolerance coefficients $\gamma_A$ and $\gamma_B$ , then an optimal risk transfer is given by:*

$$\boxed{F^* = \frac{\gamma_A}{\gamma_A+\gamma_B}X}$$

**Proof:**
This proof relies on the general result on dilated risk measures (Proposition 7 *iii*)). ■

The optimal risk transfer is, in this case, consistent with the so-called Borch's Theorem. In his paper [11], K. Borch indeed obtained, in a utility framework, optimal exchange of risk, leading, in particular when considering an exponential utility framework, to familiar linear quota-sharing of total pooled losses.

### 3.3.2 Characterization in the general framework

In a more general framework, it is not possible to find explicitly the optimal risk transfer. It is however feasible to obtain a necessary and sufficient condition to its existence in terms of an optimal probability measure.

Let us first introduce two particular definitions of optimality related to the dual representation of convex risk measures (Theorem 3).

**Definition 9** *Given a convex risk measure $\rho$ and its associated penalty function $\alpha$, we say*
*i) that the probability measure $\mathbb{Q}_\rho^\Psi$ is optimal for $(\Psi,\rho)$ if*

$$\rho(\Psi) = \sup_{\mathbb{Q}\in\mathcal{M}_1} \{\mathbb{E}_\mathbb{Q}(-\Psi) - \alpha(\mathbb{Q})\} = \mathbb{E}_{\mathbb{Q}_\rho^\Psi}(-\Psi) - \alpha\left(\mathbb{Q}_\rho^\Psi\right)$$

*ii) that the exposure $\Psi$ is optimal for $(\mathbb{Q},\alpha)$ if*

$$\alpha(\mathbb{Q}) = \sup_{\Phi\in\mathcal{X}} \{\mathbb{E}_\mathbb{Q}(-\Phi) - \rho(\Phi)\} = \mathbb{E}_\mathbb{Q}(-\Psi) - \rho(\Psi)$$

The following result is similar to a saddle point result, in convex analysis, between optima of a functional and its dual representation:

**Theorem 10** *The necessary and sufficient condition to have an optimal solution $F^*$ to the inf-convolution program*

$$R_{AB}(X) = \inf_F \{\rho_A(F) + \rho_B(X-F)\}$$

*is that there exists an optimal probability measure $\mathbb{Q}_{AB}^X$ for $(X, R_{AB})$ such that $F^*$ is optimal for $\left(\mathbb{Q}_{AB}^X, \alpha_A\right)$ and $X - F^*$ is optimal for $\left(\mathbb{Q}_{AB}^X, \alpha_B\right)$.*

**Proof:**
Some preliminary notations are first introduced: $\mathbf{Q}_{AB}^X \in \mathcal{M}_1$ is an optimal probability measure for $(X, R_{AB})$ and for the sake of simplicity, $\Psi^c = \Psi - \mathbb{E}_{\mathbf{Q}_{AB}^X}(\Psi)$ for any $\Psi \in \mathcal{X}$.
By definition of $\mathbf{Q}_{AB}^X$, we have

$$R_{AB}(X) = \mathbb{E}_{\mathbf{Q}_{AB}^X}(-X) - \alpha_A\left(\mathbf{Q}_{AB}^X\right) - \alpha_B\left(\mathbf{Q}_{AB}^X\right)$$

and

$$\begin{aligned} -R_{AB}(X^c) &= \alpha_A\left(\mathbf{Q}_{AB}^X\right) + \alpha_B\left(\mathbf{Q}_{AB}^X\right) = \sup_{F\in\mathcal{X}}\left\{-\rho_A(F^c)\right\} + \sup_{F\in\mathcal{X}}\left\{-\rho_B(X^c - F^c)\right\} \\ &\geq \sup_{F\in\mathcal{X}}\left\{-\rho_A(F^c) - \rho_B(X^c - F^c)\right\} \\ &= -\inf_{F\in\mathcal{X}}\left\{\rho_A(F^c) + \rho_B(X^c - F^c)\right\} = -R_{AB}(X^c) \end{aligned}$$

In particular,

$$\sup_{F\in\mathcal{X}}\left\{-\rho_A(F^c)\right\} + \sup_{F\in\mathcal{X}}\left\{-\rho_B(X^c - F^c)\right\} = \sup_{F\in\mathcal{X}}\left\{-\rho_A(F^c) - \rho_B(X^c - F^c)\right\}$$

Hence, $F^*$ is optimal for the inf-convolution problem, or equivalently for the program on the right-hand side of this equality, if and only if $F^*$ is optimal for both problems $\sup_{F\in\mathcal{X}}\left\{-\rho_A(F^c)\right\}$ and $\sup_{F\in\mathcal{X}}\left\{-\rho_B(X^c - F^c)\right\}$. ■

# 4 Optimal risk transfer at the interface finance-insurance

We now assume that in order to reduce their respective risk exposure, both agents may also invest in a financial market. This market plays a hedging role for the agents. Note that we use the generic terminology "financial markets" but it may cover a more general investment framework, including, for instance, some insurance investments.

## 4.1 Hedging portfolios and investment strategies

We simply consider a set $\mathcal{V}_T$ of bounded terminal gains[1] $\xi_T$, at time $T$, resulting from a self-financing investment strategy with a null initial value. The key point is that all agents in the market agree on the initial value of these strategies, in other words, the market value at time 0 of any of these strategy is null. In particular, an admissible strategy is associated with a derivative contract with bounded terminal payoff $\Phi$ only if its forward market price at time $T$, $q^m(\Phi)$, is a transaction price for all agents in the market. Then, $\Phi - q^m(\Phi)$ is the bounded terminal gain at time $T$ and is an element of $\mathcal{V}_T$. Typical example of admissible terminal gains $\xi_T$ are then the terminal wealth associated with transactions based on options.
Moreover, in order to have coherent transaction prices, we assume in the following that the market is arbitrage-free. In our framework, this can be expressed by:

$$\exists \mathbb{Q} \sim \mathbb{P} \qquad \forall \xi_T \in \mathcal{V}_T \quad \mathbb{E}_{\mathbb{Q}}(\xi_T) \leq 0 \tag{5}$$

In particular, considering the financial assets, with a terminal payoff $\Phi$ that can be sold and bought, such a condition is written as

$$q^m(\Phi) = \mathbb{E}_{\mathbb{Q}}(\Phi)$$

[1] More precisely, the net potential gain corresponds to the spread between the terminal wealth resulting from the adopted strategy and the capitalized initial wealth.

The probability measure $\mathbb{Q}$ may be viewed as a static version of the classical $\mathcal{V}_T$-martingale measures in a dynamic framework.

The set $\mathcal{V}_T$, previously defined, has to satisfy some properties to be coherent with some investment principles. The first principle, being the "minimal assumption", is the consistency with the diversification principle. In other words, any convex combination of admissible gains should also be an admissible gain. Hence, the set $\mathcal{V}_T$ is always taken as a *convex set*.
Some additional requirements may be introduced, in particular, if agents are not sensitive to the size of the transactions. In this case, $\mathcal{V}_T$ is assumed to be a *cone*. This assumption is relevant for liquid markets leading to the possibility to make the same order for any quantity.

Even if there exists a unique large underlying financial market, both agents may not have however the same access to it. Indeed, both agents may be of very different natures *a priori* and the set of hedging products to which they have access may be completely different, because of specific regulations, of usual strategies...
The set of admissible strategies for Agent $A$ (resp. Agent $B$) is also characterized by the associated terminal gains and is denoted by $\mathcal{V}_T^{(A)}$ (resp. $\mathcal{V}_T^{(B)}$). We assume at least that both $\mathcal{V}_T^{(A)}$ and $\mathcal{V}_T^{(B)}$ are convex sets. Some additional assumptions may also be imposed following the previous arguments.

## 4.2 Optimal risk transfer

### 4.2.1 Optimization program and inf-convolution

The opportunity to invest optimally in a financial market has a direct impact on the risk measure of both agents as mentioned in Subsection 2.2.1. More precisely, agent $A$ now assesses her exposure using the market modified risk measure $\rho_A^m$ defined as the inf-convolution between her initial risk measure $\rho_A$ and $\nu^{\mathcal{V}_T^{(A)}}$, while agent $B$ assesses her risk using $\rho_B^m = \rho_B \Box \nu^{\mathcal{V}_T^{(B)}}$, provided the technical assumptions $\inf_{\xi_B \in \mathcal{V}_T^{(B)}} \rho_B(\xi_B) > -\infty$ and $\inf_{\xi_A \in \mathcal{V}_T^{(A)}} \rho_A(\xi_A) > -\infty$ are satisfied.

Consequently, the optimization program related to the $F$-transaction is simply

$$\inf_{F \in \mathcal{X}, \pi} \rho_A^m(\pi + F) \qquad \text{subject to} \qquad \rho_B^m(X - F - \pi) \leq \rho_B^m(X)$$

As previously, using the cash translation invariance property and binding the constraint at the optimum, the pricing rule of the $F$-structure is fully determined by the buyer as

$$\pi^*(F) = \rho_B^m(X) - \rho_B^m(X - F) \tag{6}$$

Using again the cash translation invariance property, the optimization program simply becomes

$$\inf_{F \in \mathcal{X}} \left( \rho_A^m(F) + \rho_B^m(X - F) - \rho_B^m(X) \right)$$

We are almost in the framework of Theorem 5, apart from the constant $\rho_B^m(X)$. To deal with it, noticing that the value functional obtained in this case should be translated by the constant $-\rho_B^m(X)$ in order to obtain the value function of the previous program, we consider the reduced program

$$\boxed{\begin{array}{rcl} R_{AB}^m(X) & = & \inf_{F \in \mathcal{X}} \left( \rho_A^m(F) + \rho_B^m(X - F) \right) \\ & = & \rho_A^m \Box \rho_B^m(X) = \rho_A \Box \nu^{\mathcal{V}_T^{(A)}} \Box \rho_B \Box \nu^{\mathcal{V}_T^{(B)}}(X) \end{array}} \tag{7}$$

The value functional $R^m_{AB}$ of this program, resulting from the inf-convolution of four different risk measures, may be interpreted as the residual risk measure after all transactions. Writing the optimization program as a succession of inf-convolution problems enables us to deal with it more easily. The commutativity property of the inf-convolution allows us to consider the different programs in any order. In particular, we may consider

$$R^m_{AB}(X) = \nu^{\mathcal{V}_T^{(A)}} \Box \nu^{\mathcal{V}_T^{(B)}} \Box \rho_A \Box \rho_B (X) = \inf_{\xi \in \nu^{\mathcal{V}_T^{(A)}} + \nu^{\mathcal{V}_T^{(B)}}} \rho_A \Box \rho_B (X - \xi)$$

### 4.2.2 Optimal risk transfer

Just as previously, when no diversification opportunity was available to both agents (section 3), we first focus on the situation when both agents simply differ in their risk tolerance coefficients before considering the general framework where the optimal structure cannot be explicitly determined.

**Dilated risk measures** Let us first consider the situation when both agents simply differ in their risk tolerance coefficients. In this particular situation, the residual risk measure $R^m_{AB}(X)$ defined in terms of $\rho_A \Box \rho_B$ may be simply expressed in terms another dilated risk measure $\rho_C$ since

$$\rho_A \Box \rho_B = \rho_{\gamma_A} \Box \rho_{\gamma_B} = \rho_{\gamma_A + \gamma_B} \triangleq \rho_C$$

The problem to be solved

$$R^m_{AB}(X) = \inf_{\xi \in \nu^{\mathcal{V}_T^{(A)}} + \nu^{\mathcal{V}_T^{(B)}}} \rho_C (X - \xi)$$

corresponds to the hedging problem of a "global" agent having a dilated risk measure $\rho_C$. The ability to stay in the same family of risk measures makes the solving much easier.

**Proposition 11** *Assume that both agents have dilated risk measures associated with the respective risk tolerance coefficients $\gamma_A$ and $\gamma_B$.*
1) *If they have the same access to the financial market via a* **cone** *$\mathcal{V}_T$, then an optimal structure is given by:*

$$\boxed{F^* = \frac{\gamma_A}{\gamma_A + \gamma_B} X}$$

2) *If they do not have the same access to the financial market and if $\xi^* = \eta_A^* + \eta_B^*$ is an optimal solution of the Program*

$$\inf_{\xi \in \mathcal{V}_T^{(A)} + \mathcal{V}_T^{(B)}} \rho_C (X - \xi)$$

*with $\eta_A^* \in \mathcal{V}_T^{(A)}$ and $\eta_B^* \in \mathcal{V}_T^{(B)}$. Then*

$$\boxed{F^* = \frac{\gamma_A}{\gamma_A + \gamma_B} X + \frac{\gamma_B}{\gamma_A + \gamma_B} \eta_A^* - \frac{\gamma_A}{\gamma_A + \gamma_B} \eta_B^*}$$

*is an optimal structure.*

**Proof:**
Please refer to Barrieu-El Karoui [7]. ■

Standard diversification will also occur in exchange economies as soon as agents simply differ in their risk tolerance but assess their respective exposure using the same type of risk measures. The regulator has to impose very different rules on agents as to generate risk measures with non-proportional penalty functions if she wants to increase the diversification in the market. In other words, diversification occurs when agents are very different one from the other. This result supports for instance the intervention of reinsurance companies on financial markets in order to increase the diversification on the reinsurance market.
Note that the optimal transfer of the non-tradable risk (in other words, of the risk related to the initial exposure $X$ of agent $B$) is not modified when other investment opportunities are available for both agents. When their access to the financial market is different, there is an additional term corresponding to an exchange of financial products to which each agent has a special access. Using the trade talks, both agents take somehow the opportunity to invest on products not accessible directly for them.

**More general framework** The previous result, Theorem 10, obtained in a simple framework (section 3) when no financial market was available for both agents, may be extended to this framework by simply replacing $\rho_A$ and $\rho_B$ by their respective market modification $\rho_A^m$ and $\rho_B^m$. As previously mentioned, this Theorem 10 gives necessary and sufficient conditions to have an optimal risk transfer and some of its characteristics may be derived. It does not however give an explicit representation of $F^*$. In a non-proportional framework (when at least one risk measure is not dilated), the optimal structure has *a priori* a richer dependence relationship on the initial exposure.
Solving the optimal structure problem in a general framework is more complex. In particular, obtaining an explicit characterization of this transfer requires some technical methods. The use of dynamic programing techniques, in particular Backward Stochastic Differential Equations (BSDEs) and non-linear Partial Differential Equations (PDEs), may help to study risk measures defined by their local specifications. For more details, please refer to Barrieu-El Karoui [6].

# 5 Comments

The framework of convex risk measures enables to set additional constraints or opportunities to economic agents without changing the general framework's characteristics. In particular, a constraint imposed by another agent or the opportunity to invest on a financial market are technically equivalent as they simply lead to a transformation of the initial risk measure of the considered agent into another convex risk measure: both corresponds indeed to the solution of an inf-convolution problem. The penalty function of the generated risk measure is simply made of the sum of the penalty of the initial risk measure and the penalty associated with the constraint.
This ability to generate familiar risk measures is very interesting for the sake of economic interpretation. Modifications in the investment framework of an agent change her perception of risk and consequently generate a new risk measure. The fact that this risk measure still holds the key properties of monotonicity, convexity and translation invariance is consistent with the notion of risk measure itself.

The optimal pricing rule is fully determined by the buyer as it bonds her constraint at the optimum. The obtained price is very similar to an indifference price since it makes the buyer indifferent, from her risk measure point of view, between doing the $F$-transaction and not doing it. This type of pricing rule is usually obtained when studying the problems of replicating a terminal cash flow using a utility criterion (see, for instance, the papers of S.D. Hodges and A. Neuberger [25] or of N. El Karoui and R. Rouge [20]).

The optimal structure obtained in the particular framework of dilated risk measures is simply a proportion of the issuer's initial exposure. The proportional coefficient is the relative risk tolerance of the seller. The result may be seen as an extension of the famous Borch's theorem to convex risk measures with the possibility of alternative investments. The interpretation of this result is quite strong: when both agents have the same access to the financial market and simply differ in their risk tolerance, the underlying logic of this transaction is that of insurance and is far away from that of speculation. There will be a transaction if and only if the buyer has a risk to hedge. This is a logic of insurance and hedging. The sale of this type of contract aims to hedge a real exposure towards a non-financial risk.

All these parameters, especially the risk measures, are probably revealed during the trade talks preceding the transaction, where both agents will reveal some information concerning their anticipation (prior, exposure...) just as their attitudes towards risk. Note that the negotiation takes place at a double level: not only the price is at stake but also the structure (or equivalently, in some ways, the amount). This will lead to a higher probability to reach an agreement between both agents.

Moreover, the obtained results are interesting from a regulation point of view: standard diversification will occur in exchange economies as soon as agents simply differ in their risk tolerance. Because of their differences in nature, regulations, accounting systems... agents assess their risk using risk measures, which are inherently different. A standard proportional sharing of the risk among the different agents is not optimal. Richer, more complex and subtile structures of risk transfer should be looked at in order to improve the efficiency of hedging and the liquidity on these "new" markets.

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